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On Multiple Mixed Interior and Boundary Peak Solutions for Some Singularly Perturbed Neumann Problems

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Abstract. We consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in R^N, $\varepsilon > 0$ is a small parameter and f is a superlinear, subcritical nonlinearity. It is known that this equation possesses multiple boundary spike solutions that concentrate, as ε approaches zero, at multiple critical points of the mean curvature function $H(P)$, $P \in \partial\Omega$. It is also proved that this equation has multiple interior spike solutions which concentrate, as $\varepsilon \to 0$, at *sphere packing* points in Ω.

In this paper, we prove the existence of solutions with multiple spikes *both* on the boundary and in the interior. The main difficulty lies in the fact that the boundary spikes and the interior spikes usually have different scales of error estimation. We have to choose a special set of boundary spikes to match the scale of the interior spikes in a variational approach.

1 Introduction

Recently there is a large literature on the existence of spike layer solutions to the following singularly perturbed elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded smooth domain in R^N, $\varepsilon > 0$ is a constant, the exponent p satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 2$ and $\nu(x)$ denotes the normal derivative at $x \in \partial\Omega$.

Equation (1.1) is known as the stationary equation of the Keller-Segal system in chemotaxis. It can also be seen as the limiting stationary equation of the so-called Gierer-Meinhardt system in biological pattern formation, see [18], [21] and [23] for more details.

It is known that equation (1.1) admits boundary and interior spike solutions. For single boundary spike solutions, please see [7], [18], [21], [22], [28], *etc.* For multiple boundary

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spikes, please see [5], [10], [13], [14], [19], [23], [32], *etc.* (When $p = \frac{N+2}{N-2}$, similar results for the boundary spike layer solutions have been obtained by [1], [2], [3], [15], [20], [24], *etc.*) For single interior spikes, please see [8], [26], [27], [29], [31], *etc.* For multiple interior spikes, please see [4], [6], [17], *etc.* In particular, in [14], it was proved that at a local minimum point P_0 of the mean curvature function $H(P)$, for any positive integer $K \in \mathcal{N}$, there exists a solution of (1.1) with K boundary local maximum points $Q_\varepsilon^1, \ldots, Q_\varepsilon^K$ such that $Q_\varepsilon^i \to P_0$ as $\varepsilon \to 0$. On the other hand, in [17] it was proved that for any positive integer K, there exists a solution of (1.1) with K interior maximum points $P_\varepsilon^1, \ldots, P_\varepsilon^K$ such that

$$\varphi_K(P_\varepsilon^1, \ldots, P_\varepsilon^K) \to \max_{P_i \in \Omega} \varphi_K(P_1, \ldots, P_K)$$

where $\varphi_K(P_1, \ldots, P_K) = \min_{i \neq j,k}\big(d(P_k, \partial\Omega), \frac{1}{2}|P_i - P_j|\big)$.

In all the above papers, the boundary and interior spikes are *separated*. An interesting question is the following: can we construct multiple spike solutions with *both* boundary and interior spikes? The purpose of this paper is to construct such mixed boundary and interior spike solutions.

The main difficulty in constructing mixed boundary and interior spike solutions is that we need to deal with two completely different order of small terms. It is known that the order of boundary spike is of *algebraic* while the order of interior spike is of *exponentially small*. Since these two orders are simply incomparable, a new method should be employed so as to separate the two scales.

In fact we will consider a more general problem (as in [14] and [17])

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f \colon R^+ \to R$ is of class $C^{1+\sigma}$ and satisfies the following conditions

(f1) $f(t) \equiv 0$ for $t \leq 0$ and $f(t) \to +\infty$ as $t \to \infty$.

(f2) $f(0) = 0, f'(0) = 0$ and $f(u) = O(|u|^{p_1})$, $f'(u) = O(|u|^{p_2 - 1})$ as $|u| \to \infty$ for some $1 < p_1, p_2(\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2}$ if $N > 2; = \infty$ if $N \leq 2)$ and there exists $1 < p_3 < (\frac{N+2}{N-2})_+$ such that

$$|f_u(u + \phi) - f_u(u)| \leq \begin{cases} C|\phi|^{p_3 - 1} & \text{if } p_3 > 2 \\ C(|\phi| + |\phi|^{p_3 - 1}) & \text{if } p_3 \leq 2 \end{cases}$$

(f3) The following equation

$$\begin{cases} \triangle w - w + f(w) = 0, w > 0 & \text{in } R^N \\ w(0) = \max_{y \in R^N} w(y), w \to 0 & \text{at } \infty \end{cases} \tag{1.3}$$

has a unique solution $w(y)$ (by the results of [12], w is radial, *i.e.*, $w = w(r)$ and $w' < 0$ for $r = |y| \neq 0$) and w is nondegenerate. Namely the operator

$$L := \triangle - 1 + f'(w) \tag{1.4}$$

is invertible in the space $H^2_r(R^N) := \{u = u(|y|) \in H^2(R^N)\}$

Remark Nonlinearities satisfying (f1)–(f3) can be found in [9] and [17].

Fix any two positive integers $K_1 \in \mathbb{N}, K_2 \in \mathbb{N}$. We shall construct solutions to (1.1) with K_1 interior peaks and K_2 boundary peaks.

Our first assumption on the domain is the following:

(H1) $\exists Q^0_j \in \partial\Omega$ and $r_j > 0$ such that $H(Q^0_j) < H(Q), \forall Q \in B_{r_j}(Q^0_j) \cap \partial\Omega, j = 1, \dots, K_2$.

Remark Here we allow $Q^0_i = Q^0_j$ if $i \neq j$. Hence if the domain admits one strict local minimum point Q^0, we may take $Q^0_j = Q^0$, $j = 1, \dots, K_2, r_i = r_j$.

Set

$$\Gamma^B_j = B_{r_j}(Q^0_j) \cap \partial\Omega, \Gamma^B = \Gamma^B_1 \times \cdots \times \Gamma^B_{K_2}, H_j = H(Q^0_j),$$
$$\Lambda^B_\varepsilon = \left\{ \mathbf{Q} = (Q_1, \dots, Q_{K_2}) \in \Gamma^B, w\left(\frac{|Q_k - Q_l|}{\varepsilon}\right) < \eta\varepsilon, k, l = 1, \dots, K_2, k \neq l \right\}$$

where η is a small number and w is the unique solution of (1.3).

For any $\mathbf{P} = (P_1, \dots, P_{K_1}) \in \Omega^{K_1} = \Omega \times \Omega \times \cdots \times \Omega$, the following function was introduced in [17]

$$\varphi_{K_1}(P_1, P_2, \dots, P_{K_1}) = \min_{i,k,l=1,\dots,K_1; k\neq l} \left(d(P_i, \partial\Omega), \frac{1}{2}|P_k - P_l| \right).$$

In order to include the boundary-interior-spike interactions, we introduce

$$\tilde{\varphi}_{K_1}(P_1, P_2, \dots, P_{K_1}) = \min_{i=1,\dots,K_1, j=1,\dots,K_2} \left(\varphi_{K_1}(P_1, \dots, P_{K_1}), \frac{1}{2}|P_i - Q^0_j| \right).$$

Our second assumption on the domain is the following: there exists an open subset Λ^I of Ω^{K_1} such that

(H2) $\max_{P_1,\dots,P_{K_1}) \in \Lambda^I} \tilde{\varphi}_{K_1}(P_1, \dots, P_{K_1}) > \max_{(P_1,\dots,P_{K_1}) \in \partial\Lambda^I} \tilde{\varphi}_{K_1}(P_1, \dots, P_{K_1})$.

We emphasize that such a set Λ^I always exists. For example, we can take $\Lambda = \Omega^{K_1}$. We also observe that any such Λ^I can be modified so that for $\mathbf{P} = (P_1, \dots, P_{K_1}) \in \Lambda^I$ we have

$$d(P_i, \partial\Omega) > \delta > 0, |P_i - Q^0_j| > 2\delta > 0, |P_k - P_l| > 2\delta > 0 \tag{1.5}$$

for some sufficiently small $\delta > 0$, where $i, k, l = 1, \dots, K_1, k \neq l, j = 1, \dots, K_2$.

The main result of this paper is the following.

Theorem 1.1 *Assume that conditions (H1) and (H2) are satisfied. Let f satisfy assumptions (f1)–(f3). Then for any $K_1, K_2 \in \mathbb{N}$ and for ε sufficiently small problem (1.2) has a solution*

u_ε which possesses exactly $K = K_1 + K_2$ local maximum points $P_1^\varepsilon, \dots, P_{K_1}^\varepsilon$, $Q_1^\varepsilon, \dots, Q_{K_2}^\varepsilon$ with $\mathbf{P}^\varepsilon = (P_1^\varepsilon, \dots, P_{K_1}^\varepsilon) \in \Lambda^I$, $\mathbf{Q}^\varepsilon = (Q_1^\varepsilon, \dots, Q_{K_2}^\varepsilon) \in \Lambda^B$. Moreover $Q_j^\varepsilon \to Q_j^0$ and

$$\tilde{\varphi}_{K_1}(P_1^\varepsilon, \dots, P_{K_1}^\varepsilon) \to \max_{\mathbf{P} \in \Lambda^I} \tilde{\varphi}_{K_1}(\mathbf{P}). \tag{1.6}$$

Furthermore, we have

$$u_\varepsilon(x) \le C \exp\Big(-\frac{b \min_{i=1,\dots,K_1, j=1,\dots,K_2}(|x - P_i^\varepsilon|, |x - Q_j^\varepsilon|)}{\varepsilon}\Big) \tag{1.7}$$

for certain positive constants C, b.

Remark (1) Note that we can put $K_{2,j} \ge 0$ boundary spikes at Q_j^0 as long as we have $K_2 = \sum_{j=1}^{K_2} K_{2,j}$. So there are many possibilities on the combinations.

(2) When (H1) is satisfied, for any positive integers K_1, K_2 there exists a solution with $K_1 + K_2$ spikes which are located near the centers of spheres packed in the following way: All sphere are of largest possible equal radia, K_2 of them to be centered at Q_j^0, $j = 1, 2, \dots, K_2$ (which are defined in (H1) and could be repeated) and K_1 of them are packed inside the domain with the existence of of the above mentioned K_2 spheres. This follows from the above theorem in the simplest cases when $\Lambda^I = \Omega^{K_1}$ or its modifications such that $\mathbf{P} = (P_1, \dots, P_{K_1}) \in \Lambda^I$ implies

$$d(P_i, \partial\Omega) > \delta > 0, \quad |P_i - Q_j^0| > 2\delta > 0, \quad |P_k - P_l| > 2\delta > 0 \tag{1.8}$$

for some sufficiently small $\delta > 0$, where $i, k, l = 1, \dots, K_1$, $k \ne l$, $j = 1, \dots, K_2$.

(3) Note that there are domains which don't satisfy condition (H1). A simple example is a ball $\Omega = B_R$. In this case, one may use the symmetry to construct mixed boundary and interior spike solutions. It is an open question whether or not there always exists mixed multiple spike solutions.

We now outline the main idea of the proof of Theorem 1.1. Our idea is similar to that of [14] and [17]. However, the key step lies in separating the boundary spikes and interior spikes. As we mentioned earlier, boundary spikes are driven by algebraic order terms. Therefore, to minimize the algebraic effect of boundary spikes, one needs to find a function which approximates the boundary spikes up to *exponentially small* errors. To be more precise, we introduce some notations first.

Let w be the unique solution of (1.3). For any smooth bounded domain U, we set $\mathcal{P}_U w$ to be the unique solution of

$$\begin{cases} \Delta u - u + f(w) = 0 & \text{in } U, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } U. \end{cases} \tag{1.9}$$

For $P \in \bar{\Omega}$, we set

$$\Omega_\varepsilon = \{y : \varepsilon y \in \Omega\}, \quad \Omega_{\varepsilon,P} = \{y : \varepsilon y + P \in \Omega\},$$
$$w_{\varepsilon,P} = \mathcal{P}_{\Omega_{\varepsilon,P}} w.$$

If $P \in \Omega$, we set

$$\psi_{\varepsilon,P}(x) = -\varepsilon \log\Big(-\Big(w\big((x-P)/\varepsilon\big) - w_{\varepsilon,P}\Big)\Big)$$

$$\psi_\varepsilon(P) = \psi_{\varepsilon,P}(P).$$

(It is known (see [27]) that as $\varepsilon \to 0$, $\psi_\varepsilon(P) \to 2d(P, \partial\Omega)$.)

Associated with problem (1.2) is the following energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) - \int_\Omega F(u)$$

where $F(u) = \int_0^u f(s)\,ds$ and $u \in H^1(\Omega)$.

Let

$$S_\varepsilon(u) = \varepsilon^2 \Delta u - u + f(u).$$

Fix $\mathbf{P} = (P_1, P_2, \dots, P_{K_1}) \in \bar\Lambda^I$, we let

$$w_{\varepsilon,\mathbf{P}} = \sum_{i=1}^{K_1} w_{\varepsilon,P_i},$$

$$\mathcal{K}_{\varepsilon,\mathbf{P}} = \operatorname{span}\Big\{\frac{\partial w_{\varepsilon,P_i}}{\partial P_{i,l}}, i = 1,\dots,K_1, l = 1,\dots,N\Big\} \subset H^2(\Omega_\varepsilon)$$

$$\mathcal{C}_{\varepsilon,\mathbf{P}} = \operatorname{span}\Big\{\frac{\partial w_{\varepsilon,P_i}}{\partial P_{i,l}}, i = 1,\dots,K_1, l = 1,\dots,N\Big\} \subset L^2(\Omega_\varepsilon)$$

Similarly, if $\mathbf{Q} = (Q_1, \dots, Q_{K_2}) \in \bar\Lambda^B$, we can define

$$w_{\varepsilon,\mathbf{Q}} = \sum_{j=1}^{K_2} w_{\varepsilon,Q_j},$$

$$\mathcal{K}_{\varepsilon,\mathbf{Q}} = \operatorname{span}\Big\{\frac{\partial w_{\varepsilon,Q_j}}{\partial \tau_l(Q_j)}, l = 1,\dots,N-1, j = 1,\dots,K_2\Big\} \subset\in H^2(\Omega_\varepsilon)$$

$$\mathcal{C}_{\varepsilon,\mathbf{P}} = \operatorname{span}\Big\{\frac{\partial w_{\varepsilon,Q_j}}{\partial \tau_l(Q_j)}, l = 1,\dots,N-1, j = 1,\dots,K_2\Big\} \in L^2(\Omega_\varepsilon)$$

where $\tau_l(Q_j)$ is the l-th tangential derivatives of Q_j on $\partial\Omega$.

For mixed boundary and interior spikes, we define

$$w_{\varepsilon,\mathbf{P},\mathbf{Q}} = w_{\varepsilon,\mathbf{P}} + w_{\varepsilon,\mathbf{Q}},$$

$$\mathcal{K}_{\varepsilon,\mathbf{P},\mathbf{Q}} = \mathcal{K}_{\varepsilon,\mathbf{P}} \cup \mathcal{K}_{\varepsilon,\mathbf{Q}} \subset H^2(\Omega_\varepsilon),$$

$$\mathcal{C}_{\varepsilon,\mathbf{P},\mathbf{Q}} = \mathcal{C}_{\varepsilon,\mathbf{P}} \cup \mathcal{C}_{\varepsilon,\mathbf{Q}} \subset L^2(\Omega_\varepsilon).$$

In [14] and [17], the following two propositions are proved, respectively.

Proposition A **(See Lemma 3.4 in [14])** *For any* $(Q_1, \ldots, Q_{K_2}) \in \bar{\Lambda}^B$, *there exists a unique* $\Phi^B_{\varepsilon,\mathbf{Q}}$ *such that*

$$S_\varepsilon(w_{\varepsilon,\mathbf{Q}} + \Phi^B_{\varepsilon,\mathbf{Q}}) \in \mathcal{C}_{\varepsilon,\mathbf{Q}}, \Phi^\perp_{\varepsilon,\mathbf{Q}} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{Q}}. \tag{1.10}$$

Proposition B **(See Lemma 3.4 in [17])** *For any* $(P_1, \ldots, P_{K_1}) \in \bar{\Lambda}^I$, *there exists a unique* $\Phi^I_{\varepsilon,\mathbf{P}}$ *such that*

$$S_\varepsilon(w_{\varepsilon,\mathbf{P}} + \Phi^I_{\varepsilon,\mathbf{P}}) \in \mathcal{C}_{\varepsilon,\mathbf{P}}, \Phi^\perp_{\varepsilon,\mathbf{P}} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{P}}. \tag{1.11}$$

We now define

$$\tilde{w}_{\varepsilon,\mathbf{Q}} = w_{\varepsilon,\mathbf{Q}} + \Phi^B_{\varepsilon,\mathbf{Q}}, \tilde{w}_{\varepsilon,\mathbf{P}} = w_{\varepsilon,\mathbf{P}} + \Phi^I_{\varepsilon,\mathbf{P}}$$

Our main idea is to use $\tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{P}}$ as approximate solution. It turns out this choice separates the boundary and interior spikes.

Thus we let

$$u_\varepsilon = \tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{P}} + \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}.$$

We first solve $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}$ in $\mathcal{K}^\perp_{\varepsilon,\mathbf{P},\mathbf{Q}}$ by using the standard Liapunov-Schmidt reduction method. We show that $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}$ is C^1 in $\mathbf{P}, \mathbf{Q}$. After that, we define a new function

$$M_\varepsilon(\mathbf{P}, \mathbf{Q}) := \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{Q}} + \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}) \tag{1.12}$$

We maximize $M_\varepsilon(\mathbf{P}, \mathbf{Q})$ over $\bar{\Lambda}^B \times \bar{\Lambda}^I$. We show that the resulting solution has the properties of Theorem 1.1.

The paper is organized as follows. We present some important estimates in Section 2. Section 3 contains Liapunov-Schmidt procedure and we solve (1.2) up to approximate kernel and cokernel, respectively. We set up a maximizing problem in Section 4. Finally we show that the solution to the maximizing problem is indeed a solution of (1.2) and satisfies all properties of Theorem 1.1.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. $\delta > 0$ is a very small number.

We set

$$\gamma_1 := \int_{R^N} f(w) e^{y_1}\, dy$$

$$\Sigma := \Big\{ \int_{R^N_+} f\big(w(y)\big) e^{\langle b,y \rangle}\, dy \mid b \in R^N, |b| = 1 \Big\}$$

$$\Sigma_1 := \Big\{ \int_{R^N_+} f\big(w(y)\big) e^{\langle b,y \rangle}\, dy \mid b = (b_1, \ldots, b_N) \in R^N, b_N = 0, |b| = 1 \Big\}.$$

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2 Important Estimates

In this section, we first study boundary spikes and interior spikes separately. Then we obtain some important estimates on the interactions of boundary spikes and interior spikes. Many of the results are obtained in [14] and [17].

We introduce some notations first. For $\mathbf{P} \in \bar{\Lambda}^I$, $\mathbf{Q} \in \bar{\Lambda}^B$, we define

$$\langle u, v\rangle_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \nabla v + uv), \langle u, u\rangle_\varepsilon = \|u\|_\varepsilon^2.$$

Let

$$I(w) := \int_{R^N} \frac{1}{2}(|\nabla w|^2 + w^2) - \int_{R^N} F(w).$$

We first consider boundary spikes.

Lemma 2.1 **(See Lemma 3.4, Lemma 3.5 and Lemma 3.6 in [14])** *For any* $\mathbf{Q} = (Q_1, \dots, Q_{K_2}) \in \bar{\Lambda}^B$ *and* ε *sufficiently small, there exists a unique* $\Phi^B_{\varepsilon,\mathbf{Q}} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{Q}}$ *such that*

$$S_\varepsilon(w_{\varepsilon,\mathbf{Q}} + \Phi^B_{\varepsilon,\mathbf{Q}}) \in \mathcal{C}_{\varepsilon,\mathbf{Q}}.$$

Moreover $\Phi^B_{\varepsilon,\mathbf{Q}}$ *is* C^1 *in* $\mathbf{Q}$ *and we have*

$$J_\varepsilon(w_{\varepsilon,\mathbf{Q}} + \Phi^B_{\varepsilon,\mathbf{Q}}) = \varepsilon^N \Bigg[\frac{K_2}{2} I(w) - \varepsilon\big(\gamma_0 + o(1)\big) \sum_{i=1}^{K} H(Q_i)$$

$$- \frac{1}{2} \sum_{k,l=1,k\neq l}^{K} \big(\gamma_{kl} + o(1)\big) w\left(\frac{|Q_k - Q_l|}{\varepsilon}\right) + o(\varepsilon)\Bigg], \tag{2.1}$$

where $\gamma_0 > 0$ *is a positive number and* $\gamma_{kl} = \gamma_{lk} \in \Sigma$. *Furthermore, if* $w(\frac{|P_k - P_l|}{\varepsilon}) = \eta\varepsilon$, *we have* $\gamma_{kl} \in \Sigma_1$.

We need some properties of $\Phi^B_{\varepsilon,\mathbf{Q}}$.

Lemma 2.2 *Let* $\Phi^B_{\varepsilon,\mathbf{Q}}$ *be the solution constructed in Lemma 2.1. Then we have*

$$\Phi^B_{\varepsilon,\mathbf{Q}} = O(\varepsilon)\Big(\sum_{j=1}^{K_2} w^{1-\eta}_{\varepsilon,Q_j}\Big) \tag{2.2}$$

for any $0 < \eta < 1$.

Proof Note that $\Phi^B_{\varepsilon,\mathbf{Q}}$ satisfies

$$S_\varepsilon(w_{\varepsilon,\mathbf{Q}} + \Phi_{\varepsilon,\mathbf{Q}}) = \sum_{j=1,\dots,K_2,l=1,\dots,N-1} c^B_{jl} \frac{\partial w_{\varepsilon,Q_j}}{\partial \tau_l Q_j}$$

Moreover by Lemma 7.1 in [31], one can show that

$$c_{jl}^B = O(\varepsilon^2).$$

Hence for fixed R large, $\Phi^B_{\varepsilon,\mathbf{Q}}$ satisfies the following equation

$$\Delta\Phi^B_{\varepsilon,\mathbf{Q}} - (1-\eta)^2\Phi^B_{\varepsilon,\mathbf{Q}} + O\Big(\varepsilon\sum_{j=1}^{K_2} w\Big(y-\frac{Q_j}{\varepsilon}\Big)\Big) = 0 \quad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^{K_2} B_R(Q_j/\varepsilon)$$

$$|\Phi^B_{\varepsilon,\mathbf{Q}}| \le C\varepsilon \quad \text{on } \partial\Big(\bigcup_{j=1}^{K_2} B_R(Q_j/\varepsilon)\Big)$$

$$\frac{\partial\Phi^B_{\varepsilon,\mathbf{Q}}}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

Lemma 2.2 follows by comparison principle. ■

Next we consider the interior spikes. The following lemma is proved in [17].

Lemma 2.3 **(Lemma 2.6, Lemma 3.4 and Lemma 3.5 in [17])** *For any* $\mathbf{P} = (P_1,\dots,P_{K_1}) \in \bar\Lambda^I$ *and* ε *sufficiently small, there exists a unique* $\Phi^I_{\varepsilon,\mathbf{P}} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{P}}$ *such that*

$$S_\varepsilon(w_{\varepsilon,\mathbf{P}} + \Phi^I_{\varepsilon,\mathbf{P}}) \in \mathcal{C}_{\varepsilon,\mathbf{P}}.$$

Moreover $\Phi^I_{\varepsilon,\mathbf{P}}$ *is* C^1 *in* $\mathbf{P}$ *and we have*

$$J_\varepsilon(w_{\varepsilon,\mathbf{P}} + \Phi^I_{\varepsilon,\mathbf{P}}) = \varepsilon^N\Big[K_1 I(w) - \frac{1}{2}\big(\gamma_1 + o(1)\big)\sum_{i=1}^{K_1} e^{-\frac{1}{\varepsilon}\psi_\varepsilon(P_i)} - \big(\gamma_1 + o(1)\big)\sum_{i,l=1,i\ne l}^{K_1} w\Big(\frac{|P_i-P_l|}{\varepsilon}\Big)\Big], \tag{2.3}$$

where $\gamma_1 > 0$ *is given at the end of Section 1.*

We need some properties of $\Phi^I_{\varepsilon,\mathbf{P}}$.

Lemma 2.4 *Let* $\Phi^I_{\varepsilon,\mathbf{P}}$ *be the solution constructed in Lemma 2.3. Then we have*

$$\Phi^I_{\varepsilon,\mathbf{P}} = O(e^{-2\delta/\varepsilon})\Big(\sum_{j=1}^{K_1} w^{1-\eta}_{\varepsilon,P_j}\Big) \tag{2.4}$$

for any $0<\eta<1$.

Proof Note that $\Phi^I_{\varepsilon,\mathbf{P}}$ satisfies

$$S_\varepsilon(w_{\varepsilon,\mathbf{P}} + \Phi_{\varepsilon,\mathbf{P}}) = \sum_{i=1,\dots,K_1, l=1,\dots,N} c^I_{il} \frac{\partial w_{\varepsilon,P_i}}{\partial P_{i,l}}$$

Moreover as Lemma 7.1 in [27], one can show that (since $\mathbf{P} \in \bar{\Lambda}^I$)

$$c^I_{il} = O\big(e^{-\varphi_{K_1}(\mathbf{P})/\varepsilon}\big) = O(e^{-2\delta/\varepsilon}).$$

The rest of the proof is similar to that of Lemma 2.2. ■

Let $\tilde{w}_{\varepsilon,\mathbf{P}} = w_{\varepsilon,\mathbf{P}} + \Phi^I_{\varepsilon,\mathbf{P}}$ and $\tilde{w}_{\varepsilon,\mathbf{Q}} = w_{\varepsilon,\mathbf{Q}} + \Phi^B_{\varepsilon,\mathbf{Q}}$.

The next proposition considers the interaction between boundary and interior spikes.

Lemma 2.5

$$\int_\Omega f(\tilde{w}_{\varepsilon,\mathbf{Q}})\tilde{w}_{\varepsilon,\mathbf{P}} = \varepsilon^N\big(\gamma_{ij} + o(1)\big)\sum_{i,j} w(|P_i - Q_j|/\varepsilon), \tag{2.5}$$

$$\int_\Omega f(\tilde{w}_{\varepsilon,\mathbf{P}})w_{\varepsilon,\mathbf{Q}} = \varepsilon^N\big(\gamma_1 + o(1)\big)\sum_{i,j} w(|P_i - Q_j|/\varepsilon). \tag{2.6}$$

where $\gamma_{ij} \in \Sigma$.

Proof We first note that

$$\begin{aligned}\int_{\Omega_\varepsilon} f(w_{\varepsilon,Q_j})w_{\varepsilon,P_i} &= \big(1 + o(1)\big)\int_{R^N_+} f\big(w(y)\big)w\Big(\frac{|Q_j - P_i + \varepsilon y|}{\varepsilon}\Big)\\ &= \big(1 + o(1)\big)w(|Q_j - P_i|/\varepsilon)\int_{R^N_+} f(w)e^{-\frac{\langle Q_j - P_i, y\rangle}{|Q_j - P_i|}}\\ &= \big(\gamma_{ij} + o(1)\big)w(|Q_j - P_i|/\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_\varepsilon} f'(w_{\varepsilon,Q_j})|\Phi^B_{\varepsilon,\mathbf{Q}}||w_{\varepsilon,P_i}| &= \int_{\Omega_\varepsilon} O(\varepsilon w^{p-1+1-\eta}_{\varepsilon,Q_j} w_{\varepsilon,P_i})\\ &= o\big(w(|P_i - Q_j|/\varepsilon)\big)\end{aligned} \tag{2.7}$$

by Lemma 2.2.

Hence

$$\begin{aligned}\int_{\Omega_\varepsilon} f(\tilde{w}_{\varepsilon,\mathbf{Q}})w_{\varepsilon,\mathbf{P}} &= \sum_{i=1,\dots,K_1, j=1,\dots,K_2}\int_{\Omega_\varepsilon} f(w_{\varepsilon,Q_j})w_{\varepsilon,P_i}\\ &\quad + \int_{\Omega_\varepsilon} f'(w_{\varepsilon,\mathbf{Q}})\Phi^I_{\varepsilon,\mathbf{Q}}w_{\varepsilon,\mathbf{P}}\\ &\quad + o\Big(\sum_{i=1,\dots,K_1, j=1,\dots,K_2} w(|P_i - Q_j|/\varepsilon)\Big)\\ &= \sum_{i=1,\dots,K_1, j=1,\dots,K_2}\big(\gamma_{ij} + o(1)\big)w(|P_i - Q_j|/\varepsilon).\end{aligned}$$

This proves (2.5).

For (2.6), we observe that by Lemma 2.4

$$\int_{\Omega} f(w_{\varepsilon,Q_j})\Phi^I_{\varepsilon,\mathbf{P}} = \varepsilon^N \int_{\Omega_\varepsilon} O\Big(w^p\Big(\frac{x-Q_j}{\varepsilon}\Big)e^{-\delta/\varepsilon}w^{1-\eta}\Big(\frac{x-P_i}{\varepsilon}\Big)\Big)$$

$$= \varepsilon^N o\big(w(|P_i-Q_j|/\varepsilon)\big) \tag{2.8}$$

if η small, and

$$\int_{\Omega_\varepsilon} f(w_{\varepsilon,P_i})w_{\varepsilon,Q_j} = \big(1+o(1)\big)\int_{R^N} f\big(w(y)\big)w\Big(\frac{|Q_j-P_i-\varepsilon y|}{\varepsilon}\Big)$$

$$= \big(1+o(1)\big)w(|Q_j-P_i|/\varepsilon)\int_{R^N} f(w)e^{\frac{\langle Q_j-P_i,y\rangle}{|Q_j-P_i|}}.$$

Combining all the above estimates, we obtain the lemma. ■

Lemma 2.6

$$\varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{P}}+\tilde w_{\varepsilon,\mathbf{Q}}) = \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{Q}}) + K_1 I(w) - \frac{1}{2}\big(\gamma_1+o(1)\big)\sum_{i=1}^{K_1} e^{-\frac{1}{\varepsilon}\psi_{\varepsilon,P_i}(P_i)}$$

$$-\big(\gamma_1+o(1)\big)\sum_{i,l=1,i\neq l}^{K_1} w\Big(\frac{|P_i-P_l|}{\varepsilon}\Big)$$

$$-\sum_{i=1,\dots,K_1,j=1,\dots,K_2}\Big(\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_{ij}+o(1)\Big)w\Big(\frac{|P_i-Q_j|}{\varepsilon}\Big)$$

Proof

$$\varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{P}}+\tilde w_{\varepsilon,\mathbf{Q}}) = \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{Q}}) + \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{P}}) + \int_{\Omega_\varepsilon}[\nabla\tilde w_{\varepsilon,\mathbf{P}}\nabla\tilde w_{\varepsilon,\mathbf{Q}}+\tilde w_{\varepsilon,\mathbf{P}}\tilde w_{\varepsilon,\mathbf{Q}}]$$

$$-\int_{\Omega_\varepsilon}[F(\tilde w_{\varepsilon,\mathbf{P}}+\tilde w_{\varepsilon,\mathbf{Q}})-F(\tilde w_{\varepsilon,\mathbf{P}})-F(\tilde w_{\varepsilon,\mathbf{Q}})]$$

$$= \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{Q}}) + \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{P}})$$

$$+\int_{\Omega_\varepsilon} f(\tilde w_{\varepsilon,\mathbf{P}})\tilde w_{\varepsilon,\mathbf{Q}} + \int_{\Omega_\varepsilon} c^I_{il}\frac{\partial w_{\varepsilon,P_i}}{\partial P_{i,l}}\tilde w_{\varepsilon,\mathbf{Q}}$$

$$-\int_{\Omega_\varepsilon}[F(\tilde w_{\varepsilon,\mathbf{P}}+\tilde w_{\varepsilon,\mathbf{Q}})-F(\tilde w_{\varepsilon,\mathbf{P}})-F(\tilde w_{\varepsilon,\mathbf{Q}})]$$

$$= \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{Q}}) + \varepsilon^{-N}J_\varepsilon(\tilde w_{\varepsilon,\mathbf{P}})$$

$$-\int_{\Omega_\varepsilon} f(\tilde w_{\varepsilon,\mathbf{Q}})\tilde w_{\varepsilon,\mathbf{P}} + o\Big(\sum_{i=1,\dots,K_1,j=1,\dots,K_2} w(|P_i-Q_j|/\varepsilon)\Big)$$

By Lemma 2.5, Lemma 2.1 and Lemma 2.3, Lemma 2.6 is thus proved. ■

The following lemma will be useful in the next section.

Lemma 2.7

$$\|f(\tilde{w}_{\varepsilon,\mathbf{P}}+\tilde{w}_{\varepsilon,\mathbf{Q}})-f(\tilde{w}_{\varepsilon,\mathbf{P}})-f(\tilde{w}_{\varepsilon,\mathbf{Q}})\|_{L^2(\Omega_\varepsilon)} = O\Big(\sum_{i=1,\dots,K_1,j=1,\dots,K_2} w(|P_i-Q_j|/\varepsilon)^{(1+\eta_0)/2}\Big)$$

for any $\eta_0<\sigma=\min(1,p-1)$.

Proof Observe that

$$|f(\tilde{w}_{\varepsilon,\mathbf{P}}+\tilde{w}_{\varepsilon,\mathbf{Q}})-f(\tilde{w}_{\varepsilon,\mathbf{P}})-f(\tilde{w}_{\varepsilon,\mathbf{Q}})|\le C|f'(\tilde{w}_{\varepsilon,\mathbf{P}})|\tilde{w}_{\varepsilon,\mathbf{Q}}+|f'(\tilde{w}_{\varepsilon,\mathbf{Q}})|\tilde{w}_{\varepsilon,\mathbf{P}}.$$

Let us consider the first term on the right hand. We have

$$\begin{aligned}\int_{\Omega_\varepsilon}|f'(\tilde{w}_{\varepsilon,\mathbf{P}})|^2\tilde{w}_{\varepsilon,\mathbf{Q}}^2 &\le C\sum_{i,j}\int_{\Omega_\varepsilon}|f'(w_{\varepsilon,P_i})|^2|w_{\varepsilon,Q_j}|^2+C\int_{\Omega_\varepsilon}|f'(w_{\varepsilon,\mathbf{P}})|\,|\Phi^B_{\varepsilon,\mathbf{Q}}|^2\\ &\le C\sum_{i,j}|w(|Q_j-P_i|/\varepsilon)^{1+\sigma}|+O(\varepsilon^2)\sum_{i,j}|w(|Q_j-P_i|/\varepsilon)^{1+\sigma-2\eta}|\\ &\le C\sum_{i,j}|w(|Q_j-P_i|/\varepsilon)^{1+\sigma-2\eta}|\end{aligned}$$

where $\eta>0$ is any small number.

The second term on the right hand side can be estimated similarly. ■

3 Liapunov-Schmidt Reduction

In this section, we use the standard Liapunov-Schmidt reduction procedure to solve problem (1.2). Since this is a routine procedure, we omit most of the proofs. We refer [17] and [14] for technical details.

We first introduce some notations.

Let $H^2_N(\Omega_\varepsilon)$ be the Hilbert space defined by

$$H^2_N(\Omega_\varepsilon)=\Big\{u\in H^2(\Omega_\varepsilon)\;\Big|\;\frac{\partial u}{\partial\nu_\varepsilon}=0\text{ on }\partial\Omega_\varepsilon\Big\}.$$

Define

$$S_\varepsilon(u)=\Delta u-u+f(u)$$

for $u\in H^2_N(\Omega_\varepsilon)$. Then solving equation (1.2) is equivalent to

$$S_\varepsilon(u)=0,\,u\in H^2_N(\Omega_\varepsilon).$$

Fix $\mathbf{P}=(P_1,\dots,P_{K_1})\in\bar{\Lambda}^I$, $\mathbf{Q}=(Q_1,\dots,Q_{K_2})\in\bar{\Lambda}^B$. Let $\tilde{w}_{\varepsilon,\mathbf{P}}=w_{\varepsilon,\mathbf{P}}+\Phi^I_{\varepsilon,\mathbf{P}}$ be given by Lemma 2.3 and $\tilde{w}_{\varepsilon,\mathbf{Q}}=w_{\varepsilon,\mathbf{Q}}+\Phi^B_{\varepsilon,\mathbf{Q}}$ be given by Lemma 2.1.

In this section we solve the following equation

$$S_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{P}}+\tilde{w}_{\varepsilon,\mathbf{Q}}+\Phi)\in\mathcal{C}_{\varepsilon,\mathbf{P},\mathbf{Q}},\quad \Phi\in\mathcal{K}^\perp_{\varepsilon,\mathbf{P},\mathbf{Q}}.$$

We first have

Proposition 3.1 *For each* $(\mathbf{P},\mathbf{Q}) \in \bar{\Lambda}^I \times \bar{\Lambda}^B$, *there exists a unique* $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}} \in \mathcal{K}^{\perp}_{\varepsilon,\mathbf{P},\mathbf{Q}}$ *such that*

$$S_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{P}} + \tilde{w}_{\varepsilon,\mathbf{Q}} + \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}) \in \mathcal{C}_{\varepsilon,\mathbf{P},\mathbf{Q}} \tag{3.1}$$

Moreover,

$$\|\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}\|_{H^2(\Omega_\varepsilon)} \le C\|f(\tilde{w}_{\varepsilon,\mathbf{P}} + \tilde{w}_{\varepsilon,\mathbf{Q}}) - f(\tilde{w}_{\varepsilon,\mathbf{P}}) - f(\tilde{w}_{\varepsilon,\mathbf{Q}})\|_{L^2(\Omega_\varepsilon)} \tag{3.2}$$

and $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}$ *is* C^1 *smooth in* $\mathbf{P},\mathbf{Q}$.

Let us now define a new functional:

$$M_\varepsilon(\mathbf{P},\mathbf{Q}) = \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{P}} + \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}})$$
$$M_\varepsilon : \bar{\Lambda}^I \times \bar{\Lambda}^B \to R.$$

The following energy estimate for $M_{\varepsilon,\mathbf{P},\mathbf{Q}}$ is very important.

Proposition 3.2 *For any* $(\mathbf{P},\mathbf{Q}) \in \Lambda^B \times \Lambda^I$, *we have*

$$\begin{aligned} M_\varepsilon(\mathbf{P},\mathbf{Q}) = \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}}) + K_1 I(w) \\ - \frac{1}{2}\big(\gamma_1 + o(1)\big)\sum_{i=1}^{K_1}(e^{-\frac{1}{\varepsilon}\psi_\varepsilon(P_i)}) - \big(\gamma_1 + o(1)\big)\sum_{i,l=1,i\neq l}^{K_1} w\Big(\frac{|P_i - P_l|}{\varepsilon}\Big) \\ - \sum_{i=1,\dots,K_1,j=1,\dots,K_2}\Big(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_{ij} + o(1)\Big) w\Big(\frac{|P_i - Q_j|}{\varepsilon}\Big) \end{aligned}$$

Proof By Lemma 2.5, we just need to show that the introduction of $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}$ is negligible. Note that by Lemma 2.6 and Proposition 3.1, we have

$$\|\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}\|^2_{H^2(\Omega_\varepsilon)} = O\Big(\sum_{i,j} w^{1+\eta_0}(|P_i - Q_j|/\varepsilon)\Big).$$

Hence

$$\begin{aligned} M_\varepsilon(\mathbf{P},\mathbf{Q}) &= \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{P}}) + \langle \tilde{w}_{\varepsilon,\mathbf{P}}, \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}\rangle_\varepsilon + \langle \tilde{w}_{\varepsilon,\mathbf{Q}}, \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}\rangle_\varepsilon \\ &\quad - \varepsilon^{-N}\int_{\Omega_\varepsilon} f(\tilde{w}_{\varepsilon,\mathbf{P}} + \tilde{w}_{\varepsilon,\mathbf{Q}})\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}} + O(\|\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}\|^2_\varepsilon) \\ &= \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{P}}) + \int_{\Omega_\varepsilon}[f(\tilde{w}_{\varepsilon,\mathbf{P}} + \tilde{w}_{\varepsilon,\mathbf{Q}} \\ &\quad - f(\tilde{w}_{\varepsilon,\mathbf{P}}) - f(\tilde{w}_{\varepsilon,\mathbf{Q}})]\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}} + o\Big(\sum_{i,j} w(|P_i - Q_j|/\varepsilon)\Big) \\ &= \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}} + \tilde{w}_{\varepsilon,\mathbf{P}}) + o\big(w(|\mathbf{P} - \mathbf{Q}|/\varepsilon)\big). \end{aligned}$$

■

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4 The Reduced Problem: A Maximizing Procedure

In this section, we study a maximizing problem.

Fix $\mathbf{P} \in \bar{\Lambda}^I$, $\mathbf{Q} \in \bar{\Lambda}^B$. Let $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}$ be the solution given by Proposition 3.1. We define a new functional

$$M_\varepsilon(\mathbf{P},\mathbf{Q}) = J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{P}} + \tilde{w}_{\varepsilon,\mathbf{Q}} + \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}) : \bar{\Lambda}^I \times \bar{\Lambda}^B \to R \tag{4.1}$$

We shall prove

Proposition 4.1 *For ε small, the following maximizing problem*

$$\max\{M_\varepsilon(\mathbf{P},\mathbf{Q}) : (\mathbf{P},\mathbf{Q}) \in \bar{\Lambda}^I \times \bar{\Lambda}^B\} \tag{4.2}$$

has a solution $(\mathbf{P}^\varepsilon, \mathbf{Q}^\varepsilon) \in \Lambda^I \times \Lambda^B$.

Proof Since $M_\varepsilon(\mathbf{P},\mathbf{Q})$ is continuous in $\mathbf{P},\mathbf{Q}$, the maximizing problem has a solution. Let $M_\varepsilon(\mathbf{P}^\varepsilon,\mathbf{Q}^\varepsilon)$ be the maximum where $\mathbf{P}^\varepsilon \in \bar{\Lambda}^I$, $\mathbf{Q}^\varepsilon \in \bar{\Lambda}^B$.

We first claim that $\mathbf{Q}^\varepsilon \in \Lambda^B$.

In fact, suppose not. Note that $\partial\Lambda^B \subset \{Q_j \in \partial\Gamma_j^B \text{ or } w(\frac{|Q_k-Q_l|}{\varepsilon}) = \varepsilon\eta\}$. Hence if $\mathbf{Q} \in \partial\Lambda^B$, we have that either

$$H(Q_j) \geq \min_{P\in\partial\Gamma_j^B} H(P) \geq H_j + 2\delta_0$$

for some $j = 1,\dots,K_2$ and $\delta_0 > 0$ (by condition (H1)) or

$$\frac{1}{\varepsilon} w\Big(\frac{|Q_k-Q_l|}{\varepsilon}\Big) = \eta$$

for some $k \neq l$.

Hence if $\mathbf{Q} \in \partial\Lambda^B$ we have

$$\gamma_1 \sum_{j=1}^{K_2} H(Q_j^\varepsilon) + \frac{1}{\varepsilon}\sum_{k\neq l}\Big(\frac{1}{2}\gamma_{kl} + o(1)\Big) w\Big(\frac{|Q_k^\varepsilon - Q_l^\varepsilon|}{\varepsilon}\Big)$$
$$\geq \gamma_1 \sum_{j=1}^{K_2} H_j + \min\big(\gamma_1\delta_0, \min_{k\neq l, w(\frac{|Q_k-Q_l|}{\varepsilon})=\eta\varepsilon} \gamma_{kl}\eta\big).$$

Note that $\min_{k\neq l, w(\frac{|P_k-P_l|}{\varepsilon})=\eta\varepsilon} \gamma_{kl} \geq \inf_{\tau\in\Sigma_1} \tau \geq \gamma_0 > 0$ since for any $\tau \in \Sigma_1$, we have

$$\tau = \int_{R_+^N} f(w)e^{\langle b,y\rangle} = \frac{1}{2}\int_{R^N} f(w)e^{\langle b,y\rangle} > 0.$$

Therefore we have

$$M_\varepsilon(\mathbf{P},\mathbf{Q}) \leq \Big(K_1 + \frac{K_2}{2}\Big) I(w) - \varepsilon\Big[\gamma_1 \sum_{j=1}^{K_2} H_j + \min\big(\gamma_1\delta_0, \min_{k\neq l, w(\frac{|P_k-P_l|}{\varepsilon})=\eta\varepsilon} \gamma_{kl}\eta\big)\Big] + O(e^{-\delta/\varepsilon}).$$

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On the other hand, if we choose Q_j such that $H(Q_j) \to H_j$ and $w(\frac{|Q_k - Q_l|}{\varepsilon})\frac{1}{\varepsilon} \to 0$. We will have

$$M_\varepsilon(\mathbf{P}, \mathbf{Q}) \geq (K_1 + K_2/2)I(w) - \varepsilon\gamma_1 \sum_{i=1}^{K_2} H(Q_i^\varepsilon) - \sum_{k\neq l}\Big(\frac{1}{2}\gamma_{kl} + o(1)\Big) w\Big(\frac{|Q_k^\varepsilon - Q_l^\varepsilon|}{\varepsilon}\Big)$$
$$\geq (K_1 + K_2/2)I(w) - \varepsilon\Big[\gamma_1 \sum_{j=1}^{K_2} H_j - \delta\Big]$$

for any $\delta > 0$.

A contradiction!

Hence $\mathbf{Q}^\varepsilon \in \Lambda^B$. Moreover the above argument also shows that $Q_j^\varepsilon \to Q_j^0$ as $\varepsilon \to 0$.

We next claim that $\mathbf{P}^\varepsilon \in \Lambda^I$.

In fact suppose not, we have that $\mathbf{P}^\varepsilon \in \partial\Lambda^I$ and hence

$$\beta_\varepsilon := \tilde{\varphi}_{K_1}(\mathbf{P}^\varepsilon) \leq \max_{\mathbf{P}\in\partial\bar{\Lambda}^I} \tilde{\varphi}_{K_1}(\mathbf{P}) := \beta_1.$$

Note that since $Q_j^\varepsilon \to Q_j^0$, $|P_i^\varepsilon - Q_j^\varepsilon| \geq 2\delta + o(1)$ for $i = 1, \ldots, K_1$, $j = 1, \ldots, K_2$.

In this case, we have by Lemma 3.2 and Lemma 2.5

$$M_\varepsilon(\mathbf{P}^\varepsilon, \mathbf{Q}^\varepsilon) = \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{P}^\varepsilon} + \tilde{w}_{\varepsilon,\mathbf{Q}^\varepsilon} + \Phi_{\varepsilon,\mathbf{P}^\varepsilon,\mathbf{Q}^\varepsilon})$$
$$= \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}^\varepsilon}) + K_1 I(w) - \frac{1}{2}\big(\gamma_1 + o(1)\big) \sum_{i=1}^{K_1} e^{-\psi_\varepsilon(P_i^\varepsilon)}$$
$$- \sum_{k\neq l, k,l=1,\ldots,K_1} \big(\gamma_1 + o(1)\big) w\Big(\frac{|P_k^\varepsilon - P_l^\varepsilon|}{\varepsilon}\Big)$$
$$- \sum_{i=1,\ldots,K_1, j=1,\ldots,K_2} \Big(\frac{1}{2}\gamma + \frac{1}{2}\gamma_{ij} + o(1)\Big) w\Big(\frac{|P_i^\varepsilon - Q_j^\varepsilon|}{\varepsilon}\Big)$$
$$\leq \max_{\mathbf{Q}\in\bar{\Lambda}^B} \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}}) + K_1 I(w) - c_1 e^{-(2+o(1))\beta_\varepsilon/\varepsilon} \tag{4.3}$$

On the other hand, if we take $\mathbf{Q}_0^\varepsilon$ such that

$$\max_{\mathbf{Q}\in\bar{\Lambda}^B} \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}}) = \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}_0^\varepsilon})$$

(hence $\mathbf{Q}_0^\varepsilon \to \mathbf{Q}_0$) and $\mathbf{P}_\varepsilon$ such that it attains the following maximizing problem

$$\max_{\mathbf{P}\in\Lambda^I} \min\Big(\varphi_{K_1}(\mathbf{P}), \frac{1}{2}|P_i - (\mathbf{Q_0}^\varepsilon)_j|\Big) := d_\varepsilon \tag{4.4}$$

we obtain that

$$M_\varepsilon(\mathbf{P}_0, \mathbf{Q}^\varepsilon) \geq \max_{\mathbf{Q}\in\bar{\Lambda}^B} \varepsilon^{-N} J_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{Q}}) + K_1 I(w) - c_2 e^{-(2+o(1))\, d_\varepsilon/\varepsilon} \tag{4.5}$$

for some $c_2 > 0$.

Note that as $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} d_\varepsilon \to \max_{\mathbf{P}\in\Lambda^I} \tilde{\varphi}_{K_1}(\mathbf{P}) > \beta_1 \geq \lim_{\varepsilon\to 0} \beta_\varepsilon \tag{4.6}$$

by assumption (H2).

Since $M_\varepsilon(\mathbf{P}^\varepsilon)$ is the maximum, we have by (4.3) and (4.5)

$$d_\varepsilon \leq \beta_\varepsilon + o(1)$$

A contradiction to (4.6)!

Therefore $\mathbf{P}^\varepsilon \in \Lambda^I$.

Moreover, the same arguments show that

$$\tilde{\varphi}_{K_1}(P_1^\varepsilon, \ldots, P_{K_1}^\varepsilon) \to \max_{\mathbf{P}\in\Lambda^I} \tilde{\varphi}_{K_1}(\mathbf{P})$$

as $\varepsilon \to 0$. This completes the proof of Proposition 4.1.

5 Proof of Theorem 1.1

In this section, we apply results in Section 3 and Section 4 to prove Theorem 1.1.

Proof of Theorem 1.1 By Proposition 3.1, there exists ε_0 such that for $\varepsilon < \varepsilon_0$ we have a C^1 map which, to any $\mathbf{P} \in \bar{\Lambda}^I$, $\mathbf{Q} \in \bar{\Lambda}^B$, associates $\Phi_{\varepsilon,\mathbf{P},\mathbf{Q}} \in \mathcal{K}^\perp_{\varepsilon,\mathbf{P},\mathbf{Q}}$ such that

$$S_\varepsilon(\tilde{w}_{\varepsilon,\mathbf{P}} + \tilde{w}_{\varepsilon,\mathbf{Q}} + \Phi_{\varepsilon,\mathbf{P},\mathbf{Q}}) = \sum_{il} \alpha^I_{il} \frac{\partial w_{\varepsilon,P_i}}{\partial P_{i,l}} + \sum_{jl} \alpha^B_{jl} \frac{\partial w_{\varepsilon,Q_j}}{\partial \tau_l Q_j}$$

for some constants $\alpha^I_{il} \in R^{K_1 N}$, $\alpha^B_{jl} \in R^{K_2(N-1)}$.

By Proposition 4.1, we have $(\mathbf{P}^\varepsilon, \mathbf{Q}^\varepsilon) \in \Lambda^I \times \Lambda^B$, achieving the maximum of the maximization problem in Proposition 4.1. Let $\Phi_\varepsilon = \Phi_{\varepsilon,\mathbf{P}^\varepsilon,\mathbf{Q}^\varepsilon}$ and $u_\varepsilon = \tilde{w}_{\varepsilon,\mathbf{P}^\varepsilon} + \tilde{w}_{\varepsilon,\mathbf{Q}^\varepsilon} + \Phi_{\varepsilon,\mathbf{P}^\varepsilon,\mathbf{Q}^\varepsilon}$. Then we have

Proposition 5.1 *u_ε is a critical point of J_ε if and only if $(\mathbf{P}^\varepsilon, \mathbf{Q}^\varepsilon)$ is a critical point of M_ε.*

Proof The proof is similar to the proof of Proposition 4.1 in [17]. ■

By the above Proposition, u_ε is a critical point of J_ε. Hence u_ε satisfies

$$\varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + f(u_\varepsilon) = 0, \quad \text{in } \Omega$$
$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Multiplying the above equation by $u_\varepsilon^- = \min(0, u_\varepsilon)$, we obtain

$$\langle u_\varepsilon^-, u_\varepsilon^- \rangle_\varepsilon = \int_{\Omega_\varepsilon} f(u_\varepsilon^-) u_\varepsilon^-.$$

Hence we have

$$\Big(\int_{\Omega_\varepsilon}(u_\varepsilon^-)^{p+1}\Big)^{2/(p+1)} \le C\|u_\varepsilon^-\|_\varepsilon^2 \le C\int_{\Omega_\varepsilon}|u_\varepsilon^-|^{p+1}$$

since p is subcritical.

Thus either

$$\int_{\Omega_\varepsilon}|u_\varepsilon^-|^{p+1} \ge C$$

or

$$u_\varepsilon^- \equiv 0.$$

By our construction, it is easy to see that $\int_{\Omega_\varepsilon}|u_\varepsilon^-|^{p+1} = o(1)$. Hence $u_\varepsilon \ge 0$. By Maximum Principle $u_\varepsilon > 0$ in Ω. Moreover $\varepsilon^N J_\varepsilon(u_\varepsilon) \to (K_1 + K_2/2)I(w)$ and u_ε has only K local maximum points $\tilde{P}_1^\varepsilon, \dots, \tilde{P}_{K_1}^\varepsilon$, $\tilde{Q}_1^\varepsilon, \dots, \tilde{Q}_{K_2}^\varepsilon$. By the structure of u_ε we see that (up to a permutation) $\tilde{P}_i^\varepsilon - P_i^\varepsilon = o(1)$, $\tilde{Q}_j^\varepsilon - Q_j^\varepsilon = o(1)$. Hence $\tilde{\varphi}_{K_1}(\tilde{P}_1^\varepsilon, \dots, \tilde{P}_{K_1}^\varepsilon) \to \max_{\mathbf{P}\in\Lambda^I}\tilde{\varphi}_{K_1}(P_1, \dots, P_{K_1})$. This proves Theorem 1.1.

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