Complete manifolds with non-negative Ricci curvature
and the Caffarelli–Kohn–Nirenberg inequalities

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Abstract

In this paper, we prove that complete open Riemannian manifolds with non-negative Ricci curvature of dimension greater than or equal to three in which some Caffarelli–Kohn–Nirenberg type inequalities are satisfied are close to the Euclidean space.

1. Introduction

Let \( n \geq 3 \) be an integer and let \( a, b, \) and \( p \) be constants satisfying
\[
-\infty < a < \frac{n - 2}{2}, \quad a \leq b \leq a + 1, \quad \text{and} \quad p = \frac{2n}{n - 2 + 2(b - a)}.
\]
(1.1)

Denote by \( C_0^\infty(\mathbb{R}^n) \) the space of smooth functions with compact support in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). In [CKN84], among a much more general family of inequalities, Caffarelli, Kohn, and Nirenberg proved the following result. There exists a positive constant \( C \) depending only on \( a, b \) and \( n \) such that
\[
\left( \int_{\mathbb{R}^n} |x|^{-bp} |u|^p \, dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 \, dx \right)^{1/2},
\]
(1.2)

for all \( u \in C_0^\infty(\mathbb{R}^n) \), where \( |x| \) is the Euclidean length of \( x \in \mathbb{R}^n \). Note that the Caffarelli–Kohn–Nirenberg inequalities contain the classical Sobolev inequality \((a = b = 0)\) and the Hardy inequality \((a = 0, b = 1)\) as special cases, which have many important applications (see e.g. [Aub82, Aub98, CKN84, HLP52, Heb96, Heb99, Lie83] and references therein).

Let \( K_{a,b} \) be the best constant for the Caffarelli, Kohn, and Nirenberg inequality (1.1), that is
\[
K_{a,b}^{-1} = \inf_{u \in C_0^\infty(\mathbb{R}^n) - \{0\}} \frac{\left( \int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 \, dx \right)^{1/2}}{\left( \int_{\mathbb{R}^n} |x|^{-bp} |u|^p \, dx \right)^{1/p}}.
\]
(1.3)

For the Sobolev inequality \((a = b = 0)\), it has been proved by Aubin [Aub76] and Talenti [Tal76] that
\[
K_{0,0} = \left( \frac{1}{n(n - 2)} \right)^{1/2} \left( \frac{2\Gamma(n)}{n\omega_n \Gamma^2(n/2)} \right)^{1/n},
\]
where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \), and that a family of minimizers of (1.3) is given by
\[
u(x) = (\lambda + |x|^2)^{1-n/2}, \quad \lambda > 0.
\]

In [Lie83], Lieb considered the case \( a = 0, 0 < b < 1 \), and proved that the best constant is
\[
K_{0,b} = \left( \frac{1}{(n - 2)(n - bp)} \right)^{1/2} \left( \frac{(2 - bp)\Gamma((2n - 2bp)/(2 - bp))}{n\omega_n \Gamma^2((n - bp)/(2 - bp))} \right)^{2(n-bp)/(2-bp)},
\]
which is very close to the Euclidean space. For the particular case of the Euclidean space, \( K_{0,0} \) is the best constant for the Sobolev inequality (1.2) and is known to be the same as \( K_0 \) in the following sense:
\[
K_0 = \inf_{u \in C_0^\infty(\mathbb{R}^n) - \{0\}} \frac{\left( \int_{\mathbb{R}^n} |x|^{-2} |\nabla u|^2 \, dx \right)^{1/2}}{\left( \int_{\mathbb{R}^n} |u|^2 \, dx \right)^{1/2}},
\]
(1.4)

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and a family of minimizers is
\[ u(x) = \frac{1}{(\lambda + |x|^{2b}(n-2)/2b)} \lambda > 0. \]

Chou and Chu [CC93] studied the case \( a \geq 0, \ a \leq b < a + 1 \), and proved that the best constant is
\[ K_{a,b} = \left( \frac{1}{(n-2a-2)(n-bp)} \right)^{1/2} \left( \frac{(2-bp+2a)\Gamma((2n-2bp)/(2-bp+2a))}{n\omega_n\Gamma^2((n-bp)/(2-bp+2a))} \right)^{2(n-bp)/(2-bp+2a)}, \]
and that, for \( a > 0 \), all minimizers are non-zero constant multiples of the function
\[ u(x) = \frac{1}{(\lambda + |x|^{2b+2a}(n-2-2a)/2b)} \lambda > 0. \]

For the remaining case, the best constant \( K_{a,b} \) and the existence or non-existence of the minimizers have been studied recently in [CW01].

In this paper, we study complete manifolds with non-negative Ricci curvature in which some Caffarelli–Kohn–Nirenberg inequalities are satisfied. Now we fix some notation. For an integer \( n \geq 3 \), we will from now on let \( a \) and \( b \) be constants satisfying
\[ 0 \leq a < \frac{n-2}{2}, \quad a \leq b < a + 1, \tag{1.4} \]
and set
\[ p = \frac{2n}{n-2 + 2(b-a)}. \tag{1.5} \]

For a Riemannian manifold \( M \), we let \( dv \) be the Riemannian volume element on \( M \), denote by \( \nabla \) the gradient operator, \( C_0^\infty(M) \) the space of smooth functions on \( M \) with compact support, \( B(x,r) \) the geodesic ball with center \( x \in M \) and radius \( r \), and \( \text{vol}[B(p,r)] \) the volume of \( B(p,r) \).

Our purpose is to prove the following result.

**Theorem 1.1.** Let \( C \geq K_{a,b} \) be a constant and \( M \) be an \( n \)-dimensional \( (n \geq 3) \) complete open Riemannian manifold with non-negative Ricci curvature. Fix a point \( x_0 \in M \) and denote by \( \rho \) the distance function on \( M \) from \( x_0 \). Assume that, for any \( u \in C_0^\infty(M) \), we have
\[ \left( \int_M \rho^{-bp}|u|^p \, dv \right)^{1/p} \leq C \left( \int_M \rho^{-2a} |\nabla u|^2 \, dv \right)^{1/2}. \tag{1.6} \]

Then for any \( x \in M \), we have
\[ \text{vol}[B(x,r)] \geq (C^{-1}K_{a,b})^{n/(1+a-b)}V_0(r), \quad \forall r > 0, \tag{1.7} \]
where \( V_0(r) \) is the volume of the \( r \)-ball in \( \mathbb{R}^n \).

In the special case that \( a = b = 0 \), the above theorem has been proved in [Xia01].

The theorem has several consequences for manifolds with non-negative Ricci curvature.

The Bishop–Gromov comparison theorem (cf. [BC64, Cha93, GLP81]) implies that, if \( M \) is an \( n \)-dimensional complete Riemannian manifold with non-negative Ricci curvature, then for any \( x \in M, \text{vol}[B(x,r)] \leq V_0(r) \), with equality holding if and only if \( B(x,r) \) is isometric to an \( r \)-ball in \( \mathbb{R}^n \). Combining this fact and Theorem 1.1, one immediately gets the following rigidity theorem.

**Corollary 1.2.** An \( n \)-dimensional \( (n \geq 3) \) complete open Riemannian manifold \( M \) with non-negative Ricci curvature in which the inequality
\[ \left( \int_M \rho^{-bp}|u|^p \, dv \right)^{1/p} \leq K_{a,b} \left( \int_M \rho^{-2a} |\nabla u|^2 \, dv \right)^{1/2}, \quad \forall u \in C_0^\infty(M), \]
is satisfied, is isometric to \( \mathbb{R}^n \).
When \(a = b = 0\), Corollary 1.2 is the main theorem in [Led99].

A theorem of Cheeger and Colding [CC97] states that given integer \(n \geq 2\) there exists a constant \(\delta(n) > 0\) such that any \(n\)-dimensional complete Riemannian manifold with non-negative Ricci curvature and \(\text{vol}[B(x, r)] \geq (1 - \delta(n))V_0(r)\) for some \(p \in M\) and all \(r > 0\) is diffeomorphic to \(\mathbb{R}^n\). Thus combining the Cheeger–Colding theorem and Theorem 1.1, one deduces the following topological rigidity for manifolds with non-negative Ricci curvature.

**Corollary 1.3.** Given integer \(n \geq 3\), there exists a positive constant \(\epsilon = \epsilon(n, a, b)\) such that any \(n\)-dimensional \((n \geq 3)\) complete non-compact Riemannian manifold \(M\) with non-negative Ricci curvature in which the inequality

\[
\left( \int_M \rho^{-bp}|u|^p \, dv \right)^{1/p} \leq \left( K_{a,b} + \epsilon \right) \left( \int_M \rho^{-2a} \| \nabla u \|^2 \, dv \right)^{1/2}, \quad \forall u \in C_0^\infty(M),
\]

is satisfied, is diffeomorphic to \(\mathbb{R}^n\).

A theorem due to Li [Li86] and Anderson [And90] states that, if \(M\) is an \(n\)-dimensional complete manifold with non-negative Ricci curvature in which the inequality \(\text{vol}[B(p, r)] \geq \alpha V_0(r)\) holds for some constant \(\alpha > 0\) and all \(r > 0\), the fundamental group \(\pi_1(M)\) is finite and \(\# \pi_1(M) \leq 1/\alpha\). Thus from the Li–Anderson theorem and Theorem 1.1 we have the following corollary.

**Corollary 1.4.** Let \(C \geq K_{a,b}\) be a constant and \(M\) be an \(n\)-dimensional \((n \geq 3)\) complete open Riemannian manifold with non-negative Ricci curvature. Assume that, for any \(u \in C_0^\infty(M)\), we have

\[
\left( \int_M \rho^{-bp}|u|^p \, dv \right)^{1/p} \leq C \left( \int_M \rho^{-2a} \| \nabla u \|^2 \, dv \right)^{1/2}.
\]

Then \(M\) has finite fundamental group and the order of \(\pi_1(M)\) is bounded above by \((K_{a,b}^{-1}C)^{n/(1+a-b)}\).

One can find some related results about the topology of complete manifolds with non-negative Ricci curvature, for example, in [AG90, And90, CX00, Col98, Li86, OSY00, Ots89, SS97, She93, She96, SS01, Sor00, Xia99].

**2. A Proof of Theorem 1.1**

First notice the following fact. The Bishop–Gromov comparison theorem (cf. [BC64, Cha93, GLP81]) tells us that for any \(p \in M\) the function \(\text{vol}[B(p, r)]/V_0(r)\) is decreasing and so the limit

\[
\lim_{r \to +\infty} \frac{\text{vol}[B(p, r)]}{V_0(r)}
\]

exists. Also one can easily check that the above limit does not depend on the choice of \(p\). It then follows that if (1.7) holds for some point \(p_0 \in M\), then it is satisfied for all \(x \in M\). Now we are going to show that (1.7) holds at the point \(x_0\).

Set

\[
w = 2a - bp + 2, \quad q = \frac{(n - 2a - 2)p}{2a - bp + 2} = \frac{2p}{p - 2},
\]

and, for any \(\lambda > 0\), let

\[
F(\lambda) = \frac{p - 2}{p + 2} \int_M \frac{dv}{\rho^{bp}(\lambda + \rho^w)q}. \tag{2.2}
\]

Then, for \(\lambda > 0\), we have from the Fubini theorem (cf. [SY94]) that

\[
F(\lambda) = \frac{p - 2}{p + 2} \int_0^{+\infty} \text{vol} \left\{ x : \frac{1}{\rho^{bp}(\lambda + \rho^w)q} > s \right\} \, ds.
\]

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Making the variable change \( s = 1(h^{bp}(\lambda + h)^{q-1}) \) in the above equality, one concludes that

\[
F(\lambda) = \frac{p - 2}{p + 2} \int_0^{+\infty} \text{vol}\{x : \rho(x) < h\} \frac{h^{bp+1}(\lambda + h^q)}{(h^{bp+1}(\lambda + h^q))'} \, dh
\]

\[
= \frac{p - 2}{p + 2} \int_0^{+\infty} \text{vol}[B(x_0, h)] \frac{(h^{bp+1}(\lambda + h^q))'}{(h^{bp+1}(\lambda + h^q))} \, dh.
\]

(2.3)

Since the Bishop–Gromov comparison theorem implies that \( \text{vol}[B(x_0, h)] \leq \omega_n h^n \), we have

\[
F(\lambda) \leq \frac{\omega_n(p - 2)}{p + 2} \int_0^{+\infty} (h^{bp+1}(\lambda + h^q))' h^{n-bp-1}(\lambda + h^q)^{-q} \, dh.
\]

On the other hand, one can deduce from (1.4), (1.5), and (2.1) that

\[
n - bp - 1 > -1, \quad n - bp - 1 + w(1 - q) < -1.
\]

It then follows that \( 0 \leq F(\lambda) < +\infty, \forall \lambda > 0 \), and that \( F \) is differentiable. Also, we have

\[
F'(\lambda) = -\int_M \frac{dv}{\rho^{bp}(\lambda + \rho^w)^q}.
\]

(2.4)

Consider the function \( H_0 : (0, +\infty) \to \mathbb{R} \) defined by

\[
H_0(\lambda) = \frac{p - 2}{p + 2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp}(\lambda + |x|^w)^{q-1}}.
\]

Recall that when \( M = \mathbb{R}^n \) and \( C = K_{a,b} \), the extremal functions in the inequality (1.6) are the functions \( u_\lambda := (\lambda + |x|^w)^{-q/p}, \lambda > 0 \). That is, we have

\[
(-H_0'(\lambda))^{2/p} = \left( \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp}(\lambda + |x|^w)^q} \right)^{2/p} = \left( \frac{K_{a,b}qw}{p} \right)^2 \int_{\mathbb{R}^n} \frac{dx}{|x|^{2(1+a-w)(\lambda + |x|^w)^2+2q/p}}
\]

\[
= \left( \frac{K_{a,b}qw}{p} \right)^2 \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp-w(\lambda + |x|^w)^q}} = \left( \frac{K_{a,b}qw}{p} \right)^2 \left( H_0'(\lambda) + \frac{p + 2}{p - 2} H_0(\lambda) \right).
\]

Substituting \( H_0(\lambda) = H_0(1)\lambda^{-2/(p-2)} \) into the above equation, one gets

\[
H_0(1) = \frac{p - 2}{p + 2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp}(1 + |x|^w)^{q-1}} = 2^{2/(p-2)}(p-2)((n-2a-2)^2 K_{a,b}^2)^{-p/(p-2)}.
\]

(2.5)

By a simple approximation procedure, we can apply (1.6) to \( (\lambda + \rho^w)^{-q/p} \) for every \( \lambda > 0 \) to get

\[
\left( \int_M \frac{dv}{\rho^{bp}(\lambda + \rho^w)^q} \right)^{2/p} \leq \left( \frac{qwC}{p} \right)^2 \int_M \frac{dv}{\rho^{2(1+a-w)(\lambda + \rho^w)^2+2q/p}}
\]

\[
= \left( \frac{qwC}{p} \right)^2 \int_M \frac{dv}{\rho^{bp-w(\lambda + \rho^w)^q}}.
\]

Let \( l = (p/qwC)^2 \); then the above inequality becomes

\[
(l(-F'(\lambda))^{2/p} - \lambda F'(\lambda) \leq \frac{p + 2}{p - 2} F(\lambda).
\]

(2.6)
The idea now is to compare the solutions of (2.6) to the solutions $H$ of the following differential equality:

$$l(-H'(\lambda))^{2/p} - \lambda H'(\lambda) = \frac{p+2}{p-2} H(\lambda).$$

(2.7)

One can easily check that $H_1(\lambda)$ given by

$$H_1(\lambda) := A\lambda^{-2/(p-2)}$$

is a particular solution of (2.7), where

$$A = 2^{2/(p-2)}(p-2) \left(\frac{1}{p}\right)^{p/(p-2)}$$

$$= 2^{2/(p-2)}(p-2)((n-2a-2)^2pC^2)^{-p/(p-2)}$$

$$= (C^{-1} K_{a, b})^{2p/(p-2)}(p-2) \cdot (n-2a-2)^2pK_{a,b}^2)^{-p/(p-2)}$$

$$= (C^{-1} K_{a, b})^{2p/(p-2)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp(1 + |x|^w)^q-1}}$$

$$= (C^{-1} K_{a, b})^{n/(1+a-b)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp(1 + |x|^w)^q-1}}.$$  

(2.9)

Observe that

$$H_1(\lambda) = (C^{-1} K_{a, b})^{n/(1+a-b)} \cdot \lambda^{-2/(p-2)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp(1 + |x|^w)^q-1}}$$

$$= (C^{-1} K_{a, b})^{n/(1+a-b)} H_0(\lambda).$$  

(2.10)

Before we can conclude the proof of Theorem 1.1, we shall need the following two lemmas.

**Lemma 2.1.** If for some $\lambda_0 > 0$, $F(\lambda_0) < H_1(\lambda_0)$, then $F(\lambda) < H_1(\lambda)$ $\forall \lambda \in (0, \lambda_0]$.

**Proof.** Suppose that Lemma 2.1 is false. Set

$$\lambda_1 = \sup\{\lambda < \lambda_0; F(\lambda) = H_1(\lambda)\}.$$  

For each $\lambda > 0$, the function $\phi_\lambda : [0, +\infty) \to \mathbb{R}$ defined by

$$\phi_\lambda(s) = ls^{2/p} + \lambda s$$

is increasing. By (2.6), we have

$$\phi_\lambda(-F'(\lambda)) \leq \frac{p+2}{p-2} F(\lambda),$$

which gives

$$-F'(\lambda) \leq \phi_\lambda^{-1} \left(\frac{p+2}{p-2} F(\lambda)\right).$$

On the other hand, (2.7) implies that

$$-H_1'(\lambda) = \phi_\lambda^{-1} \left(\frac{p+2}{p-2} H_1(\lambda)\right).$$

Thus, on the subset $\{s \mid F(s) \leq H_1(s)\}$, we have

$$F'(\lambda) - H_1'(\lambda) \geq \phi_\lambda^{-1} \left(\frac{p+2}{p-2} H_1(\lambda)\right) - \phi_\lambda^{-1} \left(\frac{p+2}{p-2} F(\lambda)\right).$$

Since $(F - H_1)|_{[\lambda_1, \lambda_0]} \leq 0$, we conclude therefore that $(F - H_1)' \leq 0$ on $[\lambda_1, \lambda_0]$. Consequently, one gets

$$0 = (F - H_1)(\lambda_1) \leq (F - H_1)(\lambda_0) < 0.$$

This is a contradiction and completes the proof of Lemma 2.1.  

\[\Box\]
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Lemma 2.2. We have

\[ \liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} \geq 1. \]  
(2.11)

Proof. Fix a small \( \epsilon > 0 \). Since

\[ \lim_{u \to 0} \frac{\text{vol}[B(x_0, u)]}{V_0(u)} = 1, \]

there exists a \( \delta > 0 \) such that \( \text{vol}[B(x_0, h)] \geq (1 - \epsilon) V_0(h), \forall h \leq \delta. \)

It then follows from (2.3) that

\[
F(\lambda) \geq \frac{p - 2}{p + 2} (1 - \epsilon) \int_{0}^{\delta} V_0(h) \frac{(bp \lambda + (bp + (q - 1)w)h^w)}{h^{bp+1}(\lambda + h^w)^q} dh \\
= \frac{p - 2}{p + 2} (1 - \epsilon) \lambda^{[(n+bp)/w]+1-q} \int_{0}^{\delta/\lambda^{1/w}} V_0(s) \frac{(bp + (bp + (q - 1)w)s^w)}{s^{bp+1}(1 + s^w)^q} ds \\
= \frac{p - 2}{p + 2} (1 - \epsilon) \lambda^{-2/(p-2)} \int_{0}^{\delta/\lambda^{1/w}} V_0(s) \frac{(bp + (bp + (q - 1)w)s^w)}{s^{bp+1}(1 + s^w)^q} ds.
\]

On the other hand, it is easy to see that

\[ H_0(\lambda) = \frac{p - 2}{p + 2} \lambda^{-2/(p-2)} \int_{0}^{+\infty} V_0(s) \frac{(bp + (bp + (q - 1)w)s^w)}{s^{bp+1}(1 + s^w)^q} ds. \]

We conclude therefore that

\[ \liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} \geq 1 - \epsilon. \]

Letting \( \epsilon \to 0 \), one gets

\[ \liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} \geq 1. \]  
(2.12)

This completes the proof of Lemma 2.2. \( \square \)

Now we continue on the proof of Theorem 1.1. We separate the proof into two cases.

Case 1: \( C > K_{a,b} \). In this case, it follows from (2.10) and Lemma 2.2 that

\[ \liminf_{\lambda \to 0} \frac{F(\lambda)}{H_1(\lambda)} = \left( \frac{C}{K_{a,b}} \right)^{n/(1+a-b)} \liminf_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} \geq \left( \frac{C}{K_{a,b}} \right)^{n/(1+a-b)} > 1, \]  
(2.13)

which, combining with Lemma 2.1, implies that

\[ F(\lambda) \geq H_1(\lambda), \forall \lambda > 0. \]  
(2.14)

That is, for any \( \lambda > 0 \), we have

\[ \int_{0}^{+\infty} (\text{vol}[B(x_0, s)] - (C^{-1}K_{a,b})^{n/(1+a-b)} V_0(s)) \frac{bp \lambda + (bp + (q - 1)w)s^w}{s^{bp+1}(\lambda + s^w)^q} ds \geq 0. \]  
(2.15)

Recall that the Bishop–Gromov comparison theorem says that the function \( |B(x_0, s)|/V_0(s) \) is decreasing. Set \( d = (C^{-1}K(n,q))^{n/(1+a-b)} \) and assume that

\[ \lim_{s \to +\infty} \frac{|B(x_0, s)|}{V_0(s)} = d_0. \]

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The proof of Theorem 1.1 will be completed if we can show that \( d_0 \geq d \). We prove this fact by contradiction. Thus suppose that \( d_0 = d - \epsilon_0 \), for some \( \epsilon_0 > 0 \). Then there exists an \( N_0 > 0 \) such that

\[
\frac{\text{vol}[B(x_0, s)]}{V_0(s)} \leq d - \frac{\epsilon_0}{2}, \quad \forall s \geq N_0. \tag{2.16}
\]

By introducing (2.16) into (2.15), one derives for every \( \lambda > 0 \) that

\[
0 \leq \int_0^{\infty} \left( \frac{\text{vol}[B(x_0, s)]}{V_0(s)} - d \right) s^n (bp\lambda + (bp + (q - 1)w)s^w) s^{b\lambda+1} (\lambda + s^w)^q ds \\
\leq \int_0^{N_0} \frac{\text{vol}[B(x_0, s)]}{V_0(s)} s^n (bp\lambda + (bp + (q - 1)w)s^w) s^{b\lambda+1} (\lambda + s^w)^q ds \\
+ \int_{N_0}^{\infty} \left( d - \frac{\epsilon_0}{2} \right) s^n (bp\lambda + (bp + (q - 1)w)s^w) s^{b\lambda+1} (\lambda + s^w)^q ds \\
- d \int_0^{\infty} s^n (bp\lambda + (bp + (q - 1)w)s^w) s^{b\lambda+1} (\lambda + s^w)^q ds \\
\leq \int_0^{N_0} \left( 1 - d + \frac{\epsilon_0}{2} \right) s^n (bp\lambda + (bp + (q - 1)w)s^w) s^{b\lambda+1} (\lambda + s^w)^q ds \\
- \frac{\epsilon_0}{2\omega_n} \int_0^{\infty} V_0(s)(bp\lambda + (bp + (q - 1)w)s^w) s^{b\lambda+1} (\lambda + s^w)^q ds \\
\leq \left( 1 - d + \frac{\epsilon_0}{2} \right) \lambda^{-q} \int_0^{N_0} (bp\lambda s^{n-bp-1} + (bp + (q - 1)w)s^{n+w-bp-1}) ds \\
- \frac{\epsilon_0}{2\omega_n} \cdot \frac{p + 2}{p - 2} \cdot \lambda^{-2/(p-2)} \cdot H_0(1) \\
= \left( 1 - d + \frac{\epsilon_0}{2} \right) \lambda^{-q} \left( \frac{\lambda bp N_0^{n-bp}}{n - bp} + \frac{(bp + (q - 1)w)N_0^{n+w-bp}}{n + w - bp} \right) \\
- \frac{\epsilon_0(p + 2)H_0(1)}{2\omega_n(p - 2)} \cdot \lambda^{-2/(p-2)},
\]

which implies for any \( \lambda > 0 \) that

\[
\frac{\epsilon_0(p + 2)H_0(1)}{2\omega_n(p - 2)(1 - d + \epsilon_0/2)} \leq \lambda^{2/(p-2)-q} \left( \frac{\lambda bp N_0^{n-bp}}{n - bp} + \frac{(bp + (q - 1)w)N_0^{n+w-bp}}{n + w - bp} \right).
\]

Letting \( \lambda \to +\infty \) in the above inequality and observing that \( 2/(p-2) - q + 1 < 0 \), one obtains the desired contradiction. Thus \( d_0 \geq d \). This completes the proof of Theorem 1.1 in the case that \( C > K_{a,b} \).
Case 2: \( C = K_{a,b} \). In this case, we have for any fixed \( \delta > 0 \) that

\[
\left( \int_M \rho^{-b |u|^p} \, dv \right)^{1/p} \leq (K_{a,b} + \delta) \left( \int_M \rho^{-2a |\nabla u|^2} \, dv \right)^{1/2}.
\]

Thus for any \( x \in M \) we have from case 1 that

\[
\text{vol}[B(x, r)] \geq \left( \frac{K_{a,b}}{K_{a,b} + \delta} \right)^{n/(1 + a - b)} V_0(r), \quad \forall r > 0.
\]

Letting \( \delta \to 0 \), one obtains that

\[
\text{vol}[B(x, r)] \geq V_0(r), \quad \forall r > 0.
\]

This completes the proof of Theorem 1.1 for the case that \( C = K_{a,b} \).

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