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ROOT SYSTEMS AND CARTAN MATRICES

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1. Introduction. This paper is concerned with two things. The first is a (primarily) geometric axiomatic description for the systems of real roots of Lie algebras arising from (generalized) Cartan matrices. The description is base free and is a natural extension of the well-known axiomatic description of finite root systems. The primary component of our description is an open convex cone which, following Looijenga [**3**], we call the Tits cone. In fact it was Looijenga's paper that led to this axiomatic formulation. Unlike his construction, the dimension of the Tits cone is not tightly connected to the dimension of the Cartan matrix which it eventually yields. This leads us to the second part of the paper which concerns the construction of Cartan matrices of low row rank. We can show that if we have an $l \times l$ Cartan matrix of row rank n, then we can model an axiomatic description of it with a cone of dimension $n + 1$. We show how to construct $l \times l$ Cartan matrices for all $l \geqq 3$ (including $l = \infty$) with row rank 3, thus providing us with root systems of arbitrary rank (rank: $= l$) modelled in 4-dimensional cones.

A generalized Cartan matrix of rank $l(1 \leqq l \leqq \infty)$ is by definition an $l \times l$ matrix $A = (A_{ij})$ of integers satisfying:

$$A_{ij} = 2 \text{ for all } i$$

$$A_{ij} \leqq 0 \text{ if } i \neq j$$

$$A_{ij} = 0 \Leftrightarrow A_{ji} = 0.$$

Notice that the "rank" l is not the same as the row rank in general.

Given a finite $(l < \infty)$ Cartan matrix A we may define two l-dimensional real vector spaces V_0 and H_0 with bases $\alpha_1, \ldots, \alpha_l; \alpha_1^\vee, \ldots, \alpha_l^\vee$ respectively and a bilinear pairing

$$\langle \cdot, \cdot \rangle: V_0 \times H_0 \to \mathbf{R}$$

through

$$\langle \alpha_i, \alpha_j^\vee \rangle = A_{ji}.$$

In general this is degenerate, but it is an easy exercise to see that an

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extension of $\langle \cdot, \cdot \rangle$ to extensions V and H of V_0 and H_0 with dimensions $2l-$row rank (A) is possible so that $\langle \cdot, \cdot \rangle$: $V \times H \rightarrow \mathbf{R}$ is non-degenerate.

Let r_i: $V \rightarrow V$ $(1 \leqq i \leqq l)$ be the linear mapping

$$r_i: v \mapsto v - \langle v, \alpha_i^\vee \rangle \alpha_i.$$

Then $r_1, \ldots, r_l$ are involutions and generate a group, W, called the Weyl group of A. W is a Coxeter group with $r_1, \ldots, r_l$ as Coxeter generators and the relations $(r_i r_j)^{m_{ij}} = 1$ $(i \neq j)$ where the m_{ij} are given by the values of the products $A_{ij}A_{ji}$ according to:

$A_{ij}A_{ji}$	0	1	2	3	$\geqq 4$
m_{ij}	2	3	4	6	∞

See [**7**].

The r_i act by transpose action on H where they are given by

$$r_i: h \mapsto h - \langle \alpha_i, h \rangle \alpha_i^\vee.$$

In this way W acts on H. We have $\langle w\alpha, x \rangle = \langle \alpha, w^{-1}x \rangle$ for all $w \in W$, $\alpha \in V$, $x \in H$.

The set

$$\Delta = \bigcup_{i=1}^{l} \bigcup_{w \in W} w\alpha_i \subset V$$

is called the set of real roots of V relative to the base $\alpha_1, \ldots, \alpha_l$. The terminology arises from the theory of Lie algebras where these elements index certain important one-dimensional subspaces. Obviously Δ depends on the choice of the basis $\alpha_1, \ldots, \alpha_l$, and one of our objects is to provide an axiomatic description of Δ without reference to this particular basis. We might note that there is obviously a set $\Delta^\vee$ of "coroots" obtained by W acting on $\alpha_1^\vee, \ldots, \alpha_l^\vee$.

Each $\alpha \in \Delta$ has an involution r_α associated with it; namely if $\alpha = w\alpha_i$, $r_\alpha = wr_iw^{-1}$. This is independent of the representation of α in the form $w\alpha_i$. So is $\alpha^\vee := w\alpha_i^\vee$, and

$$r_\alpha: \phi \mapsto \phi - \langle \phi, \alpha^\vee \rangle \alpha, \ r_\alpha: x \mapsto x - \langle \alpha, x \rangle \alpha^\vee$$

describes the action of r_α on V and H respectively.

Let

$$C = \{x \in H | \ \langle \alpha_i, x \rangle > 0, \ i = 1, \ldots, l\}.$$

Then C is an open convex cone. Following [**3**] we let

$$\wedge = \text{int} \left(\bigcup_{w \in W} w \, \bar{C} \right),$$

the interior of the union of the $W-$ translates of the closure of C. $\wedge$ is an open convex cone [**3**], stable by W, called the Tits cone.

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Among the properties of Δ and $\wedge$ we single out those listed under I, II, III in Section 2. These are proved in [**3**]. The purpose of Sections 2 and 3 is to prove that they suffice to characterize a Cartan matrix A and Δ as the translates of a base under the action of a Coxeter group W, precisely as we have described it above.

The relation of the dimension of V to the rank of A is rather flexible and does not emerge from the axiomatization as in the construction above. Indeed the dimension of V can be reduced to row rank $(A) + 1$. A problem that arose in our work was that we were unable to prove that rank A is finite even though we assume that the dimension of V is finite. The question thus arose: Do there exist infinite ($l = \infty$) Cartan matrices of finite rank? In attempting to answer this question we produced the construction in Section 6 which produces non-symmetric Cartan matrices of row rank 3 and arbitrarily large l. After the paper was submitted for publication George Maxwell showed us a beautiful construction of infinite symmetric matrices of row rank 3. This is described in Section 9. We are very grateful to Professor Maxwell for this contribution.

2. The set-up. Let V, H be two finite dimensional real vector spaces and suppose that $\langle \cdot, \cdot \rangle : V \times H \to \mathbf{R}$ is a non-degenerate pairing.

Let $0 \neq \alpha \in V$. A *symmetry* in α is an endomorphism r of V having a point-wise fixed hyperplane L and satisfying $r\alpha = -\alpha$. For such a symmetry there is a unique $\alpha^\vee \in H$ such that $\langle L, \alpha^\vee \rangle = 0$, $\langle \alpha, \alpha^\vee \rangle = 2$. In terms of this, the action of r on V is

$$\phi \mapsto \phi - \langle \phi, \alpha^\vee \rangle \alpha.$$

Clearly $r^2 = \mathrm{id}_V$.

Let Δ be a non-empty subset of $V - \{0\}$. We assume that:

(I) For each $\alpha \in \Delta$ there is a symmetry r_α in α such that $r_\alpha \Delta \subset \Delta$.

(II) For all $\alpha, \beta \in \Delta$, $\langle \alpha, \beta^\vee \rangle \in \mathbf{Z}$.

Let $W \subset GL(V)$ be the group generated by the symmetries r_α. For each $\alpha \in \Delta$ let

$$H_\alpha : = \{x \in H | \langle \alpha, x \rangle = 0\}.$$

W acts by transpose action on H and thereby r_α is seen to induce the symmetry in $\alpha^\vee$ with fixed hyperplane H_α.

Use the hyperplanes H_α, $\alpha \in \Delta$, to introduce the standard equivalence relation $\sim$ on H: $x \sim y$ if and only if for each $\alpha \in \Delta$ either x, y lie on H_α or on the same side of H_α [**1**]. The equivalence classes are called *facettes*, those facettes with non-empty interiors are *chambers*, and those facettes which support a hyperplane H_α are called *faces*. Since $r_\alpha H_\beta = H_{r_\alpha \beta}$ for all α, $\beta \in \Delta$, the facettes are permuted by W.

Next we assume:

(III) There exists a W-invariant non-empty open convex cone $\wedge$ which is a union of facettes such that

(a) if $x, y \in \wedge$ then there is a cover by finitely many facettes of the line segment $[x, y]$ in H.

(b) for all $\alpha \in \Delta$ the point-wise stabilizer of H_α in W is finite.

Later we will see that, with the other assumptions, III (a), (b) are equivalent to

(III)′ W acts properly discontinuously on $\wedge$.

$\wedge$ is called a *Tits cone*.

Let $\alpha \in \Delta$. Then for $x \in \wedge$ the line segment joining x and $r_\alpha x$ lies in $\wedge$ and meets H_α. Hence H_α meets $\wedge$.

For $\alpha \in \Delta$, r_α is the only involution in W with H_α as its 1-eigenspace. Indeed if s is such an involution sr_α fixes H_α pointwise, has finite order and hence is semi-simple (III(b)), and has determinant 1, whence $sr_\alpha = \mathrm{id}$. Thus for $\alpha, \beta \in \Delta$, $H_\alpha = H_\beta \Rightarrow \beta \in \mathbf{R}\alpha$. As usual $\mathbf{R}\alpha \cap \Delta \subset \{\pm\frac{1}{2}\alpha, \pm\alpha, \pm 2\alpha\}$ [**1**, VI § 1].

Let

$$\Delta_{\mathrm{red}} : = \{\alpha \in \Delta | \ \alpha/2 \notin \Delta\}.$$

Δ_{red} satisfies I, II, III with the same cone $\wedge$.

THEOREM 1. *Suppose that Δ satisfies I, II, III.*

(1) *W is simply transitive on the chambers of $\wedge$.*

(2) *Let C be a chamber in $\wedge$ and let*

$$\Delta_{\mathrm{red}}{}^{+} : = \{\alpha \in \Delta_{\mathrm{red}} | \ \alpha \textit{ is positive on } C\}.$$

Let

$$\Pi : = \{\alpha \in \Delta_{\mathrm{red}}{}^{+} | \ H_\alpha \textit{ is a wall of } C\}$$

(see below for definition). Π is countable. For $\alpha, \beta \in \Pi$ let $A_{\alpha\beta} = \langle \beta, \alpha^\vee \rangle$. Then $A : = (A_{\alpha\beta})$ is a Cartan matrix. Let Δ' be the root system defined by A on a base $\Pi' : = \{\alpha' | \ \alpha \in \Pi\}$ with Weyl group W' generated by the reflections $r_{\alpha'}$. Then W is generated by the symmetries r_α, $\alpha \in \Pi$ and $W' \cong W$ through $r_{\alpha'} \mapsto r_\alpha$. Furthermore there is a unique (W', W) – equivariant bijective mapping $\Delta' \to \Delta_{\mathrm{red}}$ such that $\alpha' \mapsto \alpha$ for all $\alpha' \in \Pi'$.

3. Proof of theorem 1. It is a straightforward consequence of III(a) that

(a) The decomposition of a closed interval $I \subset \wedge$ by facettes appears as a finite set of points separated by open intervals.

(b) For each $x \in \wedge$ there is a ball B about x in $\wedge$ such that for all $\alpha \in \Delta$,

$$H_\alpha \cap B \neq \emptyset \Rightarrow x \in H_\alpha.$$

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To see this, let $I_1, \ldots, I_n$ ($n = \dim V$) be closed line segments in $\wedge$ in independent directions with $x \in \mathring{I}_j$ for each I. By (a) there is an open interval J_j about x in I_j such that any H_α meeting J_j meets it in x. Let B be an open ball about x in the interior of the convex hull of the $\bar{J}_j$.

(c) For each $x \in \wedge$ there are only finitely many H_α through x. For any $x, y \in \wedge$ there are only finitely many hyperplanes H_α separating x and y.

For the first statement let S be a solid simplex such that $x \in \mathring{S} \subset S \subset \wedge$.

Any H_α passing through x is supported by its set of cuts with the vertices and edges of S of which there are only finitely many.

(d) Δ is countable.

To see this cover $\wedge$ by open balls of the type in (b) and take a countable subcover.

There are at least two chambers in $\wedge$. Let C be one. For each $\alpha \in \Delta$, α as a function on C takes values of constant sign. Partition Δ, Δ_{red} according to sign to get $\Delta^+(C)$, $\Delta_{\text{red}}{}^+(C)$, $-\Delta^+(C)$, and $-\Delta_{\text{red}}{}^+(C)$. A face F is called a *face of* C if $F \cap \bar{C}$ supports F. A face of C supports a unique H_α which is called a *wall of* C. It should be noted that the facettes lying in the closure of a chamber C of $\wedge$ are not in general in $\wedge$. However,

(e) The faces of a chamber in $\wedge$ are in $\wedge$.

Let $F \subset H_\alpha$ be a face of the chamber $C \subset \wedge$. It suffices to see that $F \cap \wedge \neq \emptyset$. Let $x_0 \in C$ and let x_1 be a point of $F \cap \bar{C}$. Then

$$[x_0, x_1) \subset C \text{ and } [r_\alpha x_0, x_1) \subset r_\alpha C$$

which is a chamber in $\wedge$ on the opposite side of H_α. Points close to x_1 on these two intervals have joins in $\wedge$ which meet H_α in F. We note that F is a face of $r_\alpha C$.

(f) Let C be a chamber of $\wedge$ and let $F_\alpha \subset H_\alpha$ be a face of C, $\alpha \in \Delta_{\text{red}}{}^+(C)$. Then

$$F_\alpha = \{x \in H | \langle \beta, x\rangle > 0 \text{ if } \beta \in \Delta_{\text{red}}{}^+(C) - \{\alpha\} \text{ and } \langle \alpha, x\rangle = 0\}.$$

The only other chamber with F_α as a face is $r_\alpha C$.

(g) Let C be a chamber of $\wedge$ and let $x \in C$, $y \in \wedge - \bar{C}$. If B is any ball about y in $\wedge - \bar{C}$ then the cone of rays $\cup_{b \in B} R(x, b)$, where $R(x, b)$ is the ray through x towards b, cuts at least one face of C in an open subset of that face.

Choose one point $b' \in B$ on each ray above and let this set of b''s be denoted by B_0. For each $b' \in B_0$ the interval $[x, b']$ is covered by finitely

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many facettes of which x lies in C whereas $b' \notin C$. Let $F_{b'}$ be the first facette cutting $[x, b']$ after C. This cut is a point $c(b') \in \wedge$ and there is a hyperplane $H_{\alpha(b')}$, $\alpha(b') \in \Delta$, through $c(b')$. Seeing as Δ is countable whereas B_0 is uncountable, $\alpha(b')$ is some fixed $\alpha \in \Delta$ for infinitely many b'. Furthermore α may be taken so that H_α is supported by the $c(b')$ lying in it, for otherwise a countable number of affine spaces of dimension less than n are required to cover our cone of rays which has non-empty interior.

Thus affinely independent $c(b_1), \ldots, c(b_n)$ exist on some H_α. The open simplex S in H_α of which they are the vertices is an open subset of $H_\alpha \cap \wedge$. Now $S \subset \bar{C}$, for if not some hyperplane H_β separates x from some $z \in S$. This hyperplane meets at least one segment $(x, c(b_i))$ contrary to the choice of $c(b_i)$. This proves (g).

(h) Let C be a chamber in $\wedge$ and let $x \in C$. Let

$$y \in \wedge - \bigcup_{\alpha \in \Delta} H_\alpha.$$

Then there is a ball B about y in $\wedge$ such that for all $z \in B$ the number $N_x(z)$ of hyperplanes separating x and z is dominated by $N_x(y)$.

Cover $[x, y]$ with finitely many balls B_j of the type described in (b). For all z close to y, $[x, z] \subset \bigcap B_j$ and $N_x(z) \leqq N_x(y)$.

Let C be a fixed chamber in $\wedge$. Let

$$\Pi = \{\alpha \in \Delta_{\text{red}}{}^+(C) \mid H_\alpha \text{ is a wall of } C\}.$$

By (g), $\Pi \neq \emptyset$. Let W_C be the subgroup of W generated by the r_α, $\alpha \in \Pi$.

(i) W_C is transitive on the chambers of $\wedge$.

Let C' be a chamber of $\wedge$. Let $x \in C$ and let

$$N_x(C') := \min \{N_x(y) \mid y \in C'\} \text{ and}$$

$$N = N(C, C') := \min \{N_x(C') \mid x \in C\}.$$

Use induction on N to show that there is a $w \in W_C$ with $wC = C'$. There is nothing to do if $N = 0$. Assume $N > 0$. Fix $x_0 \in C$, $y \in C'$ for which $N_{x_0}(y) = N$. Put a ball B about y in C' such that $N_{x_0}(z) = N$ for all $z \in B$ (see (h)). By (g) there is a $z \in B$ such that $[x_0, z]$ is on a ray meeting a face F_α of C. Let $x_1, \ldots, x_k$ be the (interior) points of $[x_0, z]$ at which hyperplane cuts occur. By assumption $x_1 \in F_\alpha$. Let $u \in (x_1, x_2)$ (or (x_1, z) if $k = 1$). Then u necessarily lies in $r_\alpha C$. Now $[u, z]$, hence $[r_\alpha u, r_\alpha z]$, cuts only $N - 1$ hyperplanes, so there is a $w \in W_C$ such that $wC = r_\alpha C'$.

(j) $\Delta_{\text{red}} = W\Pi$, $W = W_C$.

For the first statement take any $\beta \in \Delta_{\text{red}}$ and any $x \in H_\beta \cap \wedge$ such that x lies on no other hyperplane. Then x lies in the face F_β of some chamber $C' = wC$ $(w \in W_C)$. Note $w^{-1}F_\beta \subset w^{-1}H_\beta = H_{w^{-1}\beta}$ is a face of C, so $w^{-1}\beta \in \pm\alpha$ for some $\alpha \in \Pi$. For the second statement, note that $w^{-1}r_\beta w$ is a symmetry in H_α hence is r_α (see Section 2). Thus $r_\beta \in W_C$ for arbitrary $\beta \in \Delta_{\text{red}}$ and $W = W_C$.

(k) If $\alpha, \beta \in \Pi$, $\alpha \neq \beta$ then $\langle \alpha, \beta^\vee \rangle \leqq 0$.

Let $x_0 \in C$. Then $r_\beta x_0 \in r_\beta C$ and $r_\beta C$ is defined by the inequalities $\langle \gamma, x \rangle > 0$ for all $\gamma \in \Delta_{\text{red}}^+(C) - \{\beta\}$ and $\langle \beta, x \rangle < 0$. In particular $\langle \alpha, r_\beta x_0 \rangle > 0$ so

$$\langle \alpha, x_0 - \langle \beta, x_0 \rangle \beta^\vee \rangle = \langle \alpha, x_0 \rangle - \langle \beta, x_0 \rangle \langle \alpha, \beta^\vee \rangle > 0.$$

Let x_0 approach face F_α and conclude that

$$-\langle \beta, x_0 \rangle \langle \alpha, \beta^\vee \rangle \geqq 0.$$

With $\langle \beta, x_0 \rangle > 0$ we have $\langle \alpha, \beta^\vee \rangle \leqq 0$.

(l) For $\alpha, \beta \in \Delta$, $\langle \beta, \alpha^\vee \rangle = 0 \Leftrightarrow \langle \alpha, \beta^\vee \rangle = 0$:

$$\langle \beta, \alpha^\vee \rangle = 0 \Rightarrow r_\alpha \beta = \beta \Rightarrow r_\alpha H_\beta = H_\beta \Rightarrow r_\alpha r_\beta r_\alpha = r_\beta.$$

Computing $r_\alpha r_\beta \alpha = r_\beta r_\alpha \alpha$ in two ways gives $\langle \alpha, \beta^\vee \rangle = 0$.

After (k), (l), $A := (\langle \beta, \alpha^\vee \rangle)\ \alpha, \beta \in \Pi$ is a Cartan matrix, though not necessarily finitely dimensioned. The quantity card(Π) is usually called the *rank* of the root system Δ. This obviously leaves something to be desired and in order to avoid confusion we will use the term *row rank* for the maximum number of independent rows.

The rest of the argument follows a well-worn trail. The proof of Bourbaki's Theorem 1, Chapter V, § 3 [**1**] can be taken without change to give:

(m) W is a Coxeter group with Coxeter generators r_α, $\alpha \in \Pi$, and W is simply transitive on the chambers of $\wedge$.

For H_α a wall of C, define

$$P_\alpha = \{w \in W \mid wC \text{ and } C \text{ are on the same side of } H_\alpha\},$$

and define A_α to be the open half space in H defined by H_α and containing C. Then

$$P_\alpha = \{w \in W \mid l(r_\alpha w) > l(w)\}$$

[**1**, Chapter IV, § 1, Proposition 6] and hence for $\alpha \in \Pi$,

$$l(r_\alpha w) < l(w) \Leftrightarrow r_\alpha w \in P_\alpha \Leftrightarrow r_\alpha wC \subset A_\alpha \Leftrightarrow wC \subset r_\alpha A_\alpha.$$

With this equivalence, one may apply Proposition 5 of Chapter V, § 4 [**1**].

Thus for each $X \subset \Pi$ define

$$C_X = \bigcap_{\alpha \in X} H_\alpha \cap \bigcap_{\alpha \in \Pi - X} {}^{\tau}A_\alpha \subset \bar{C}.$$

Once that we know that $C_\phi = C$ (which we show below (p)), C_X can be seen to lie in $\bar{C}$ by considering the line segment joining any point of C_X to any point of C.

(n) For $X, X' \subset \Pi$, $wC_X \cap C_{X'} \neq \emptyset \Rightarrow X = X'$, $C_X = C_{X'}$, $w \in W_X: = \langle r_\alpha | \alpha \in X \rangle$.

(o) For all $x \in \wedge$, the stabilizer $\mathrm{stab}_W(x)$ of x in W is finite.

Considering a small ball about x we see that there are only finitely many chambers whose closures contain x. These are necessarily permuted by $\mathrm{stab}_W(x)$ which is then finite by (m).

In particular notice that (n) and (o) show that the stabilizer of any facette lying in $\wedge$ is finite. This is false for facettes of $\bar{\wedge} - \wedge$.

To finish off the theorem we need to look at the abstract root system Δ' based on $\Pi' = \{\alpha' | \alpha \in \Pi\}$ in the real space V' whose basis is Π' (see Section 1). Let $\pi: V' \to V$ be the linear mapping defined by $\pi(\alpha') = \alpha$, $\alpha \in \Pi$. The Weyl group W' of Δ' is generated by $\{r_{\alpha'} | \alpha' \in \Pi'\}$ where

$$r_{\alpha'} \beta' = \beta' - \langle \beta, \alpha^\vee \rangle \alpha'.$$

Now $W' \cong W$ through $r_{\alpha'} \mapsto r_\alpha$. Then π is a (W', W)-equivariant mapping and in particular

$$\pi(\Delta') = \pi(W'\Pi') = W\Pi = \Delta_{\mathrm{red}}.$$

Finally $\pi|_{\Delta'}$ is injective since

$$\begin{aligned} w_1\alpha_1 = w_2\alpha_2 &\Rightarrow w r_{\alpha_1} w^{-1} = r_{\alpha_2} \text{ (where } w = w_2^{-1}w_1) \\ &\Rightarrow w' r'_{\alpha_1} w'^{-1} = r_{\alpha_2'} \Rightarrow w_1' \alpha_1' = \pm w_2' \alpha_2' \\ &\qquad \text{(where } w_i' \leftrightarrow w_i,\ w' = w_2'^{-1} w_1'). \end{aligned}$$

If $w_1'\alpha_1' = -w_2'\alpha_2'$, then $w_2\alpha_2 = w_1\alpha_1 = -w_2\alpha_2$, which is absurd. Thus $w_1'\alpha_1' = w_2'\alpha_2'$.

(p) $C_\phi = C$.

Let $\beta \in \Delta_{\mathrm{red}}$ and let β' be its preimage in Δ'. Since Δ' is an abstract root system based on Π', β' is a finite sum $\sum n_{\alpha'} \alpha'$ where the $\alpha' \in \Pi'$ and the $n_{\alpha'} \in \mathbf{Z}$ and have a constant sign [**6**]. Thus $\beta = \sum n_{\alpha'} \alpha$, $\alpha = \pi(\alpha')$. If $\beta \in {\Delta_{\mathrm{red}}}^+$ then all the $n_{\alpha'} \geqq 0$ as is seen by computing $\langle \beta, x \rangle$ for some $x \in C$. Now if $x \in C_\phi$ then $\langle \alpha, x \rangle > 0$ for all $\alpha \in \Pi$, so $\langle \beta, x \rangle > 0$ too. Thus $C_\phi \subset C$. The reverse inclusion is obvious.

This concludes the proof of the theorem.

(III)$'$. W acts properly discontinuously on $\wedge$.

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Let U, V be compact subsets of $\wedge$. We have to show that $\{w \in W \mid wU \cap V \neq \emptyset\}$ is finite. Only finitely many facettes in $\wedge$ meet U and V and the relation $wU \cap V \neq \emptyset$ indicates the existence of a pair of facettes in $\wedge$: F with $F \cap U \neq \emptyset$, F' with $F' \cap V \neq \emptyset$ such that $wF = F'$. The same pair can only occur for finitely many $w \in W$ because of the remark after (o).

It is rather easy to see that III′ implies III (a), (b).

4. The dimension of V. We have already pointed out that card(Π) need not be equal to dim V. In this section we show:

THEOREM 2. *Let A be an $l \times l$ Cartan matrix of row rank $n < l$. Then there exist V, H, $\langle\cdot,\cdot\rangle: V \times H \to \mathbf{R}$, and Δ satisfying the axioms of Section 2 and determining A with* dim V = dim $H = n + 1$.

In particular since we can construct Cartan matrices of row rank 3 and arbitrarily large l (see Sections 6, 8, 9) we can find models of these rank l root systems with 4-dimensional cones.

Proof. Let A be an $l \times l$ Cartan matrix of row rank n, $n < l$. Let $\langle\cdot,\cdot\rangle: \tilde{V} \times \tilde{H} \to \mathbf{R}$ be constructed from A as in Section 1. Thus

$$\dim \tilde{V} = \dim \tilde{H} = l + (l - n) \text{ and}$$
$$\tilde{C} = \{x \in \tilde{H} \mid \langle \alpha, x\rangle > 0, \alpha \in \Pi\}$$

determines a Tits cone $\tilde{\wedge} = \operatorname{int}\left(\bigcup_{w \in \tilde{W}} w(\overline{\tilde{C}})\right)$. Let V_0 be the span of Π in $\tilde{V}$ and

$$V_0^{\perp} = \{h \in \tilde{H} \mid \langle V_0, h\rangle = 0\}.$$

Then we have a non-degenerate pairing

$$\langle\cdot,\cdot\rangle: V_0 \times \tilde{H}/V_0^{\perp} \to \mathbf{R}.$$

We claim that Δ, V_0, $\tilde{H}/V_0^{\perp}$ and the image $\wedge_0$ of $\tilde{\wedge}$ in $\tilde{H}/V_0^{\perp}$ satisfy the axioms. Indeed this is all trivial with the possible exception of III (b). For that, let us note that $\tilde{W}$ induces a group W on $\tilde{H}/V_0^{\perp}$. Let $w \in W$ pointwise fix the reflecting hyperplane $H_\alpha{}^0$ in $\tilde{H}/V_0^{\perp}$ for some $\alpha \in \Delta$. Let $x \in H_\alpha{}^0 \cap \wedge_0$ and let $\tilde{x}$ be a preimage of x in $\tilde{\wedge}$. Then

$$wx = x \Rightarrow \tilde{w}\tilde{x} \equiv \tilde{x} \bmod V_0^{\perp},$$

where $\tilde{w}$ is some preimage of w in $\tilde{W}$. Thus for all $\beta \in \Delta$, $\langle \beta, \tilde{w}\tilde{x}\rangle = \langle \beta, \tilde{x}\rangle$ so that $\tilde{x}$ and $\tilde{w}\tilde{x}$ lie in the same facette of $\wedge$. By (n) $\tilde{w}\tilde{x} = \tilde{x}$ and $\tilde{w} \in \operatorname{stab}(\tilde{x})$ which is finite. This proves III(b). Clearly

$$\dim \tilde{H}/V_0^{\perp} = 2l - n - ((2l - n) - l) = l.$$

Let $\tilde{U} \subset \tilde{H}$ be the span of the coroots $\alpha^\vee$, $\alpha \in \Delta$ and let U be its image in $\tilde{H}/V_0^{\perp}$ under the quotient map $-: \tilde{H} \to \tilde{H}/V_0^{\perp}$. Now let $x_0 \in \wedge_0 - U$

be chosen so that distinct elements of Δ take distinct values on x_0. Let $H = \mathbf{R}x_0 + U$ and let $V = V_0/H^{\perp}$, where $H^{\perp} = \{v \in V_0 | \langle v, H\rangle = 0\}$. We have a non-degenerate pairing $\langle \cdot, \cdot \rangle: V \times H \rightarrow \mathbf{R}$. Since

$$r_\alpha x_0 = x_0 - \langle \alpha, x_0 \rangle \overline{\alpha^\vee} \in H$$

and U is W-invariant, we see that H is W-invariant.

By the choice of x_0, Δ is mapped injectively into V. Set $\wedge = \wedge_0 \cap H$. Then $\wedge$ is a W-invariant open convex cone and is the union of facettes (of H). In fact, regarding the facettes of H, given any $x, y \in H$ and preimages $\tilde{x}, \tilde{y} \in \tilde{H}$, $x \sim y$ if and only if $\tilde{x} \sim \tilde{y}$. III(a) and (b) are then clear.

We have $\dim V = \dim H = 1 + \dim U$. Since $\dim \tilde{U} = l$, and $\dim \tilde{U} \cap V_0^{\perp} = l - n$, we find $\dim U = n$. This completes the proof of Theorem 2.

5. Comments. There is a well-known set of axioms for finite root systems (see [**1**], [**8**]). These are simply (I) and (II) together with the assumptions that Δ is finite and Δ spans V. Obviously (III) is satisfied with $\wedge = H$.

In the following, notation is as in Theorem 1.

THEOREM 3 ([**3**]). *Δ is finite if and only if $\wedge = H$.*

Proof. If $\wedge = H$ then $0 \in \wedge$ and $W = \mathrm{stab}_W(0)$ is finite, so Δ is finite.

If Δ is finite then by Theorem 1, Δ is the image of a finite root system Δ'. If C is a chamber of $\wedge$, then the opposite involution $w_0 \in W$ maps C into $-C \subset \wedge$, so $0 \in \wedge$ and $\wedge = H$.

In [**4**] I. G. Macdonald gave an axiomatic description of affine root systems. These occur as affine linear functionals acting on certain affine spaces. The explanation of this in our model is that $\wedge$ is an open half-space and Macdonald's affine space is an affine hyperplane in $\wedge$ (parallel to the defining hyperplane of $\wedge$).

More precisely, we have the following result. (In order to keep matters simple, we have restricted ourselves to indecomposable root systems.)

THEOREM 4. *Suppose that $\wedge$ is an open half-space and the chambers of $\wedge$ have only finitely many faces and define an indecomposable Cartan matrix. Then Δ and A are Euclidean. Conversely if Δ and A are Euclidean with null root ν then $\wedge = \{x \in H | \langle \nu, x\rangle > 0\}$, which is an open half-space.*

Proof. The converse is proved in [**3**].

Suppose that $\wedge$ is an open-half space, say $\{x \in H | \langle \nu, x \rangle > 0\}$ for some $\nu \in V$. Since $r_\alpha \wedge = \wedge$ for all α, and r_α has only ± 1 as eigenvalues, we see that $r_\alpha \nu = \nu$ and in particular $\langle \nu, \alpha^\vee \rangle = 0$. By Theorem 3, Δ is not finite. Suppose that Δ is not Euclidean. By a result of Kac [**2**, 5,

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Lemma 7] there is an "imaginary root" $\phi = \sum_{\alpha \in \Pi} m_\alpha \alpha^\vee$, $m_\alpha \in \mathbf{N}$ such that $\langle \alpha, \phi \rangle < 0$ for all $\alpha \in \Pi$. Then $-\phi \in C \subset \wedge$ and $0 < \langle \nu, -\phi \rangle = 0$, a contradiction. Thus Δ is Euclidean.

As an exercise for the reader we leave:

THEOREM 5. (Notation as in Theorem 1). *Let $\alpha, \beta \in \Delta$ be linearly independent. Then $(\mathbf{R}\alpha + \mathbf{R}\beta) \cap \Delta$ is a root system of rank* 2.

COROLLARY. *If $\alpha, \beta \in \Delta$ then*

$$\langle \alpha, \beta^\vee \rangle > 0 \Leftrightarrow \langle \beta, \alpha^\vee \rangle > 0.$$

Proof. Rank 2 root systems are symmetrizable [**6**]. Thus there is a symmetric bilinear form σ for which the sign of $\langle \alpha, \beta^\vee \rangle$ is the same as the sign of $\sigma(\alpha, \beta)$.

6. Cartan matrices of row rank 3.

Definition. Let A_0 be a Cartan matrix. A Cartan matrix A is called an *extension* of A_0 if A is of the form

$$A = \left(\begin{array}{c|c} A_0 & * \\ \hline * & * \end{array}\right).$$

LEMMA 1. *Let A_0 be an $n \times n$ regular (i.e.* det $A_0 \neq 0$) *Cartan matrix. Let $A = \begin{pmatrix} A_0 & X \\ Y & A_1 \end{pmatrix}$ be an $l \times l$ Cartan matrix which is an extension of A_0. Then row rank $(A) = n$ if and only if $YA_0^{-1}X = A_1$.*

Proof. If row rank $(A) = n$, there exists an $n \times (l - n)$ matrix X' such that

$$\begin{pmatrix} A_0 \\ Y \end{pmatrix} X' = \begin{pmatrix} X \\ A_1 \end{pmatrix}$$

i.e., $A_0X' = X$, $YX' = A_1$. Then

$$X' = A_0^{-1}X, \; A_1 = YX' = YA_0^{-1}X.$$

The converse follows from

$$\begin{pmatrix} E_n \\ YA_0^{-1} \end{pmatrix} A_0(E_nA_0^{-1}X) = A,$$

where E_n is the $n \times n$ unit matrix.

Let A_0 be the 3×3 regular Cartan matrix

$$A_0 = \begin{pmatrix} 2 & -2 & 0 \\ -2(k+1) & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \text{ where } k \in \mathbf{N}.$$

$$\det A_0 = -8k, \; A_0^{-1} = -\frac{1}{2k} \begin{pmatrix} 1 & 1 & 0 \\ k+1 & 1 & 0 \\ 0 & 0 & -k \end{pmatrix}.$$

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THEOREM 6. *There exist Cartan matrices A which have the following properties:*

(i) *A is an extension of A_0,*

(ii) *row rank (A) = row rank $(A_0) = 3$,*

(iii) *A is a $(k + 3) \times (k + 3)$ matrix.*

Proof. Let $A = \begin{pmatrix} A_0 & X \\ Y & A_1 \end{pmatrix}$ be a $(k + 3) \times (k + 3)$ Cartan matrix which is an extension of A_0. Let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$, ${}^tY = (\mathbf{y}_1, \ldots, \mathbf{y}_k)$, where

$${}^t\mathbf{x}_i = (-x_{1i}, -x_{2i}, -x_{3i})$$
$${}^t\mathbf{y}_j = (-y_{1j}, -y_{2j}, -y_{3j})$$
$$A_1 = (A_{ij}) \ (1 \leqq i, j \leqq k),$$

x_{ij}, y_{ij} are non-negative integers and A_1 is a Cartan matrix. From Lemma 1, row rank $(A) = 3$ if and only if

$$(1) \quad YA_0^{-1}X = A_1.$$

Therefore, to construct a Cartan matrix which satisfies (i), (ii) and (iii), we have to find non-negative integers x_{ij}, y_{ij} which satisfy (1) where $A_1 = (A_{ij})$ must be a Cartan matrix.

To simplify the problem, we consider the case where x_{ij}, y_{ij} and A_{ij} are all non-zero and we assume so hereafter.

Let $-\frac{1}{2k}\,{}^t\mathbf{y}_j{}'$ be the j th row of YA_0^{-1}, i.e.,

$${}^t\mathbf{y}_j{}' = (-\{y_{1j} + (k + 1)y_{2j}\}, -(y_{1j} + y_{2j}), ky_{3j}).$$

We rewrite (1) as follows:

$$(1)_{ij} \quad {}^t\mathbf{y}_j{}' \cdot \mathbf{x}_i = (-2k)A_{ji}, \ (1 \leqq i, j \leqq k).$$

$(1)_{ii}$ becomes

$$(2)_{ii} \quad \{y_{1i} + (k + 1)y_{2i}\}x_{1i} + (y_{1i} + y_{2i})x_{2i} - ky_{3i}x_{3i} = -4k.$$

As $y_{3i}x_{3i} \neq 0$, we have

$$(3)_i \quad x_{3i} = \frac{1}{ky_{3i}}[\{y_{1i} + (k + 1)y_{2i}\}x_{1i} + (y_{1i} + y_{2i})x_{2i} + 4k]$$
$$= \frac{1}{k}[\{\bar{y}_{1i} + (k + 1)\bar{y}_{2i}\}x_{1i} + (\bar{y}_{1i} + \bar{y}_{2i})x_{2i}] + \frac{4}{y_{3i}},$$

where

$$\bar{y}_{1i} = y_{1i}/y_{3i}, \ \bar{y}_{2i} = y_{2i}/y_{3i}.$$

As $A_{ji} < 0$ $(j \neq i)$, from $(1)_{ij}$ we have

$$(2)_{ij} \quad \{y_{1j} + (k + 1)y_{2j}\}x_{1i} + (y_{1j} + y_{2j})x_{2i} - ky_{3j}x_{3i} > 0.$$

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Considering $ky_{3j} > 0$, we have

$$(3)_{ij}\ x_{3i} < \frac{1}{k}[\{\bar{y}_{1j} + (k+1)\bar{y}_{2j}\}x_{1i} + (\bar{y}_{1j} + \bar{y}_{2j})x_{2i}].$$

From $(3)_i$ and $(3)_{ij}$, we have

$$(4)_{ij}\ \frac{x_{1i}}{k}\{(\bar{y}_{1j} - \bar{y}_{1i}) + (k+1)(\bar{y}_{2j} - \bar{y}_{2i})\} + \frac{x_{2i}}{k}\{(\bar{y}_{1j} - \bar{y}_{1i}) + (\bar{y}_{2j} - \bar{y}_{2i})\} - \frac{4}{y_{3i}} > 0.$$

Notice that $(4)_{ij}$ is equivalent to $A_{ji} < 0$ when x_{3i} is given by $(3)_i$.

Now let

$$(5)\ \bar{y}_{1i} = \frac{i(i+1)}{2}k,\ \bar{y}_{2i} = (k+1-i)k,\ y_{3i} = 2 \text{ for } 1 \leqq i \leqq k.$$

Then all y_{ji} are positive integers and the ith row of YA_0^{-1} is in $\mathbf{Z}^3$. Furthermore, from $(3)_i$ we have

$$x_{3i} = \left\{\frac{i(i+1)}{2} + (k+1)(k+1-i)\right\}x_{1i} + \left\{\frac{i(i+1)}{2} + (k+1-i)\right\}x_{2i} + 2.$$

So, to prove the theorem it suffices to show the existence of positive integers x_{1i}, x_{2i} $(1 \leqq i \leqq k)$ which satisfy $(4)_{ij}$. For then $YA_0^{-1}X$ defines integral A_1 with the required signs. Substituting (5) in $(4)_{ij}$, we have

$$(4)_{ij}\ \tfrac{1}{2}(j-i)(j+i-2k-1)x_{1i} + \tfrac{1}{2}(j-i)(j+i-1)x_{2i} - 2 > 0;$$

i.e., if $j > i$,

$$x_{2i} > \frac{1}{(j+i-1)}\left\{(2k+1-j-i)x_{1i} + \frac{4}{(j-i)}\right\} = \left(\frac{2k}{j+i-1} - 1\right)x_{1i} + \frac{4}{(j+i-1)(j-i)};$$

if $j < i$,

$$x_{2i} < \left(\frac{2k}{j+i-1} - 1\right)x_{1i} - \frac{4}{(j+i-1)(i-j)}.$$

Let us observe the coefficients of x_{1i} and the constant terms:

If $j > j'$,

$$\frac{2k}{j'+i-1} - 1 > \frac{2k}{j+i-1} - 1.$$

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If $j > j' > i$,

$$\frac{4}{(j'+i-1)(j'-i)} > \frac{4}{(j+i-1)(j-i)}.$$

If $i > j > j'$,

$$\frac{4}{(j+i-1)(i-j)} > \frac{4}{(j'+i-1)(i-j')}.$$

Thus the validity of the inequalities $(4)_{ij}$ follows from $(4)_{i,i-1}$ and $(4)_{i,i+1}$ with the special cases $(4)_{12}$ and $(4)_{k,k-1}$.

For $i = 1$, $(4)_{12}$ becomes $x_{21} > (k-1)x_{11} + 2$. So take $x_{11} = 1$, $x_{21} = k + 2$. For $i = k$, $(4)_{k,k-1}$ becomes $(k-1)x_{2k} + 2 < x_{1k}$. So take $x_{2k} = 1$, $x_{1k} = k + 2$.

For $i \neq 1, k$, from $(4)_{ii-1}$ and $(4)_{ii+1}$, we have

$$(6)_i \quad \left(\frac{k}{i-1} - 1\right)x_{1i} - \frac{2}{i-1} > x_{2i} > \left(\frac{k}{i} - 1\right)x_{1i} + \frac{2}{i}.$$

Comparing both sides of $(6)_i$, it is necessary that

$$x_{1i} > 2(2i-1)/k.$$

The length of the interval

$$\left[\left(\frac{k}{i} - 1\right)x_{1i} + \frac{2}{i}, \left(\frac{k}{i-1} - 1\right)x_{1i} - \frac{2}{i-1}\right]$$

is equal to

$$\frac{k}{i(i-1)}x_{1i} - \frac{2(2i-1)}{i(i-1)},$$

which increases as x_{1i} increases. And the left side $(k/i - 1)\,x_{1i} + 2/i$ of the interval is positive for $x_{1i} > 0$. So there exist positive integers x_{1i} and x_{2i} which satisfy $(6)_i$. This concludes the proof of the theorem.

7. The minimal row rank of Cartan matrices. If A is an $l \times l$ Cartan matrix of row rank 1 or row rank 2, we will see that, by simultaneously permuting the rows and the columns (if necessary), A becomes an extension of a regular Cartan matrix A_0 of same row rank as A, and that $1 \leqq l \leqq 2$ or $2 \leqq l \leqq 4$ respectively.

We begin with the following result.

LEMMA 2. *If $X = (x_{ij})$, $Y = (y_{ij})$ are non-negative (i.e., $x_{ij} \geqq 0$, $y_{ij} \geqq 0$) $n \times n$ matrices such that $A = YX$ is a Cartan matrix, then $A = 2E_n$ and there exists $\sigma \in \mathscr{S}_n$ such that $y_{ij} = 0$ if $i \neq \sigma(j)$, $x_{ij} = 0$. if $j \neq \sigma(i)$.*

Proof. From $A = YX$, if $j \neq i$, $\sum_k y_{ik}x_{kj} \leqq 0$. On the other hand

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$\sum_k y_{ik}x_{kj} \geqq 0$. So $\sum_k y_{ik}x_{kj} = 0$, i.e., $A = 2E_n$. In particular X and Y are regular matrices. Suppose $y_{1i_1} \neq 0$. From $\sum_k y_{1k}x_{kj} = 0$ $(j \neq 1)$, $x_{i_1 j} = 0$ for all $j \neq 1$. As X is regular, $x_{i_1 l} \neq 0$ and there exists only one i_1 such that $y_{1i_1} \neq 0$. We can repeat this argument.

THEOREM 7. *Let A_0 be an $n \times n$ regular Cartan matrix $\neq 2E_n$. Assume that the coefficients of A_0^{-1} are all non-negative. If an $l \times l$ Cartan matrix A is an extension of A_0 such that row rank (A) = row rank (A_0), then $l \leqq 2n - 1$.*

Proof. We prove the following; if there exists a $2n \times 2n$ Cartan matrix A of row rank n which is an extension of an $n \times n$ regular Cartan matrix A_0, then $A_0 = 2E_n$.

Let

$$A = \begin{pmatrix} A_0 & X \\ Y & A_1 \end{pmatrix}.$$

From Lemma 1, $YA_0^{-1}X = A_1$. Applying Lemma 2 to $-YA_0^{-1}, -X$, we have $A_1 = 2E_n$. Then $A_0^{-1} = 2Y^{-1}X^{-1}$, $A_0 = 2^{-1}XY$. As each component of $2^{-1}XY$ is non-negative, $A_0 = 2E_n$. X, Y are of the form described in Lemma 2.

We recall that the condition on A_0 in Theorem 7 always holds if A_0 is of classical type $\neq 2E_n$.

For $A_0 = 2E_n$, there exist $2n \times 2n$ Cartan matrices A_σ of row rank n which are extensions of A_0. A_σ is given as follows.

$$A_\sigma = \begin{pmatrix} 2E_n & X \\ Y & 2E_n \end{pmatrix}$$

where $X = (x_{ij})$, $Y = (y_{ij})$ and $\sigma \in \mathscr{S}_n$ such that $x_{ij} = 0$ for $j \neq \sigma(i)$, $x_{i\sigma(i)} = -a_i$, $y_{ij} = 0$ for $i \neq \sigma(j)$ and $y_{\sigma(i)i} = -4/a_i$ $(a_i, 4/a_i \in \mathbf{N})$. There exists no $(2n + 1) \times (2n + 1)$ Cartan matrix which is of row rank n and an extension of A_σ.

Now, if A is an $l \times l$ Cartan matrix of row rank 1, A is an extension of $A_0 = (2)$, and $l = 1$ or 2. Let us consider the Cartan matrices of row rank 2. We remark that a 3×3 Cartan matrix all of whose principal 2×2 submatrices are of row rank 1 is regular. So we can restrict ourselves to those $l \times l$ Cartan matrices of row rank 2 which are extensions of 2×2 regular Cartan matrices A_0. If $\det A_0 < 0$, all the components of A_0^{-1} are negative, so there exists no such extension by Lemma 1. If $\det A_0 > 0$ and $A_0 \neq 2E_2$, then by Lemma 2, $l \leqq 3$. Therefore we have $2 \leqq l \leqq 4$.

8. Examples. Here are two examples of Cartan matrices of row rank 3 which were constructed by the method of Section 6.

$$\left[\begin{array}{rrr|rrrr} 2 & -2 & 0 & -1 & -3 & -5 & -6 \\ -10 & 2 & 0 & -6 & -5 & -3 & -1 \\ 0 & 0 & 2 & -53 & -86 & -106 & -103 \\ \hline -8 & -32 & -2 & 2 & -2 & -14 & -28 \\ -24 & -24 & -2 & -1 & 2 & -2 & -11 \\ -48 & -16 & -2 & -11 & -2 & 2 & -1 \\ -80 & -8 & -2 & -28 & -14 & -2 & 2 \end{array}\right] \quad (k = 4)$$

$$\left[\begin{array}{rrr|rrrrr} 2 & -2 & 0 & -1 & -2 & -3 & -5 & -7 \\ -12 & 2 & 0 & -7 & -5 & -3 & -2 & -1 \\ 0 & 0 & 2 & -75 & -91 & -101 & -136 & -165 \\ \hline -10 & -50 & -2 & 2 & -1 & -10 & -31 & -58 \\ -30 & -40 & -2 & -1 & 2 & -1 & -13 & -31 \\ -60 & -30 & -2 & -12 & -2 & 2 & -2 & -12 \\ -100 & -20 & -2 & -31 & -13 & -1 & 2 & -1 \\ -150 & -10 & -2 & -58 & -31 & -10 & -1 & 2 \end{array}\right] \quad (k = 5)$$

9. Maxwell's examples. Let $A = (A_{ij})$ be a symmetric $l \times l$ generalized Cartan matrix with the property that (1) removal of any two rows and corresponding columns leaves a generalized Cartan matrix each of whose connected components is finite or Euclidean, and (2) "two" cannot be replaced by "one" in (1). Maxwell [**5**] proves that such a matrix is hyperbolic in the sense that the quadratic form defined by A is of signature $(l - 1, 1)$. Let $V = V_0$ be as in Section 1 and let $(\cdot, \cdot)$: $V \times V \rightarrow \mathbf{R}$ be defined by $(\alpha_i, \alpha_j) = A_{ij}$. Let $\omega_1, \ldots, \omega_l$ be the dual basis to $\alpha_1, \ldots, \alpha_l$, so that $(\omega_i, \alpha_j) = \delta_{ij}$. Maxwell shows [**5**, Theorem 1.6] that for all $i, j \in \{1, \ldots, l\}$ and for all $w, w' \in W$, $(w\omega_i, w'\omega_j) \leqq 0$ unless $w\omega_i = w'\omega_j$.

Consider now the special case:

$$A = \begin{pmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & -2 & -2 \\ -2 & -2 & 2 & -2 \\ -2 & -2 & -2 & 2 \end{pmatrix}.$$

The matrix (ω_i, ω_j) is given by A^{-1} which in this case is $2^{-4}(A)$. The lattice $L: = 4 \sum \mathbf{Z}\, \omega_i$ is W-stable and $(\cdot, \cdot)$ is integral and in fact even valued on L. Now suppose that $x_1, x_2, \ldots$ are distinct elements of the set $W(4\omega_1)$. Then

$$(x_i, x_i) = 16(\omega_1, \omega_1) = 2 \text{ for all } i \text{ and}$$

$$(x_i, x_j) \in \mathbf{Z}_{\leqq 0} \text{ if } i \neq j.$$

Thus the matrix $B = ((x_i, x_j))$ is a symmetric Cartan matrix. It is easy to see that the proof of Theorem 1.6 [**5**] actually gives $(x_i, x_j) < 0$

for all $i \neq j$. Since $x_1, x_2, \ldots$ lie in V the row rank of B is at most 4. In fact if we use the sequence

$$\{4\omega_1, 4r_1\omega_1, 4r_2r_1\omega_1, 4r_1r_2r_1\omega_1, 4r_2r_1r_2r_1\omega_1, \ldots\} = \{4\omega_1, 4(\omega_1 - \alpha_1), 4(\omega_1 - \alpha_1 - 2\alpha_2), 4(\omega_1 - 4\alpha_1 - 2\alpha_2), 4(\omega_1 - 4\alpha_1 - 6\alpha_2), \ldots\}$$

the vectors all lie in $R\omega_1 + R\alpha_1 + R\alpha_2$ and the row rank is 3.

Here is the beginning of the corresponding matrix:

$$\begin{bmatrix} 2 & -14 & -14 & -62 & -62 & \cdots \\ -14 & 2 & -62 & -14 & -142 & \cdots \\ -14 & -62 & 2 & -142 & -14 & \cdots \\ -62 & -14 & -142 & 2 & -254 & \cdots \\ -62 & -142 & -14 & -254 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

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