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Compositio Math. **151** (2015), 416–434.

[doi:10.1112/S0010437X14007441](https://doi.org/10.1112/S0010437X14007441)



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LONDON  
MATHEMATICAL  
SOCIETY  
150 YEARS



# Selmer groups and class groups

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## ABSTRACT

Let  $A$  be an abelian variety over a global field  $K$  of characteristic  $p \geq 0$ . If  $A$  has nontrivial (respectively full)  $K$ -rational  $l$ -torsion for a prime  $l \neq p$ , we exploit the fppf cohomological interpretation of the  $l$ -Selmer group  $\text{Sel}_l A$  to bound  $\#\text{Sel}_l A$  from below (respectively above) in terms of the cardinality of the  $l$ -torsion subgroup of the ideal class group of  $K$ . Applied over families of finite extensions of  $K$ , the bounds relate the growth of Selmer groups and class groups. For function fields, this technique proves the unboundedness of  $l$ -ranks of class groups of quadratic extensions of every  $K$  containing a fixed finite field  $\mathbb{F}_{p^n}$  (depending on  $l$ ). For number fields, it suggests a new approach to the Iwasawa  $\mu = 0$  conjecture through inequalities, valid when  $A(K)[l] \neq 0$ , between Iwasawa invariants governing the growth of Selmer groups and class groups in a  $\mathbb{Z}_l$ -extension.

## 1. Introduction

Fix a prime  $l$ , a number field  $K$ , an abelian variety  $A \rightarrow \text{Spec } K$  of dimension  $g > 0$ , and let  $L/K$  range in some family of finite extensions. Our goal is to relate, in favorable situations, the growth of the  $l$ -torsion subgroup  $\text{Pic}(\mathcal{O}_L)[l]$  of the ideal class group of  $L$  and that of the  $l$ -Selmer group  $\text{Sel}_l A_L$ . Concrete expectations in the case of quadratic  $L/K$  are provided by folklore conjectures.

CONJECTURE 1.1. As  $L/K$  ranges over quadratic extensions,  $\#\text{Pic}(\mathcal{O}_L)[l]$  is unbounded.

CONJECTURE 1.2. As  $L/K$  ranges over quadratic extensions,  $\#\text{Sel}_l A_L$  is unbounded.

*Remarks.*

- 1.3. Conjecture 1.1 is known for  $l = 2$  due to the genus theory of Gauss, but is open for every pair  $(K, l)$  with  $l$  odd; in the  $K = \mathbb{Q}$  case, much more precise predictions are available through the Cohen–Lenstra heuristics [CL84]. The conjectured (but not universally believed) unboundedness of  $\text{rk } A(L)$  would imply Conjecture 1.2, which is known for  $l = 2$  if  $g = 1$  (see [CS10, Theorem 3])<sup>1</sup> and for  $l = 2$  in certain  $g > 1$  cases (see Remarks 1.6 and 4.4), but is open for every pair  $(A, l)$  with  $l$  odd.
- 1.4. If Conjecture 1.1 (respectively 1.2) is known for  $(K, l)$ , it follows for  $(K', l)$  for every finite extension  $K'/K$ , see Lemma 4.5 (respectively 4.6).

We relate the conjectures by proving their equivalence after replacing  $K$  by a finite extension.

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Received 8 August 2013, accepted in final form 28 February 2014, published online 11 November 2014.

2010 Mathematics Subject Classification 11G10 (primary), 11R23, 11R29, 11R58 (secondary).

Keywords: Selmer group, class group, fppf cohomology, Iwasawa theory.

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<sup>1</sup>The case when  $A$  does not have potential complex multiplication is due to Bölling [Böl75, pp. 170–171]. Both papers concern the (stronger) unboundedness of cardinalities of 2-torsion subgroups of Shafarevich–Tate groups.

THEOREM 1.5 (Corollary 4.8).

- (a) If  $A$  has  $\mathbb{Z}/l\mathbb{Z}$  or  $\mu_l$  as a  $K$ -subgroup, then Conjecture 1.1 for  $K$  implies Conjecture 1.2 for  $A$ .
- (b) If  $A[l]$  has a filtration by  $K$ -subgroups with subquotients isomorphic to  $\mathbb{Z}/l\mathbb{Z}$  or  $\mu_l$ , then Conjecture 1.2 for  $A$  implies Conjecture 1.1 for  $K$ .

Remarks.

- 1.6. The known  $l = 2$  case of Conjecture 1.1 therefore proves the  $l = 2$  and  $A(K)[2] \neq 0$  case of Conjecture 1.2. Restricting further to  $g = 1$ , this combines with the unboundedness of  $\#\text{Sel}_2 A_L$  proved by Klagsbrun, Mazur, and Rubin [Kla11, 1.2] under the  $A(K)[2] = 0$  assumption to reprove Conjecture 1.2 in the  $(g, l) = (1, 2)$  case.
- 1.7. Even though the idea that Selmer groups and class groups are related is not new (compare, e.g., [Sch96]), the relationship furnished by Theorem 1.5 is sharper than those available previously. Moreover, it is specific neither to quadratic  $L/K$  nor to number fields: § 4, containing its proof, works in the setting of bounded degree extensions of any fixed global field  $K$ .

### 1.8 The method of the proof

Under the assumptions of (a) (respectively (b)) of Theorem 1.5, we prove lower (respectively upper) bounds for  $\#\text{Sel}_l A$  in terms of  $\#\text{Pic}(\mathcal{O}_K)[l]$  in § 2 (respectively § 3), which we apply after base change to  $L$ . As for the bounds themselves, the fppf cohomological interpretation of Selmer groups provides the idea. To explain it, assume for simplicity that  $A[l] \cong (\mathbb{Z}/l\mathbb{Z})^g \oplus \mu_l^g$  over  $K$ , and let  $S$  be the spectrum of the ring of integers of  $K$  and  $\mathcal{A} \rightarrow S$  the Néron model of  $A$ . The Néron property of  $\mathcal{A}[l]_{S[\frac{1}{l}]}$  (see [Čes14, B.6]) forces  $\mathcal{A}[l]_{S[\frac{1}{l}]} \cong (\mathbb{Z}/l\mathbb{Z})^g \oplus \mu_l^g$ . Passing to cohomology, both  $\#H^1(S[\frac{1}{l}], \mathbb{Z}/l\mathbb{Z})$  and  $\#H^1(S[\frac{1}{l}], \mu_l)$  relate to  $\#\text{Pic}(S)[l]$  (see Lemmas B.1 and B.2), whereas  $H^1(S[\frac{1}{l}], \mathcal{A}[l]) \subset H^1(K, A[l])$  is defined by local conditions [Čes14, 4.2], which at finite places of good reduction agree with those defining  $\text{Sel}_l A \subset H^1(K, A[l])$  [Čes14, 2.5]; it remains to quantify the resulting relation between  $\#H^1(S[\frac{1}{l}], \mathcal{A}[l])$  and  $\#\text{Sel}_l A$ .

### 1.9 The function field case

The argument sketched in 1.8 continues to work for a global function field  $K$  of positive characteristic  $p \neq l$ . For such  $K$ , the analogue of Conjecture 1.2 is known in the case of a constant supersingular elliptic curve:  $\text{rk } A(L)$  is unbounded due to the work of Tate and Shafarevich [TŠ67]. With this input, we prove the analogue of Conjecture 1.1 for every  $K$  containing a fixed finite field  $\mathbb{F}_{p^n}$  (depending on  $l$ ) and consequently, for such  $K$ , also the analogue of Conjecture 1.2 for  $A$  that have  $\mathbb{Z}/l\mathbb{Z}$  or  $\mu_l$  as a  $K$ -subgroup. For precise statements, see Theorem 5.1 and Corollary 5.5. As in the number field case discussed in Remark 1.3, no case of the analogue of Conjecture 1.1 was previously known for odd  $l$  (for  $l = 2$ , see [Mad72, Theorem 3]).

### 1.10 Applications to Iwasawa theory

The bounds mentioned in 1.8 lead to inequalities of Propositions 7.1 and 7.3 between the Iwasawa invariants governing the growth of Selmer groups and class groups in the layers of a  $\mathbb{Z}_p$ -extension. These inequalities imply our main result concerning Iwasawa theory (for a detailed discussion and other results see §§ 6–8).

THEOREM 1.11 (Theorem 8.4). *For a prime  $p$  and a number field  $K$ , to prove the Iwasawa  $\mu = 0$  conjecture for the cyclotomic  $\mathbb{Z}_p$ -extension  $K_\infty/K$ , it suffices to find an abelian  $K$ -variety  $A$  such that:*

- (i)  $A$  has good ordinary reduction at all places above  $p$ ;
- (ii)  $A$  has  $\mathbb{Z}/p\mathbb{Z}$  as a  $K$ -subgroup;
- (iii)  $\text{Hom}(\text{Sel}_{p^\infty} A_{K_\infty}, \mathbb{Q}_p/\mathbb{Z}_p)$  is a torsion module over the Iwasawa algebra and has  $\mu$ -invariant zero.

*Remark 1.12.* In fact, it suffices to find such an  $A$  after replacing  $K$  by a finite extension, see Lemma 7.7. It is not clear, however, how to take advantage of the apparent flexibility of choice: for arbitrary  $K$  and  $p$ , Theorem 1.11(iii) alone seems nontrivial to fulfill. For  $K = \mathbb{Q}$  and  $p = 5$ , the elliptic curve 11A3 satisfies Theorem 1.11(i)–(iii) [Gre99, pp. 120–124]; with this  $A$ , Theorem 1.11 reproves an easy case of the Ferrero–Washington theorem (which is not used in *loc. cit.*, so the argument is not circular).

### 1.13 The contents of the paper

The bounds discussed in 1.8 are essential for all subsequent applications and are proved in §§ 2–3. These technical sections rely on (standard but crucial) auxiliary computations of Appendices A and B. Theorem 1.5 is proved in § 4, which applies the inequalities of §§ 2–3 in families of bounded degree extensions of  $K$ . Both §§ 2–4 and Appendix B work under the assumption that  $K$  is a global field. Special cases of function field analogues of Conjectures 1.1 and 1.2 are proved in § 5. The remaining §§ 6–8 discuss Iwasawa theory (and assume that  $K$  is a number field). The introductory § 6 records how Iwasawa invariants control the growth of  $\text{Pic}(\mathcal{O}_K)[p^m]$  and  $\text{Sel}_{p^m} A$ ; this deviates from the standard discussion that concerns  $\text{Pic}(\mathcal{O}_K)[p^\infty]$  and  $\text{Sel}_{p^\infty} A$ . Inequalities between Iwasawa invariants of class groups and Selmer groups result from the bounds of §§ 2–3 and are the subject of § 7. The final § 8 summarizes the conclusions for the cyclotomic  $\mathbb{Z}_p$ -extension (§§ 6–7 allow an arbitrary  $\mathbb{Z}_p$ -extension).

### 1.14 Notation

The notation set in this paragraph is in place for the rest of the paper; deviations, if any, are recorded in the beginning of each section. Let  $l$  be a prime,  $m$  a positive integer, and  $K$  a global field. If  $\text{char } K = 0$ , let  $S$  be the spectrum of the ring of integers of  $K$ ; if  $\text{char } K > 0$ , let  $S$  be the smooth proper curve over a finite field such that the function field of  $S$  is  $K$ . Let  $v$  be a place of  $K$  and  $K_v$  the corresponding completion; if  $v \nmid \infty$ , then  $v$  identifies with a closed point of  $S$ , and  $\mathcal{O}_v$  and  $\mathbb{F}_v$  denote the ring of integers and the residue field of  $K_v$ . Let  $r_1$  and  $r_2$  be the number of real and complex places of  $K$ . Let  $A \rightarrow \text{Spec } K$  be an abelian variety of dimension  $g > 0$  and  $\mathcal{A} \rightarrow S$  its Néron model. For  $v \in S$ , let  $\Phi_v$  be the étale  $\mathbb{F}_v$ -group scheme of connected components of  $\mathcal{A}_{\mathbb{F}_v}$ . For a finite extension  $L/K$ , the formation of  $S$ ,  $\mathcal{A}$ ,  $\Phi_v$  is *not* compatible with base change, and we denote by  $S^L$ ,  $\mathcal{A}^L$ ,  $\Phi_w^L$  their analogues over  $L$  (note that  $S^L$  is the normalization of  $S$  in  $L$ ).

### 1.15 Conventions

To simplify the computations,  $\gtrsim_{A,L,\dots}$  and  $\lesssim_{A,L,\dots}$  denote inequalities up to implied constants that depend only on the indicated parameters (note that  $A$ , being a morphism  $A \rightarrow \text{Spec } K$ , includes dependence on  $K$ ); when no parameters are indicated, those used last are taken. Also,  $\sim$  stands for ‘ $\gtrsim$  and  $\lesssim$ ’. When needed (e.g. for forming composita or intersections), a choice of a separable closure  $\overline{F}$  of a field  $F$  is made implicitly (and compatibly for overfields). The étale fundamental group of an integral scheme is based at a geometric generic point. Fppf cohomology is denoted by  $H^i$ ; when the coefficient sheaf is a smooth group scheme, the identification with étale cohomology [Gro68, 11.7 1°]) is implicit and similarly for further identifications with Galois

cohomology. Fppf cohomology with compact supports that takes into account infinite primes [Mil06, III.0.6(a)] is denoted by  $H_c^i$ . All quotients are taken in the big fppf topos, and  $X_{\text{fppf}}$  denotes the big fppf site of the scheme  $X$ . The  $l^m$ -Selmer group  $\text{Sel}_{l^m} A$  is the preimage of  $\prod_v A(K_v)/l^m A(K_v) \subset \prod_v H^1(K_v, A[l^m])$  in  $H^1(K, A[l^m])$ , where fppf cohomology is necessary if  $l = \text{char } K$ . For a nonempty open  $U \subset S$ , the number of closed points of  $S$  not in  $U$  is  $\#(S \setminus U)$ . If  $\text{char } K = 0$ , then  $\text{Pic}_+(S)$  is the narrow ideal class group of  $K$ ; if  $\text{char } K > 0$ , then  $\text{Pic}_+(S) := \text{Pic}(S)$ . For an integer  $n$  and a scheme  $X$ , the open subscheme on which  $n$  is invertible is  $X[\frac{1}{n}]$ .

**2. Lower bounds for Selmer groups in terms of class groups**

Mimicking [Mil06, before II.3.4], for a nonempty open  $U \subset S$  and a sheaf  $\mathcal{F}$  on  $U_{\text{fppf}}$ , we define

$$D^1(U, \mathcal{F}) := \text{Im}(H_c^1(U, \mathcal{F}) \rightarrow H^1(U, \mathcal{F})).$$

PROPOSITION 2.1. *If  $U \subset S$  is a nonempty open subscheme for which  $A$  has semiabelian reduction at all  $v \in U$  with  $\text{char } \mathbb{F}_v = l$ , then*

$$\begin{array}{ccc} D^1(U, \mathcal{A}[l^m]) & \hookrightarrow & H^1(K, A[l^m]) \\ \downarrow & & \downarrow \\ \prod_{v \in U} H^1(\mathcal{O}_v, \mathcal{A}[l^m]) \times \prod_{v \notin U} 0 & \hookrightarrow & \prod_v H^1(K_v, A[l^m]) \end{array}$$

is Cartesian. If, moreover,  $l \neq \text{char } K$  or  $U = S$ , then, taking intersections inside  $H^1(K, A[l^m])$ ,

$$\begin{aligned} \# \left( \frac{D^1(U, \mathcal{A}[l^m])}{D^1(U, \mathcal{A}[l^m]) \cap \text{Sel}_{l^m} A} \right) &\leq \prod_{v \in U} \frac{\#\Phi_v(\mathbb{F}_v)}{\#(l^m \Phi_v)(\mathbb{F}_v)}, \\ \# \left( \frac{\text{Sel}_{l^m} A}{D^1(U, \mathcal{A}[l^m]) \cap \text{Sel}_{l^m} A} \right) &\leq \prod_{v \in U} \frac{\#\Phi_v(\mathbb{F}_v)}{\#(l^m \Phi_v)(\mathbb{F}_v)} \cdot \prod_{v \in S \setminus U} (l^{mg[K_v:\mathbb{Q}_l]} \cdot \#A(K_v)[l^m]) \cdot \prod_{\substack{\text{real } v \\ l=2}} \#\pi_0(A(K_v)), \end{aligned}$$

where  $[K_v : \mathbb{Q}_l] := 0$  unless  $K_v$  is a finite extension of  $\mathbb{Q}_l$ .

*Proof.* For the diagram, use the similar description of  $H^1(U, \mathcal{A}[l^m]) \subset H^1(K, A[l^m])$  [Čes14, 4.2 and B.5] and the compactly supported cohomology exact sequence [Mil06, III.0.6(a)]. For the inequalities, compare the defining local conditions by means of [Čes14, 2.5(a)] and Proposition A.1.  $\square$

THEOREM 2.2. *Suppose that  $A[l^m]$  has a  $K$ -subgroup  $G \cong \bigoplus_{i \in I} \mathbb{Z}/l^{a_i} \mathbb{Z} \oplus \bigoplus_{j \in J} \mu_{l^{b_j}}$  with  $a_i, b_j \geq 1$ .*

(a) *Set  $r := r_1$  if  $l = 2$ , and  $r := 0$  if  $l \neq 2$ ; also  $\{v \mid l\} := \emptyset$  if  $\text{char } K > 0$ . If  $l \neq \text{char } K$ , then*

$$\#\text{Sel}_{l^m} A \gtrsim_{g,l,m} \frac{\prod_i \#\text{Pic}(S[\frac{1}{l}])[l^{a_i}] \prod_j \#\text{Pic}_+(S)[l^{b_j}]}{2^{r \cdot \#J} \cdot l^{r_2 \sum_j b_j} \cdot \prod_{v \in S[\frac{1}{l}]} (\#\Phi_v(\mathbb{F}_v) / \#(l^m \Phi_v)(\mathbb{F}_v)) \cdot \prod_j \prod_{v \mid l} \#\mu_{l^{b_j}}(K_v)}.$$

(b) *If  $J = \emptyset$  and  $A$  has semiabelian reduction at all  $v$  with  $\text{char } \mathbb{F}_v = l$ , then*

$$\#\text{Sel}_{l^m} A \gtrsim_{g,l,m} \frac{\prod_i \#\text{Pic}(S)[l^{a_i}]}{\prod_{v \nmid \infty} (\#\Phi_v(\mathbb{F}_v) / \#(l^m \Phi_v)(\mathbb{F}_v))}.$$

*Proof.* We give the similar proofs together. For (a), set  $U := S[\frac{1}{l}]$ ; for (b), set  $U := S$ . By Proposition (b),  $\# \text{Sel}_l^m A \geq \# D^1(U, \mathcal{A}[l^m]) \cdot (\prod_{v \in U} (\#\Phi_v(\mathbb{F}_v)/\#(l^m\Phi_v)(\mathbb{F}_v)))^{-1}$ . Let  $\mathcal{G} \rightarrow U$  be the group smoothening of the schematic image of  $G \rightarrow \mathcal{A}_U$ ; by [BLR90, 7.1/6],  $\mathcal{G}$  is the Néron model of  $G$ , hence  $\mathcal{G} \cong \bigoplus_i \mathbb{Z}/l^{a_i}\mathbb{Z} \oplus \bigoplus_j \mu_{l^{b_j}}$ . The  $U$ -homomorphism  $\mathcal{G} \xrightarrow{f} \mathcal{A}[l^m]$  has generic fiber  $G \hookrightarrow A[l^m]$ ; moreover,  $H^1(U, \mathcal{G}) \subset H^1(K, G)$  and  $H^1(U, \mathcal{A}[l^m]) \subset H^1(K, A[l^m])$  (see [Čes14, A.5 and B.5]). Therefore,  $\# \text{Ker } H^1(f) \lesssim 1$ , giving  $\# D^1(U, \mathcal{A}[l^m]) \gtrsim \# D^1(U, \mathcal{G})$ . The conclusion follows by combining the obtained inequalities with Lemmas B.3 and B.4 and the exact sequence [Mil06, III.0.6(a)].  $\square$

### 3. Upper bounds for Selmer groups in terms of class groups

Assume in this section that  $l \neq \text{char } K$ . In contrast to the lower bounds in Theorem 2.2, we do not use implied constants in the upper bounds in Theorem 3.1. This makes the inequalities less pleasant but has the advantage of providing explicit lower bounds on the cardinalities of  $l$ -torsion subgroups of class groups when Theorem 3.1 is applied to an abelian variety of high rank. For instance, one may hope for a practical approach to Theorem 1.5: by finding an elliptic curve  $E \rightarrow \text{Spec } \mathbb{Q}$  for which  $E(\mathbb{Q})[l] \neq 0$  with  $l$  odd and a quadratic  $F/\mathbb{Q}$  for which  $\text{rk } E(F)$  is large, one would get a quadratic number field with large class group  $l$ -rank  $r_l := \dim_{\mathbb{F}_l} \text{Pic}(S^F)[l]$ . The current records (among quadratic  $F$ )  $r_3 = 6$  [Que87] and  $r_5 = 4$  [Sch83] exploit relations with elliptic curves.

**THEOREM 3.1.** *Fix a nonempty open  $U \subsetneq S[\frac{1}{l}]$  for which  $\mathcal{A}_U \rightarrow U$  is an abelian scheme. Set  $r := r_1$  if  $l = 2$ , and  $r := 0$  if  $l \neq 2$ ; also  $[K : \mathbb{Q}] := 0$  if  $\text{char } K > 0$ . If  $A[l^m]$  has a filtration by  $K$ -subgroups  $N_j$  with subquotients isomorphic to  $\mathbb{Z}/l^{a_i}\mathbb{Z}$  or  $\mu_{l^{b_j}}$  with  $a_i, b_j \geq 1$ , then*

$$\begin{aligned} \# \text{Sel}_l^m A \leq & \prod_i \#(\text{Pic}_+ S/l^{a_i} \text{Pic}_+ S) \prod_j \# \text{Pic}(U)[l^{b_j}] \cdot l^{[K:\mathbb{Q}]\sum_i a_i + (r_1+r_2+\#(S \setminus U)-1)\sum_j b_j} \\ & \cdot \prod_j \# \mu_{l^{b_j}}(K) \cdot \prod_i \prod_{v \in S \setminus U} \# \mu_{l^{a_i}}(K_v), \end{aligned}$$

and also

$$\begin{aligned} \# \text{Sel}_l^m A \leq & \prod_i \#(\text{Pic } U/l^{a_i} \text{Pic } U) \\ & \cdot \prod_j \#(\text{Pic}_+ S/l^{b_j} \text{Pic}_+ S) \cdot l^{mg[K:\mathbb{Q}] + (\#(S \setminus U)-1)\sum_i a_i + (r_1+r_2-1)\sum_j b_j} \\ & \cdot \prod_i 2^r \cdot \prod_{\substack{\text{real } v \\ l=2}} \# \pi_0(A(K_v))^{-1} \cdot \# A[l^m](K) \cdot \prod_j \prod_{v \in S \setminus U} \# \mu_{l^{b_j}}(K_v). \end{aligned}$$

*Proof.* Let  $\mathcal{N}_j$  be the schematic image of  $N_j \rightarrow \mathcal{A}[l^m]_U$ . By [EGAI, 9.5.5–6], [EGAIV(2), 2.8.5–6], [TO70, Lemma 5, p. 17], and finiteness of  $\mathcal{A}[l^m]_U$ , the  $\mathcal{N}_j$  filter  $\mathcal{A}[l^m]_U$  by finite étale  $U$ -subgroups. Due to finiteness, the étale subquotients  $\mathcal{N}_{j+1}/\mathcal{N}_j$  are the Néron models of the  $N_{j+1}/N_j$  and hence identify with  $\mathbb{Z}/l^{a_i}\mathbb{Z}$  or  $\mu_{l^{b_j}}$ . Therefore, Lemmas B.1 to B.4 bound  $\# H^1(U, \mathcal{A}[l^m])$  and  $\# H_c^1(U, \mathcal{A}[l^m])$  through cohomology sequences, and the claimed inequalities follow by combining these bounds with the following observations.

- (i) For the first inequality: by [Čes14, 2.5(d) and 4.2],  $\# \text{Sel}_{l^m} A \leq \# H^1(U, \mathcal{A}[l^m])$ .
- (ii) For the second: by [Mil06, III.0.6(a)] and Proposition 2.1, writing  $\widehat{H}^i$  for Tate cohomology,

$$\# D^1(U, \mathcal{A}[l^m]) \leq \# H_c^1(U, \mathcal{A}[l^m]) \cdot \prod_{v \in S \setminus U} \# A(K_v)[l^m]^{-1} \cdot \prod_{v | \infty} \# \widehat{H}^0(K_v, A[l^m])^{-1} \cdot \# A[l^m](K),$$

and

$$\frac{\# \text{Sel}_{l^m} A}{\# D^1(U, \mathcal{A}[l^m])} \leq l^{mg[K:\mathbb{Q}]} \cdot \prod_{v \in S \setminus U} \# A(K_v)[l^m] \cdot \prod_{\substack{\text{real } v \\ l=2}} \# \pi_0(A(K_v));$$

moreover, if  $l = 2$  and  $v$  is real, then, by Proposition A.1(c) and [GH81, 1.3],

$$\# \widehat{H}^0(K_v, A[l^m]) = \# H^1(K_v, A[l^m]) = \# \pi_0(A(K_v))^2. \quad \square$$

*Remarks.*

- 3.2. The two bounds are incomparable in general; they yield different bounds in Proposition 7.3.
- 3.3. When  $\mathbb{Z}/l^{\alpha_i}\mathbb{Z} \cong \mu_{l^{\alpha_i}}$  over  $K$ , the two interpretations of the corresponding subquotient result in different right-hand sides of the inequalities of Theorem 3.1, and hence also in the flexibility of choosing the best bound. Similarly for  $\mu_{l^{\beta_j}}$ .

#### 4. Growth of Selmer groups and class groups in extensions of bounded degree

THEOREM 4.1. *Let  $L/K$  be an extension of degree at most  $d$ .*

(a) *If either:*

- (i)  *$A$  has  $\mathbb{Z}/l\mathbb{Z}$  or  $\mu_l$  as a  $K$ -subgroup, and  $l \neq \text{char } K$ ; or*
- (ii)  *$A$  has everywhere semiabelian reduction and  $\mathbb{Z}/l\mathbb{Z}$  as a  $K$ -subgroup,*

*then*

$$\# \text{Sel}_{l^m} A_L \gtrsim_{A,d,l} \# \text{Pic}(S^L)[l].$$

(b) *If  $l \neq \text{char } K$  and  $A[l]$  has a filtration with subquotients isomorphic to  $\mathbb{Z}/l\mathbb{Z}$  or  $\mu_l$ , then*

$$\# \text{Sel}_l A_L \lesssim_{A,d,l} \# \text{Pic}(S^L)[l]^{2g}.$$

*Proof.*

(a) This follows from Theorem 2.2 since, letting  $w$  denote a place of  $L$ , we have:

- (1)  $\# \text{Pic}(S^L[\frac{1}{l}])[l] \sim_{K,d,l} \# \text{Pic}(S^L)[l]$  if  $\text{char } K \neq l$ , because  $\#(S^L \setminus S^L[\frac{1}{l}])$  is bounded;
- (2)  $\# \text{Pic}_+(S^L)[l] \sim_{K,d} \# \text{Pic}(S^L)[l]$ , because the number of real  $w$  is bounded;
- (3) there is a bounded number of  $w$ 's of bad reduction for  $A$ ; moreover, for each such  $w$ ,
  - ( $\alpha$ ) if  $\text{char } K = 0$ , up to isomorphism there are only finitely many possibilities for  $A_{L_w}$ ,
  - ( $\beta$ ) in general,  $\# \Phi_w^L(\mathbb{F}_w) / \#(l\Phi_w^L)(\mathbb{F}_w) \leq \# \Phi_w^L[l]$ , and, if  $l \neq \text{char } \mathbb{F}_w$  or the reduction is semiabelian, then  $\# \Phi_w^L[l] \leq \# \mathcal{A}^L[l]_{\mathbb{F}_w} \sim_{g,l} 1$ , as is seen by inspecting the finite part [EGAIV(4), 18.5.11 c)] of the quasi-finite separated  $\mathcal{A}^L[l]_{\mathcal{O}_w}$ .

(b) This follows from (either part of) Theorem 3.1: one argues as in parts (1) and (2) and uses

$$(4) \#(\text{Pic}_+ S^L / l \text{Pic}_+ S^L) \sim_{K,d,l} \# \text{Pic}(S^L)[l]. \quad \square$$

COROLLARY 4.2. *If either (i) or (ii) of Theorem 4.1(a) hold, then  $\#\text{Sel}_l A_L$  is unbounded as  $L/K$  ranges over degree  $l$  extensions.*

*Proof.* Indeed,  $\#\text{Pic}(S^L)[l]$  is unbounded [Mad72, Theorem 3]. □

COROLLARY 4.3. *If  $l \neq \text{char } K$ , then  $\#\text{Sel}_l A_L$  is unbounded as  $L/K$  ranges over extensions of degree at most  $l^{2g+1} - l$ .*

*Proof.* Indeed,  $A$  acquires a nontrivial  $l$ -torsion point over an extension of degree at most  $l^{2g} - 1$ . □

Remark 4.4. There are several results in the literature concerned with proving the unboundedness of  $\#\text{III}(A_L)[l]$  (and hence that of  $\#\text{Sel}_l A_L$ ) as  $L$  ranges over degree  $l$  extensions of  $K$ : [CS10, Theorem 3] treats the case  $\dim A = 1$  and  $l \neq \text{char } K$ , whereas [Cre11, Theorem 1.1], improving [Cla04, Theorem 7], allows arbitrary dimension but imposes restrictions (which are satisfied after passing to a finite extension) on the Néron–Severi group of  $A$ . In contrast, Corollary 4.2 has no dimension or Néron–Severi assumptions but constrains  $A[l]$  and only gives Selmer growth.

If  $l \neq \text{char } K$ , the assumptions of (a) and (b) in Theorem 4.1 are satisfied after passing to a suitable finite extension  $K'/K$ ; standard lemmas 4.5 and 4.6, which are also used in §7, clarify in Corollary 4.8 how this affects the unboundedness questions.

LEMMA 4.5. *Let  $L$  be a global field and  $L'/L$  an extension of degree at most  $d$ . Then*

$$\#\text{Pic}(S^{L'})[n] \gtrsim_{d,n} \#\text{Pic}(S^L)[n].$$

*Proof.* For number fields, the claim is clear from the theory of the Hilbert class field: if  $H/L$  is an unramified abelian extension with Galois group killed by  $n$ , then so is  $HL'/L'$ , for which  $[HL' : L'] \geq (1/d)[H : L]$ . The proof in the function field case is the same—the link to unramified abelian extensions is provided by Lemma B.1(a) applied to the prime factors of  $n$ :  $\#\text{Pic}(S^L)[n] \sim_n \#H^1(S^L, \mathbb{Z}/n\mathbb{Z}) = \#\text{Hom}(\pi_1^{\text{ét}}(S^L), \mathbb{Z}/n\mathbb{Z}) = [H_L : L]$  where  $H_L/L$  is the maximal (in  $\bar{L}$ ) unramified abelian extension with Galois group killed by  $n$ , and similarly for  $L'$ . □

LEMMA 4.6. *Let  $L$  be a global field,  $A$  a  $g$ -dimensional abelian variety over  $L$ , and  $L'/L$  an extension of degree at most  $d$ . If  $\text{char } L \nmid n$ , then*

$$\#\text{Sel}_n A_{L'} \gtrsim_{d,g,n} \#\text{Sel}_n A.$$

*Proof.* Let  $R_{L'/L}$  denote the restriction of scalars. By [CGP10, A.5.1–2, A.5.4(1), A.5.7],

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A[n] & \longrightarrow & A & \xrightarrow{n} & A & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & R_{L'/L}(A[n]_{L'}) & \longrightarrow & R_{L'/L}(A_{L'}) & \xrightarrow{n} & R_{L'/L}(A_{L'}) & \longrightarrow & 0
 \end{array} \tag{4.6.1}$$

is a morphism of short exact (in the big étale site of  $L$ ) sequences of smooth  $L$ -group schemes. Moreover,  $R_{L'/L}(A[n]_{L'})$  is finite étale with  $\#R_{L'/L}(A[n]_{L'}) \sim 1$ : for separable  $L'/L$ , this is evident after base change to  $L'$ , [CGP10, A.5.13] handles the purely inseparable case, and in general one uses the transitivity of  $R_{L'/L}$ . Consequently,  $\#\text{Ker } H_{\text{ét}}^1(a) \sim 1$ , and since



$H_{\text{ét}}^i(L, R_{L'/L}(A[n]_{L'})) \cong H_{\text{ét}}^i(L', A[n])$  (see [SGA4 $\frac{1}{2}$ , p. 24, II.3.6]), it remains to see that  $H_{\text{ét}}^1(a)$  respects the  $n$ -Selmer subgroups. This is evident from the compatibility of the formation of (4.6.1) with any base change and the well-known  $L' \otimes_L L_v \cong \prod_{w|v} L'_w$  (see [Ser79, II. § 3, Theorem 1(iii)]) for a place  $v$  of  $L$ .  $\square$

*Remark 4.7.* For separable  $L'/L$ , one reduces to the Galois case and applies the inflation–restriction sequence in Galois cohomology to obtain another proof of Lemma 4.6.

**COROLLARY 4.8.** *Let  $L/K$  range in a family of finite extensions of bounded degree.*

- (a) *For a finite extension  $K'/K$  for which either (i) or (ii) of Theorem 4.1(a) hold, if  $\#\text{Pic}(S^L)[l]$  is unbounded, then so is  $\#\text{Sel}_l A_{K'L}$ .*
- (b) *Assume that  $l \neq \text{char } K$ . For a finite extension  $K'/K$  for which  $A[l]_{K'}$  has a filtration with subquotients isomorphic to  $\mathbb{Z}/l\mathbb{Z}$  or  $\mu_l$ , if  $\#\text{Sel}_l A_L$  is unbounded, then so is  $\#\text{Pic}(S^{K'L})[l]$ .*

*Proof.* Combine Theorem 4.1 with Lemmas 4.5 and 4.6.  $\square$

### 5. Special cases of the function field analogues of Conjectures 1.1 and 1.2

For this section, fix a prime  $p$  and suppose that  $\text{char } K = p$ , i.e.,  $K$  is a finite extension of  $\mathbb{F}_p(t)$ . The analogues in question assume that  $l \neq p$  and predict that  $\#\text{Pic}(S^L)[l]$  and  $\#\text{Sel}_l A_L$  should be unbounded as  $L$  ranges over quadratic extensions of  $K$ . We show that this is indeed the case if one replaces  $K$  by a finite extension depending on  $l$  (and also on  $A$  in the Selmer group case). The key input is the work of Tate and Shafarevich [TŠ67] on unboundedness of ranks of quadratic twists of a constant supersingular elliptic curve.

**THEOREM 5.1.** *For each prime power  $l^m$  with  $l \neq p$ , there is a  $q = p^{n(l,m)}$  such that if  $\mathbb{F}_q \subset K$ , then the number of  $\mathbb{Z}/l^m\mathbb{Z}$ -summands of  $\text{Pic}(S^L)[l^m]$  is unbounded as  $L/K$  ranges over quadratic extensions of the form  $L = L'K$  for quadratic extensions  $L'/\mathbb{F}_p(t)$ . In particular, with  $n := n(l, 1)$ , the analogue of Conjecture 1.1 holds for  $l$  and every global field containing  $\mathbb{F}_{p^n}$ .*

*Proof.* Take a supersingular elliptic curve  $E \rightarrow \text{Spec } \mathbb{F}_p$  (see [Wat69, 4.1(5)] for its existence proved by Deuring). Let  $q$  be such that  $E_{\mathbb{F}_q}[l^m] \cong \mathbb{Z}/l^m\mathbb{Z} \oplus \mu_{l^m}$ , and hence also

$$E_{S^L}[l^m] \cong \mathbb{Z}/l^m\mathbb{Z} \oplus \mu_{l^m}$$

for every  $L$ . By [Čes14, 5.4(c)],  $H^1(S^L, E_{S^L}[l^m]) = \text{Sel}_{l^m} E_L$ , and by the result of Shafarevich and Tate [Ulm07, 1.4],  $\text{rk } E(L)$  and hence also the number of  $\mathbb{Z}/l^m\mathbb{Z}$ -summands of  $\text{Sel}_{l^m} E_L$  are unbounded. It remains to note that by the proofs of Lemmas B.1 and B.2,

$$\text{Sel}_{l^m} E_L \cong H^1(S^L, \mathbb{Z}/l^m\mathbb{Z} \oplus \mu_{l^m})$$

admits a map to  $\text{Hom}(\text{Pic}(S^L)/l^m \text{Pic}(S^L), \mathbb{Z}/l^m\mathbb{Z}) \oplus \text{Pic}(S^L)[l^m]$  with kernel of bounded size.  $\square$

*Remarks.*

5.2. We expect that the conclusion of Theorem 5.1 holds already with  $n(l, m) = 1$ .

5.3. For a composite  $l_1^{m_1} \cdots l_k^{m_k}$  prime to  $p$ , the proof gives a  $q = p^{n(l_1, m_1, \dots, l_k, m_k)}$  such that for every finite extension  $K/\mathbb{F}_q(t)$ , the unbounded growth of the number of  $\mathbb{Z}/l_i^{m_i}\mathbb{Z}$ -summands of  $\text{Pic}(S^L)[l_i^{m_i}]$  is simultaneous as  $L/K$  ranges over quadratic extensions (of the form  $L = L'K$  as in Theorem 5.1).

5.4. A possible choice for  $n(l, m)$  is  $2n$  with  $(-p)^n \equiv 1 \pmod{l^m}$  (e.g.  $n(l, m) := 2l^{m-1}(l - 1)$ ): in the proof take the supersingular  $E \rightarrow \text{Spec } \mathbb{F}_p$  which has  $x^2 + p$  as the characteristic polynomial of the  $p$ -power Frobenius  $\text{Frob}_p$ , so  $\text{Frob}_{p^{2n}}$  fixes  $E[l^m]$ .

COROLLARY 5.5. *If  $l \neq p$ , then there is a finite extension  $K'/K$  (depending on  $l$  and  $A$ ) such that the analogue of Conjecture 1.2 holds for  $A_{K'}$  and  $l$  for every finite extension  $K''/K'$ , i.e.  $\# \text{Sel}_l A_L$  is unbounded as  $L/K''$  ranges over quadratic extensions.*

*Proof.* Due to Theorems 4.1(a) and 5.1, it suffices to choose  $K'$  to contain  $\mathbb{F}_{p^n}$  with  $n = n(l, 1)$  and satisfy either  $\mathbb{Z}/l\mathbb{Z} \subset A[l]_{K'}$  or  $\mu_l \subset A[l]_{K'}$ . □

*Remark 5.6.* Similar techniques to those of this section have been used by Ochiai and Trihan in [OT09, § 3] to investigate  $\mu$ -invariants of constant abelian varieties over global function fields.

### 6. Iwasawa theory of class groups and Selmer groups

To keep the discussion focused, we assume in this and the next two sections that  $K$  is a number field, even though the question of function field analogues is an interesting one. Likewise, we set aside the possibility of more general  $p$ -adic Lie extensions and fix a  $\mathbb{Z}_p$ -extension  $K_\infty/K$ . Concretely,  $K_\infty/K$  is Galois with  $\text{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ ; we fix a choice of the latter isomorphism, which identifies the Iwasawa algebra  $\Lambda$  of  $K_\infty/K$  with  $\mathbb{Z}_p[[T]]$ . We denote by  $v_1, \dots, v_k$  the places of  $K$  ramified in  $K_\infty$ , so  $k \geq 1$  and  $v_i \mid p$ , and by  $K_n$  the subfield of  $K_\infty$  fixed by  $p^n \mathbb{Z}_p$ .

#### 6.1 Iwasawa theory of class groups

Let  $M$  be the maximal unramified abelian pro- $p$  extension of  $K_\infty$ . Set  $X := \text{Gal}(M/K_\infty)$ , which is a finitely generated torsion  $\Lambda$ -module (cf. [Ser58, Theorem 5 and § 5]). The structure theory of such  $\Lambda$ -modules gives a  $\Lambda$ -homomorphism

$$X \rightarrow \bigoplus_i \Lambda / f_i^{l_i} \Lambda \oplus \bigoplus_j \Lambda / p^{m_j} \Lambda,$$

with finite kernel and cokernel (i.e. a *pseudo-isomorphism*) for uniquely determined  $m_j \in \mathbb{Z}_{>0}$ , monic polynomials  $f_i \in \mathbb{Z}_p[[T]]$  that are monomials mod  $p$ , and  $l_i \in \mathbb{Z}_{>0}$ . The  $\lambda$ - and  $\mu$ -invariants of  $K_\infty/K$  are

$$\lambda_{\text{Pic}} := \sum l_i \deg f_i, \quad \mu_{\text{Pic}} := \sum m_j.$$

We also set  $\mu_{\text{Pic}}^{(m)} := \sum_j \min(m_j, m)$  for  $m \geq 0$ , which is of interest because it governs the growth of  $\# \text{Pic}(S^{K_n})[p^m]$  (as opposed to the customary in Iwasawa theory  $\# \text{Pic}(S^{K_n})[p^\infty]$ ).

PROPOSITION 6.2. *We have  $\# \text{Pic}(S^{K_n})[p^m] \sim_{K, K_\infty, m} p^{\mu_{\text{Pic}}^{(m)} p^n}$ .*

Before giving the proof we record a trivial lemma that clarifies implicit computations in subsequent arguments involving pseudo-isomorphisms; the lemma will be used without explicit notice.

LEMMA 6.3. *Let  $R$  be a commutative ring, and let  $X \xrightarrow{f} Y$  be a homomorphism of  $R$ -modules with finite kernel and cokernel. For  $r \in R$ , the induced  $X/rX \xrightarrow{f/r} Y/rY$  and  $X[r] \xrightarrow{f[r]} Y[r]$  satisfy*

$$\begin{aligned} \# \text{Ker } f/r &\leq \# \text{Ker } f \cdot \# \text{Coker } f, & \# \text{Coker } f/r &\leq \# \text{Coker } f, \\ \# \text{Ker } f[r] &\leq \# \text{Ker } f, & \# \text{Coker } f[r] &\leq \# \text{Ker } f \cdot \# \text{Coker } f. \end{aligned}$$

*Proof.* Apply the snake lemma twice. □

*Proof of Proposition 6.2.* Replacing  $K$  by  $K_n$  has the effect of multiplying  $\mu_{\text{Pic}}^{(m)}$  by  $p^n$  (since  $\mathbb{Z}_p[[T]]$  is replaced by  $\mathbb{Z}_p[[ (T+1)^{p^n} - 1 ]]$ ). By choosing  $n$  large, we are therefore reduced to the case when each  $v_i$  is totally ramified in  $K_\infty$ .

In this case, by [Ser58, Theorem 4], as  $\mathbb{Z}_p$ -modules,  $\text{Pic}(S^{K_n})[p^\infty]$  is isomorphic to the quotient of the finitely generated  $X/((T+1)^{p^n} - 1)X$  by a submodule generated by  $k$  elements. Hence,

$$\# \text{Pic}(S^{K_n})[p^m] \sim \#(X/((T+1)^{p^n} - 1)X)[p^m] \sim \prod_j \#(\Lambda/(p^{m_j}, (T+1)^{p^n} - 1))[p^m] = p^{\mu_{\text{Pic}}^{(m)} p^n}. \quad \square$$

### 6.4 Iwasawa theory of Selmer groups

The  $p^\infty$ -Selmer group of  $A_{K_n}$  is

$$\text{Sel}_{p^\infty} A_{K_n} := \varinjlim_m \text{Sel}_{p^m} A_{K_n},$$

and that of  $A_{K_\infty}$  is

$$\text{Sel}_{p^\infty} A_{K_\infty} := \varinjlim_n \text{Sel}_{p^\infty} A_{K_n}.$$

For the compact Pontryagin dual  $X' := \text{Hom}(\text{Sel}_{p^\infty} A_{K_\infty}, \mathbb{Q}_p/\mathbb{Z}_p)$ , one knows

CLAIM 6.4.1. *The  $\Lambda$ -module  $X'$  is finitely generated.*

*Proof.* Fix a nonempty open  $U \subset S[\frac{1}{p}]$  for which  $\mathcal{A}_U \rightarrow U$  is an abelian scheme. Finiteness of  $H_{\text{ét}}^1(U, \mathcal{A}[p])$  (see [Mil06, II.2.13]) implies that of  $H_{\text{ét}}^1(U, \mathcal{A}[p^\infty])[p]$ : the exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{A}[p]_U \rightarrow \mathcal{A}[p^n]_U \xrightarrow{p} \mathcal{A}[p^{n-1}]_U \rightarrow 0, \\ 0 \rightarrow \mathcal{A}[p^{n-1}]_U \rightarrow \mathcal{A}[p^n]_U \xrightarrow{p^{n-1}} \mathcal{A}[p]_U \rightarrow 0 \end{aligned}$$

give  $\#H_{\text{ét}}^1(U, \mathcal{A}[p^n])[p] \leq \#H_{\text{ét}}^1(U, \mathcal{A}[p]) \cdot \#A(K)[p]$ . Consequently,  $H_{\text{ét}}^1(U, \mathcal{A}[p^\infty])$  is  $\mathbb{Z}_p$ -cofinitely generated.

Let  $U_\infty := \varprojlim U_{S^{K_n}}$  be the normalization of  $U$  in  $K_\infty$ . Since  $U_\infty/U$  is pro-(finite étale Galois), the Hochschild–Serre spectral sequence

$$H^i(\text{Gal}(K_\infty/K), H_{\text{ét}}^j(U_\infty, \mathcal{A}[p^\infty])) \Rightarrow H_{\text{ét}}^{i+j}(U, \mathcal{A}[p^\infty])$$

shows that  $H_{\text{ét}}^1(U_\infty, \mathcal{A}[p^\infty])^{\text{Gal}(K_\infty/K)}$  is  $\mathbb{Z}_p$ -cofinitely generated. Therefore, so is

$$(\text{Sel}_{p^\infty} A_{K_\infty})^{\text{Gal}(K_\infty/K)} \subset H_{\text{ét}}^1(U_\infty, \mathcal{A}[p^\infty])^{\text{Gal}(K_\infty/K)}.$$

Pontryagin duality then gives the finiteness of  $X'/(T, p)$ , and it remains to invoke the relevant version of Nakayama’s lemma [Ser58, Lemme 4]. □

Claim 6.4.1 and the structure theory of finitely generated  $\Lambda$ -modules give a pseudo-isomorphism

$$X' \rightarrow \Lambda^\rho \oplus \bigoplus_s \Lambda/f'_s \Lambda \oplus \bigoplus_t \Lambda/p^{m'_t} \Lambda \tag{6.4.2}$$

as in 6.1 (with similar uniqueness claims). However, unlike  $X$ , the  $\Lambda$ -module  $X'$  need not be torsion, i.e.  $\rho > 0$  is possible. As for class groups, set  $\mu_{\text{Sel}}^{(m)} := \sum_t \min(m'_t, m)$  for  $m \geq 0$ .

**6.5 Controlled growth**

We say that *the control theorem holds* for  $A$  and  $K_\infty$ , if

$$\text{Sel}_{p^\infty} A_{K_n} \rightarrow (\text{Sel}_{p^\infty} A_{K_\infty})^{\text{Gal}(K_\infty/K_n)} \quad \text{for } n \geq 0$$

has finite kernel and cokernel of order bounded independently of  $n$ . The first result of this type is due to Mazur [Maz72, 6.4 (i)]; it has subsequently been generalized by Greenberg [Gre03, 5.1]: potential good ordinary reduction of  $A$  at all  $v \mid p$  is sufficient for the control theorem to hold. Such results play a purely axiomatic role in our computations.

PROPOSITION 6.6. *We have  $\#\text{Sel}_{p^m} A_{K_n} \sim_{A, K_\infty, m} p^{(\rho m + \mu_{\text{Sel}}^{(m)})p^n}$  if the control theorem holds for  $A$  and  $K_\infty$ .*

To replace  $\text{Sel}_{p^m} A_{K_n}$  by  $(\text{Sel}_{p^\infty} A_{K_n})[p^m]$  we need a quantitative version of [BKLPR13, 5.9].

LEMMA 6.7. *Let  $A \rightarrow \text{Spec } K$  be an abelian variety over a global field,  $p$  a prime, and  $a, b \in \mathbb{Z}_{>0}$*

- (a) *The kernel and cokernel of  $\text{Sel}_a A \rightarrow (\text{Sel}_{ab} A)[a]$  are of size at most  $\#A[a](K)$ .*
- (b) *The kernel and cokernel of  $\text{Sel}_{p^m} A \rightarrow (\text{Sel}_{p^\infty} A)[p^m]$  are of size at most  $\#A[p^m](K)$ .*

*Proof.* Part 6.7 is obtained from part 6.7 by taking direct limits. As for part 6.7, the cohomology sequence of  $0 \rightarrow A[a] \rightarrow A[ab] \xrightarrow{a} A[b] \rightarrow 0$  gives the kernel claim since  $(\#A[a](K) \cdot \#A[b](K)) / \#A[ab](K) \leq \#A[a](K)$ . Selmer groups consist of  $H^1$ -classes that vanish in every  $H^1(K_v, A)$ , so  $(\text{Sel}_{ab} A)[a] / \text{Im}(\text{Sel}_a A) \hookrightarrow H^1(K, A[ab])[a] / \text{Im}(H^1(K, A[a]))$ , and the cokernel claim results from the injection

$$\frac{H^1(K, A[ab])[a]}{\text{Im}(H^1(K, A[a]))} \hookrightarrow \text{Ker}(H^1(K, A[b]) \rightarrow H^1(K, A[ab])). \quad \square$$

*Proof of Proposition 6.6.* By Lemma 6.7, the control theorem, and Pontryagin duality,

$$\begin{aligned} \#\text{Sel}_{p^m} A_{K_n} &\sim \#(\text{Sel}_{p^\infty} A_{K_n})[p^m] \sim \#(\text{Sel}_{p^\infty} A_{K_\infty})^{\text{Gal}(K_\infty/K_n)}[p^m] \\ &\sim \#(X' / (p^m, (T + 1)^{p^n} - 1)). \end{aligned}$$

Therefore, the desired conclusion results from equation (6.4.2) (and Lemma 6.3). □

**7. Relations between the Iwasawa invariants of Selmer groups and class groups**

We keep the setup of § 6 and denote by  $\text{ord}_p$  the  $p$ -adic valuation normalized by  $\text{ord}_p p = 1$ .

PROPOSITION 7.1. *Suppose that the control theorem holds for  $A$  and  $K_\infty$ , and let  $\Sigma$  be the set of finite places of  $K$  that decompose completely in  $K_\infty$ .*

- (a) *If  $A[p^m]$  has  $\bigoplus_i \mathbb{Z}/p^{a_i}\mathbb{Z} \oplus \bigoplus_j \mu_{p^{b_j}}$  with  $a_i, b_j \geq 1$  as a  $K$ -subgroup,  $p \neq 2$ , and each  $v \mid p$  is finitely decomposed in  $K_\infty$ , then*

$$\rho m + \mu_{\text{Sel}}^{(m)} \geq \sum_i \mu_{\text{Pic}}^{(a_i)} + \sum_j \mu_{\text{Pic}}^{(b_j)} - r_2 \sum_j b_j - \sum_{v \in \Sigma} \text{ord}_p \left( \frac{\#\Phi_v(\mathbb{F}_v)}{\#(p^m \Phi_v)(\mathbb{F}_v)} \right).$$

- (b) *If  $A[p^m]$  has  $\bigoplus_i \mathbb{Z}/p^{a_i}\mathbb{Z}$  with  $a_i \geq 1$  as a  $K$ -subgroup and  $A$  has semiabelian reduction at all  $v \mid p$ , then*

$$\rho m + \mu_{\text{Sel}}^{(m)} \geq \sum_i \mu_{\text{Pic}}^{(a_i)} - \sum_{v \in \Sigma} \text{ord}_p \left( \frac{\#\Phi_v(\mathbb{F}_v)}{\#(p^m \Phi_v)(\mathbb{F}_v)} \right).$$

*Proof.* We begin with some preliminary observations:

- (1) we have  $\# \text{Pic}(S^{K_n}[\frac{1}{p}])[p^{a_i}] \sim \# \text{Pic}(S^{K_n})[p^{a_i}]$  in part (a), since  $\#(S^{K_n} \setminus S^{K_n}[\frac{1}{p}])$  is bounded;
- (2) the number of complex places of  $K_n$  is  $r_2 p^n$ ;
- (3) since  $S^{K_n}[\frac{1}{p}] \rightarrow S[\frac{1}{p}]$  is étale,  $\prod_{\substack{w \nmid p\infty \\ w \text{ not above } \Sigma}} \# \Phi_w^{K_n} \sim 1$  where  $w$  denotes a place of  $K_n$ ;
- (4) for a place  $w$  of semiabelian reduction for  $A_{K_n}$ , one has

$$\frac{\# \Phi_w^{K_n}(\mathbb{F}_w)}{\#(p^m \Phi_w^{K_n})(\mathbb{F}_w)} \leq \# \Phi_w^{K_n}[p^m] \leq p^{2mg}$$

where the last step uses surjectivity of multiplication by  $p^m$  on  $(\mathcal{A}^{K_n})^0(\overline{\mathbb{F}}_w)$  and the consideration of the finite part [EGAIV(4), 18.5.11 c)] of the quasi-finite separated  $(\mathcal{A}^{K_n}[p^m])_{\mathcal{O}_w}$ .

Combining Propositions 6.2 and 6.6 with Theorem 2.2 and using observations (1)–(4), we get

$$p^{(\rho m + \mu_{\text{Sel}}^{(m)})p^n} \gtrsim_{A, K_\infty, m} p^{(\sum_i \mu_{\text{Pic}}^{(a_i)} + \sum_j \mu_{\text{Pic}}^{(b_j)} - r_2 \sum_j b_j)p^n} \cdot \left( \prod_{v \in \Sigma} \frac{\# \Phi_v(\mathbb{F}_v)}{\#(p^m \Phi_v)(\mathbb{F}_v)} \right)^{-p^n}$$

and  $p^{(\rho m + \mu_{\text{Sel}}^{(m)})p^n} \gtrsim_{A, K_\infty, m} p^{(\sum_i \mu_{\text{Pic}}^{(a_i)})p^n} \cdot \left( \prod_{v \in \Sigma} \frac{\# \Phi_v(\mathbb{F}_v)}{\#(p^m \Phi_v)(\mathbb{F}_v)} \right)^{-p^n}$

in cases (a) and (b), respectively; the claimed inequalities follow by taking  $n$  large enough.  $\square$

*Remark 7.2.* The control theorem can hold in the presence of completely decomposed places of bad reduction for  $A$ , see [Gre03, 5.1].

**PROPOSITION 7.3.** *Set  $r := r_1$  if  $p = 2$ , and  $r := 0$  if  $p \neq 2$ . Suppose that the control theorem holds for  $A$  and  $K_\infty$ , and every place  $v$  above  $p$  or of bad reduction for  $A$  is finitely decomposed in  $K_\infty$ . If  $A[p^m]$  has a filtration by  $K$ -subgroups with subquotients isomorphic to  $\mathbb{Z}/p^{a_i}\mathbb{Z}$  or  $\mu_{p^{b_j}}$  with  $a_i, b_j \geq 1$ , then*

$$\rho m + \mu_{\text{Sel}}^{(m)} \leq 2mg[K : \mathbb{Q}] - r_2 \sum_j b_j + \sum_i (\mu_{\text{Pic}}^{(a_i)} + r) + \sum_j \mu_{\text{Pic}}^{(b_j)},$$

and also

$$\rho m + \mu_{\text{Sel}}^{(m)} \leq mg[K : \mathbb{Q}] + (r_1 + r_2) \sum_j b_j + \sum_i (\mu_{\text{Pic}}^{(a_i)} + r) + \sum_j (\mu_{\text{Pic}}^{(b_j)} + r) - \sum_{\substack{\text{real } v \\ p=2}} \text{ord}_2(\#\pi_0(A(K_v))).$$

*Proof.* We begin with some preliminary observations:

- (1) each infinite place of  $K$  is completely decomposed in  $K_\infty$ ;
- (2) we have  $\#(\text{Pic}_+(S^{K_n})/p^{a_i} \text{Pic}_+(S^{K_n})) \leq 2^r p^n \cdot \# \text{Pic}(S^{K_n})[p^{a_i}]$ ;
- (3) we have  $\# \text{Pic}(U_{S^{K_n}})[p^{b_j}] \sim_{A, K_\infty, m} \# \text{Pic}(S^{K_n})[p^{b_j}]$ , since  $\#(S^{K_n} \setminus U_{S^{K_n}})$  is bounded.

Combining Propositions 6.2 and 6.6 with Theorem 3.1 applied to  $U_{S^{K_n}}$ , where  $U$  is the largest open subscheme of  $S[\frac{1}{p}]$  for which  $\mathcal{A}_U \rightarrow U$  is an abelian scheme, and using observations (1)–(3),

we get

$$p^{(\rho m + \mu_{\text{Sel}}^{(m)})p^n} \lesssim_{A, K_\infty, m} p^{(\sum_i (\mu_{\text{Pic}}^{(a_i)} + r) + \sum_j \mu_{\text{Pic}}^{(b_j)} + [K:\mathbb{Q}] \sum_i a_i + (r_1 + r_2) \sum_j b_j)p^n}$$

$$p^{(\rho m + \mu_{\text{Sel}}^{(m)})p^n} \lesssim_{A, K_\infty, m} p^{(\sum_i (\mu_{\text{Pic}}^{(a_i)} + r) + \sum_j (\mu_{\text{Pic}}^{(b_j)} + r) + mg[K:\mathbb{Q}] + (r_1 + r_2) \sum_j b_j)p^n} \left( \prod_{\substack{\text{real } v \\ p=2}} \#\pi_0(A(K_v)) \right)^{-p^n}.$$

The claimed inequalities follow by taking  $n$  large enough. □

**COROLLARY 7.4.** *Suppose that the control theorem holds for  $A$  and  $K_\infty$ , and every place  $v$  above  $p$  or of bad reduction for  $A$  is finitely decomposed in  $K_\infty$ . If  $A[p]$  has a filtration by  $K$ -subgroups with  $a$  subquotients isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  and  $b$  subquotients isomorphic to  $\mu_p$  with  $a + b = 2g$ , then*

$$\rho \leq g[K : \mathbb{Q}] + ar + 2g\mu_{\text{Pic}}^{(1)} + \min(g[K : \mathbb{Q}] - r_2b, b(r_1 + r_2 + r)).$$

*Proof.* Since  $0 \subset A[p] \subset A[p^2] \subset \dots \subset A[p^m]$  has subquotients  $A[p]$ , Proposition 7.3 applies. □

*Remark 7.5.* Remark 3.3 applies equally well to Proposition 7.3 and Corollary 7.4.

**7.6** The assumptions on  $A[p^m]$  in Propositions 7.1 and 7.3 are satisfied after replacing  $K$  by a finite extension  $K'$ . We record how this affects the Iwasawa invariants involved in the obtained inequalities. Set  $K'_\infty := K'K_\infty$ , and write  $K'_n, \mu'_{\text{Pic}}, \rho', \mu'_{\text{Sel}}$ , etc. for  $K'_\infty/K'$  analogues of the familiar notation.

**LEMMA 7.7.** *One has  $\mu_{\text{Pic}}^{(m)} \leq \mu'^{(m)}_{\text{Pic}}$  for all  $m \geq 0$ . In particular,  $\mu_{\text{Pic}} \leq \mu'_{\text{Pic}}$ .*

*Proof.* If  $K' \cap K_\infty = K_n$ , then Lemma 4.5 and Proposition 6.2 give  $p^n \mu_{\text{Pic}}^{(m)} \leq \mu'^{(m)}_{\text{Pic}}$ . □

**LEMMA 7.8.** *Suppose that the control theorem holds for  $A$  and  $K_\infty$  and also for  $A_{K'}$  and  $K'_\infty$ . Then  $\rho m + \mu_{\text{Sel}}^{(m)} \leq \rho' m + \mu'^{(m)}_{\text{Sel}}$  for all  $m \geq 0$ . In particular,  $\rho \leq \rho'$ , and if  $\rho' = 0$ , then  $\mu_{\text{Sel}} \leq \mu'_{\text{Sel}}$ .*

*Proof.* If  $K' \cap K_\infty = K_n$ , then Lemma 4.6 and Proposition 6.6 give

$$p^n(\rho m + \mu_{\text{Sel}}^{(m)}) \leq \rho' m + \mu'^{(m)}_{\text{Sel}}. \quad \square$$

### 8. Conclusions for the cyclotomic $\mathbb{Z}_p$ -extension

Keeping the setup of § 6, we now assume that  $K_\infty/K$  is the cyclotomic  $\mathbb{Z}_p$ -extension, i.e. the unique  $\mathbb{Z}_p$ -subextension of  $K(\mu_{p^\infty})/K$ . No anomalies occur: every finite  $v$  is finitely decomposed in  $K_\infty$ , and for a finite extension  $K'/K$ , the compositum  $K'_\infty := K'K_\infty$  is the cyclotomic  $\mathbb{Z}_p$ -extension of  $K'$ .

**CONJECTURE 8.1** (Iwasawa [Iwa71, p. 392] and [Iwa73, p. 11]). We have  $\mu_{\text{Pic}} = 0$ .

**CONJECTURE 8.2** (Mazur [Maz72, p. 184]). If  $A$  has good ordinary reduction at all  $v \mid p$ , then  $\rho = 0$ .

**8.3 Status of Conjectures 8.1 and 8.2**

Conjecture 8.1 is known for abelian  $K/\mathbb{Q}$  by [FW79]; Conjecture 8.2 is known for  $A = E_K$ , if  $p$  is odd,  $E \rightarrow \text{Spec } \mathbb{Q}$  is an elliptic curve with good ordinary reduction at  $p$ , and  $K/\mathbb{Q}$  is abelian by [Kat04, 17.4] and [Roh84], and also for  $A$  with finite  $\text{Sel}_{p^\infty} A$ , as the control theorem shows. Examples with  $\mu_{\text{Sel}} > 0$  are known, and in fact  $\mu_{\text{Sel}}^{(1)}$  can be arbitrarily large when  $K$  is allowed to vary, as Example 8.8 shows.

The inequalities of § 7 allow one to relate Conjectures 8.1 and 8.2.

**THEOREM 8.4.** *If  $\rho + \mu_{\text{Sel}} = 0$ , the control theorem holds for  $A$  and  $K_\infty$ , and:*

- (i)  $A$  has  $\mathbb{Z}/p\mathbb{Z}$  as a  $K$ -subgroup and semiabelian reduction at all  $v \mid p$ ; or
- (ii)  $p$  is odd, and  $A$  has  $\mathbb{Z}/p\mathbb{Z}$  as a  $K$ -subgroup; or
- (iii)  $p$  is odd,  $K$  is totally real, and  $A$  has  $\mu_p$  as a  $K$ -subgroup;

then  $\mu_{\text{Pic}} = 0$ .

*Proof.* The conclusion is immediate from Conjecture 7.1, because  $\Sigma = \emptyset$ . □

Adopting the notation of § 7.6, one can use the results of § 7 to study boundedness questions.

**THEOREM 8.5.** *If  $K = \mathbb{Q}$ , the reduction of  $A$  at  $p$  is good ordinary, and  $A[p]$  has a filtration by  $K$ -subgroups with subquotients isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  or  $\mu_p$ , then  $\rho' \lesssim_{d,g} 1$  and  $\mu_{\text{Sel}}^{(1)} \lesssim_{d,g} 1$  for an abelian extension  $K'/\mathbb{Q}$  of degree  $d$ .*

*Proof.* Indeed,  $\mu'_{\text{Pic}} = 0$  (cf. § 8.3), so Proposition 7.3 gives the claim. □

*Remarks.*

8.6. If one assumes Conjecture 8.1, then the abelian restriction on  $K'/\mathbb{Q}$  is not needed; in fact, one can then also drop the assumption on  $A[p]$  and get the conclusion  $\rho', \mu_{\text{Sel}}^{(1)} \lesssim_{d,g,p} 1$  with the help of Lemma 7.8. Conversely, due to Proposition 7.1 and Lemma Conjecture 7.8, such a conclusion for all  $d$  and a single  $A$  with good ordinary reduction at  $p$  would give  $\mu_{\text{Pic}}^{(1)} \lesssim_{d,p} 1$ . Owing to Proposition 7.3 and Lemma 7.7, this would in turn imply  $\rho', \mu_{\text{Sel}}^{(1)} \lesssim_{d,g,p} 1$  for every  $A$  with good ordinary reduction at  $p$ . Is there a way to prove  $\rho', \mu_{\text{Sel}}^{(1)} \lesssim_{d,g,p} 1$  for a single such  $A$  without restricting to abelian  $K'/\mathbb{Q}$  and relying on Conjecture 8.1?

8.7. If  $d = g = 1$  and the reduction of  $A$  at  $p$  is good ordinary (but no assumption on  $A[p]$ ), then Greenberg has conjectured that  $\mu_{\text{Sel}}^{(1)} \leq 1$  (see [Gre99, 1.11 and p. 118 Remark]). We show that  $\mu_{\text{Sel}}^{(1)}$  can grow unboundedly as  $d$  grows.

*Example 8.8.* Suppose that  $\mathcal{A}[p] \cong (\mathbb{Z}/p\mathbb{Z})^g \oplus \mu_p^g$  over  $S$  and  $A$  has good reduction at all  $v \mid p$ . Then  $\mathcal{A}^{K'}[p] \cong (\mathbb{Z}/p\mathbb{Z})^g \oplus \mu_p^g$  over  $S^{K'}$  for every finite extension  $K'/K$  (see [Čes14, 3.4 and the proof of 3.3]). For instance, this is the case for  $K = \mathbb{Q}$  and  $A = X_0(11)$  with  $p = 5$  (see [Čes14, 1.12]).

Assume that  $p > 2$ . By [Čes14, 5.5 and the proof of 5.4] and Lemmas B.1 and B.2,

$$\# \text{Sel}_p A_{K_n} \sim_{A, K_\infty} \# H^1(S^{K_n}, (\mathbb{Z}/p\mathbb{Z})^g \oplus \mu_p^g) \sim \# \text{Pic}(S^{K_n})[p]^{2g} \cdot p^{gp^n(r_1+r_2)}. \tag{8.8.1}$$

If the reduction is ordinary at all  $v \mid p$ , then (8.8.1) combines with Propositions 6.2 and 6.6 to give

$$\rho + \mu_{\text{Sel}}^{(1)} = 2g\mu_{\text{Pic}}^{(1)} + g(r_1 + r_2).$$

The same reasoning applies with  $K$  replaced by a finite extension  $K'$ . In particular, if  $p > 2$ , the reduction of  $A$  at all  $v \mid p$  is good ordinary, and  $\mathcal{A}[p] \cong (\mathbb{Z}/p\mathbb{Z})^g \oplus \mu_p^g$ , then

$$\rho' + \mu'_{\text{Sel}}(1) = 2g\mu'_{\text{Pic}}(1) + g(r'_1 + r'_2)$$

for every finite extension  $K'/K$ . In particular, under Conjectures 8.1 and 8.2,  $\mu'_{\text{Sel}}(1) = g(r'_1 + r'_2)$ , and for  $K = \mathbb{Q}$ ,  $A = X_0(11)$ ,  $p = 5$ , and  $K'/\mathbb{Q}$  abelian, the same holds unconditionally (cf. § 8.3).

ACKNOWLEDGEMENTS

I thank Bjorn Poonen for many helpful discussions and suggestions. I thank Julio Brau, Pete Clark, Tim Dokchitser, Jordan Ellenberg, Zev Klagsbrun, Barry Mazur, Filip Najman, Karl Rubin, Doug Ulmer, Jeanine van Order, Larry Washington, and David Zureick-Brown for helpful conversations or correspondence regarding the material of the paper. I thank the referee for helpful suggestions. Part of the research presented here was carried out during the author’s stay at the Centre Interfacultaire Bernoulli (CIB) in Lausanne during the course of the program ‘Rational points and algebraic cycles’. I thank CIB, NSF, and the organizers of the program for a lively semester and the opportunity to take part.

**Appendix A. Cardinalities of the images of local Kummer homomorphisms**

Let  $K$  be a local field,  $A$  a  $g$ -dimensional abelian variety over  $K$ , and  $l$  a prime. Proposition A.1 summarizes standard computations in the form needed for the bounds of §§ 2–3.

PROPOSITION A.1. *Fix an  $m \in \mathbb{Z}_{>0}$ . If  $K$  is nonarchimedean, let  $\mathbb{F}_K$  be its residue field.*

- (a) *If  $K$  is nonarchimedean and  $l \neq \text{char } \mathbb{F}_K$ , then  $\#(A(K)/l^m A(K)) = \#A(K)[l^m]$ .*
- (b) *If  $K$  is a finite extension of  $\mathbb{Q}_l$ , then  $\#(A(K)/l^m A(K)) = l^{mg[K:\mathbb{Q}_l]} \cdot \#A(K)[l^m]$ .*
- (c) *If  $K \cong \mathbb{R}$  and  $l = 2$ , then  $A(K)/l^m A(K) \cong \pi_0(A(K))$  (component group for the archimedean topology) and  $\#\pi_0(A(K)) \leq 2^g$ . In all other archimedean cases,  $A(K)/l^m A(K) = 0$ .*

*Proof.* (a) Let  $\mathcal{O}_K$  be the ring of integers of  $K$  and  $\mathcal{A} \rightarrow \text{Spec } \mathcal{O}_K$  the Néron model of  $A$ . Since  $\mathcal{A}$  is smooth over the Henselian  $\mathcal{O}_K$ , the reduction homomorphism  $\mathcal{A}(\mathcal{O}_K) \rightarrow \mathcal{A}(\mathbb{F}_K)$  is surjective [BLR90, 2.2/14]; once we show that its kernel is uniquely divisible by  $l^m$ , the conclusion follows from the snake lemma because  $\#(\mathcal{A}(\mathbb{F}_K)/l^m \mathcal{A}(\mathbb{F}_K)) = \#\mathcal{A}(\mathbb{F}_K)[l^m]$  due to finiteness of  $\mathcal{A}(\mathbb{F}_K)$ . Since  $\mathcal{A} \xrightarrow{l^m} \mathcal{A}$  is separated étale [BLR90, 7.3/2(b)], so is its pullback over each  $P \in \mathcal{A}(\mathcal{O}_K)$ , and the claimed unique divisibility follows from [EGAIV(4), 18.5.12].

(b) The finite index inclusion  $\mathbb{Z}_l^{g[K:\mathbb{Q}_l]} \subset A(K)$  of [Mat55, Theorem 7] with the snake lemma give

$$\frac{\# \text{Coker}(A(K) \xrightarrow{l^m} A(K))}{\# \text{Ker}(A(K) \xrightarrow{l^m} A(K))} = \frac{\# \text{Coker}(\mathbb{Z}_l^{g[K:\mathbb{Q}_l]} \xrightarrow{l^m} \mathbb{Z}_l^{g[K:\mathbb{Q}_l]})}{\# \text{Ker}(\mathbb{Z}_l^{g[K:\mathbb{Q}_l]} \xrightarrow{l^m} \mathbb{Z}_l^{g[K:\mathbb{Q}_l]})} = l^{mg[K:\mathbb{Q}_l]}.$$

- (c) We have  $H^1(K, A[l^m]) = 0$  unless  $K \cong \mathbb{R}$  and  $l = 2$ , in which case [GH81, 1.1(3)] applies. □

*Remark A.2.* Finiteness of quotients  $A(K)/l^m A(K)$  fails for  $K$  of characteristic  $l$ : for instance, for the Tate elliptic curve  $\mathbb{G}_m/q^{\mathbb{Z}}$ , combine the snake lemma with the well-known infinitude of  $K^\times/K^{\times l^m}$  (see [Iwa86, (2.2) and 2.8]).



**Appendix B. The flat cohomology of  $\mathbb{Z}/l^a\mathbb{Z}$  and  $\mu_{l^b}$**

Fix a nonempty open  $U \subset S$ . We work out the cardinalities of the (compactly supported) flat cohomology groups of  $U$  with  $\mathbb{Z}/l^a\mathbb{Z}$  or  $\mu_{l^b}$  coefficients, which are needed in §§ 2–3.

LEMMA B.1. We have:

- (a)  $\#H^1(S, \mathbb{Z}/l^a\mathbb{Z}) = \#(\text{Pic}_+ S/l^a \text{Pic}_+ S)$ ;
- (b) if  $l \neq \text{char } K$ , then, interpreting  $[K_v : \mathbb{Q}_l]$  as 0 unless  $\text{char } K = 0$  and  $v \mid l$ ,

$$\#(\text{Pic}_+ S/l^a \text{Pic}_+ S) \leq \#H^1(U, \mathbb{Z}/l^a\mathbb{Z}) \leq \#(\text{Pic}_+ S/l^a \text{Pic}_+ S) \cdot \prod_{v \in S \setminus U} (\#\mu_{l^a}(K_v) \cdot l^{a[K_v:\mathbb{Q}_l]}).$$

*Proof.* (a) Since  $H^1(S, \mathbb{Z}/l^a\mathbb{Z}) \cong \text{Hom}(\pi_1^{\text{ét}}(S), \mathbb{Z}/l^a\mathbb{Z})$ , the theory of the narrow Hilbert class field gives the claim in the number field case. For function fields, one can (alternatively) use duality: by [Mil06, III.8.2],  $H^1(S, \mathbb{Z}/l^a\mathbb{Z}) \cong H^2(S, \mu_{l^a})^*$ , so, due to  $0 \rightarrow \mu_{l^a} \rightarrow \mathbb{G}_m \xrightarrow{l^a} \mathbb{G}_m \rightarrow 0$  that is exact in  $S_{\text{fppf}}$ , the vanishing of the Brauer group of  $S$  gives the claim.

(b) The exact

$$0 \rightarrow H^1(S, \mathbb{Z}/l^a\mathbb{Z}) \rightarrow H^1(U, \mathbb{Z}/l^a\mathbb{Z}) \rightarrow \prod_{v \in S \setminus U} H^1(K_v, \mathbb{Z}/l^a\mathbb{Z})/H^1(\mathcal{O}_v, \mathbb{Z}/l^a\mathbb{Z}),$$

Lemma B.1(a), and local class field theory give the bounds, because

$$H^1(K_v, \mathbb{Z}/l^a\mathbb{Z}) \cong \text{Hom}(K_v^\times / K_v^{\times l^a}, \mathbb{Z}/l^a\mathbb{Z})$$

and

$$H^1(\mathcal{O}_v, \mathbb{Z}/l^a\mathbb{Z}) \cong \text{Hom}(\pi_1^{\text{ét}}(\mathcal{O}_v), \mathbb{Z}/l^a\mathbb{Z}) \cong \mathbb{Z}/l^a\mathbb{Z}. \quad \square$$

LEMMA B.2. We have  $\#H^1(U, \mu_{l^b}) = \#\text{Pic}(U)[l^b] \cdot l^{b \cdot \max(r_1+r_2+\#(S \setminus U)-1, 0)} \cdot \#\mu_{l^b}(K)$ .

*Proof.* Since  $0 \rightarrow \mu_{l^b} \rightarrow \mathbb{G}_m \xrightarrow{l^b} \mathbb{G}_m \rightarrow 0$  is exact in  $U_{\text{fppf}}$ , its long exact cohomology sequence together with the unit theorem [AW45, Theorem 6, p. 491] give the claim.  $\square$

LEMMA B.3. Set  $r := r_1$  if  $l = 2$ , and  $r := 0$  if  $l \neq 2$ .

- (a) If  $a \geq 1$ , then  $\#H_c^1(S, \mathbb{Z}/l^a\mathbb{Z}) = \#(\text{Pic } S/l^a \text{Pic } S) \cdot 2^{\max(r-1, 0)}$ .
- (b) If  $a \geq 1$  and  $U \neq S$ , then  $\#H_c^1(U, \mathbb{Z}/l^a\mathbb{Z}) = \#(\text{Pic } U/l^a \text{Pic } U) \cdot l^{a(\#(S \setminus U)-1)} \cdot 2^r$ .

*Proof.* By duality [Mil06, III.3.2, III.8.2],  $\#H_c^1(U, \mathbb{Z}/l^a\mathbb{Z}) = \#H^2(U, \mu_{l^a})$ , and the claim follows from the cohomology sequence of

$$0 \rightarrow \mu_{l^a} \rightarrow \mathbb{G}_m \xrightarrow{l^a} \mathbb{G}_m \rightarrow 0$$

since the Brauer group of  $U$  is understood from the exact sequence

$$0 \rightarrow \text{Br } U \rightarrow \bigoplus_{v \notin U} \text{Br}(K_v) \xrightarrow{\sum \text{inv}_v} \mathbb{Q}/\mathbb{Z}$$

(see [Mil06, II.2.1]).  $\square$

LEMMA B.4. *If  $b \geq 1$  and  $l$  is invertible on  $U \neq S$ , then*

$$\begin{aligned} \#H_c^1(U, \mu_{l^b}) &\geq \#(\text{Pic}_+ S/l^b \text{Pic}_+ S) \cdot l^{-b(r_2+1)}, \\ \#H_c^1(U, \mu_{l^b}) &\leq \#(\text{Pic}_+ S/l^b \text{Pic}_+ S) \cdot l^{b(r_1+r_2-1)} \cdot \prod_{v \in S \setminus U} \#\mu_{l^b}(K_v). \end{aligned}$$

*Proof.* We replace compactly supported flat cohomology by its étale counterpart [Mil06, II.2.3 and the preceding subsection]: by [Mil06, II.3.3, III.3.2, III.8.1] and [Gro68, 11.7 1°)], the two meanings of  $H_c^i(U, \mu_{l^b})$  agree.

By the Euler characteristic formula [Mil06, II.2.13 (b)] and duality [Mil06, II.3.3],

$$\#H_c^1(U, \mu_{l^b}) = \frac{\#H_c^0(U, \mu_{l^b}) \cdot \#H_c^2(U, \mu_{l^b})}{2^r l^{br_2} \cdot \#H_c^3(U, \mu_{l^b})} = \frac{\#H_c^0(U, \mu_{l^b}) \cdot \#H^1(U, \mathbb{Z}/l^b \mathbb{Z})}{2^r l^{b(r_2+1)}}$$

with  $r$  as in Lemma B.3. By [Mil06, II.2.3(a)] (we use the  $U \neq S$  assumption to discard  $H^0(U, \mu_{l^b})$ ),

$$H_c^0(U, \mu_{l^b}) \cong \bigoplus_{v|\infty} \widehat{H}^{-1}(K_v, \mu_{l^b}) \cong \bigoplus_{\text{real } v} H^1(K_v, \mu_{l^b}) \cong \bigoplus_{\text{real } v} \mathbb{R}^\times / \mathbb{R}^{\times l^b}$$

where  $\widehat{H}^i$  denotes Tate cohomology. It remains to take into account Lemma B.1(b).  $\square$

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