A REMARK ABOUT A CERTAIN CLASS OF DISTRIBUTION SPACES

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1. Introduction

The object of this note is to exhibit a certain class of distribution spaces as being c-admissible in the sense of [1] and [2]. Throughout the terminology and notation are the same as in [1] and [2]. The only addition to this is that if \( x \in \mathbb{R}^n \) then \( \tau_x \) will denote the translation operator which carries each distribution \( u \) onto the distribution \( u_x \).

2. c-admissibility of certain spaces

We shall prove the following result.

**Proposition.** Let \( E \) be an admissible space which is barrelled and \( B_r \)-complete, and which is a module over \( S \) with respect to convolution.

Consider the following two hypotheses:

(i) \( E \) is translation-invariant and for each \( u \in E \) the mapping \( x \rightarrow u_x \) of \( \mathbb{R}^n \) into \( E \) is bounded on compact subsets of \( \mathbb{R}^n \).

(ii) \( E \) is dilation-invariant and for each \( u \in E \) the mapping \( x \rightarrow u^x \) of \( \mathbb{R}^\# \) into \( E \) is bounded on compact subsets of \( \mathbb{R}^\# \).

Then the conclusions are:

(a) If (i) holds then \( E \) is c-admissible.

(b) If both (i) and (ii) hold then \( E \) is a dilation space.

**Proof.** We shall begin with the proof of assertion (a). Thus we assume that (i) holds.

Our first task is to show that for each \( x \in \mathbb{R}^n \) the mapping \( u \rightarrow u_x \) of \( E \) into itself is continuous. Since \( E \) is both barrelled and \( B_r \)-complete, the closed graph theorem (Theorem 8.9.4 in Edwards [3] and the first Remark following it) tells us that it is sufficient to show that the linear operator \( \tau_x \) (considered as a mapping of \( E \) into itself) has a closed graph. To this end
we assume that \( x \in R^n \) is fixed, and that \((u_i)\) is a net in \( E \) such that \( \lim_i u_i = u \) in \( E \) and \( \lim \tau_x u_i = w \) in \( E \). Then for each \( \phi \in S \) it follows (because of relation (2.1) in [1]) that

\[
w * \phi(0) = \lim_i (\tau_x u_i) * \phi(0) = \lim_i u_i * \phi_x(0) = u * \phi_x(0) = (\tau_x u) * \phi(0).
\]

Thus \( w = \tau_x u \) and the graph of \( \tau_x \) is indeed closed.

Our second requirement is to show that for each \( u \in E \) and each \( v \in E' \) the mapping \( x \mapsto \langle u_x, v \rangle \) defines a continuous function on \( R^n \). With this end in mind, let \( b \in R^n \) be arbitrary but fixed. Let \( K \) be the set \( \{ x \in R^n : |x-b| \leq 1 \} \) and consider the set \( \{ \tau_x : x \in K \} \) of continuous linear mappings of \( E \) into itself. Write \( E_0 \) for the set of all \( u \in E \) for which \( \lim_{x \to b} \tau_x u \) exists in \( E \) and define the mapping \( T \) of \( E_0 \) into \( E \) by \( Tu = \lim_{x \to b} \tau_x u(u \in E_0) \). We notice that for each fixed \( \phi \in S \), the mapping \( x \in R^n \to \phi_x \in S \to \phi_x \in E \) is continuous; and hence that

\[
(2.1) \quad \lim_{x \to b} \tau_x \phi = \tau_b \phi \quad \text{in} \quad E.
\]

It follows from this that \( E_0 \) contains \( S \), which is dense in \( E \). Secondly, our assumption that (i) holds entails that the set \( \{ \tau_x : x \in K \} \) of continuous linear mappings is bounded at each point of \( E \). Thus, since \( E \) is \( B_r \)-complete and hence quasi-complete, we may refer to Corollary 7.1.4 in Edwards [3] and deduce that \( E_0 = E \) and that \( T \) is a continuous linear mapping of \( E \) into itself. But relation (2.1) shows that \( T \) coincides with the continuous linear mapping \( \tau_b \) on the dense vector subspace \( S \) of \( E \); whence it follows that the two mappings are identical. Thus \( \lim_{x \to b} \tau_x u = \tau_b u \) for each \( u \in E \).

Since \( b \in R^n \) is arbitrary, we now infer that for each \( u \in E \), the mapping \( x \to u_x \) is continuous from \( R^n \) into \( E \). It follows immediately that for each \( u \in E \) and each \( v \in E' \), the mapping \( x \to \langle u_x, v \rangle \) defines a continuous function on \( R^n \), which is what we wished to prove.

Next consider a fixed \( \phi \in S \). We claim that the mapping \( u \to u * \phi \) of \( E \) into itself is continuous. To verify this assertion, it is sufficient to show that the graph of this mapping is closed; the desired conclusion will then follow from the closed graph theorem. Thus let \( (u_i) \) be a net in \( E \) such that \( \lim_i u_i = w \) in \( E \) and \( \lim_i u_i * \phi = w \) in \( E \). Then for each \( \psi \in S \) we have \( w * \psi(0) = \lim_i u_i * \phi * \psi(0) = u * \phi * \psi(0) \). Hence \( w = u * \phi \) and the mapping \( u \in E \to u * \phi \in E \) is closed, as required.

If we recall that \( S \) is barrelled, then a similar argument shows that for each \( u \in E \), the mapping \( \phi \to u * \phi \) of \( S \) into \( E \) is continuous.
We shall now complete the proof of part (a) of the Proposition. Let \( u \in E \) and \( v \in E' \) be arbitrary but fixed. We must show that the continuous function \( x \to \langle u_x, v \rangle \) \((x \in \mathbb{R}^n)\) generates a temperate distribution on \( \mathbb{R}^n \). In view of the last paragraph, the mapping \( \phi \to \langle u * \phi, v \rangle \) \((\phi \in S)\) defines a temperate distribution, which we denote by \( s \). We shall show that the function \( x \to \langle u_x, v \rangle \) \((x \in \mathbb{R}^n)\) generates precisely this distribution \( s \). To do this it is sufficient to show that for each \( \psi \in D \)

\[
\int_{\mathbb{R}^n} \langle u_x, v \rangle \psi(-x) \, dx = s \ast \psi(0).
\]

Let \( \psi \in D \) be arbitrary. Choose a net \( (\phi_i) \) in \( S \) such that \( \lim_i \phi_i = u \) in \( E \). We notice that, because of Theorem 2.2(a) in [1] and the continuity of the functions \( x \to \langle u_x, v \rangle \) on \( \mathbb{R}^n \), the mapping \( x \to u_x \) is continuous from \( \mathbb{R}^n \) into \( E' \) for the weak topology on \( E' \). Therefore the set \( \{v_x : x \in \text{supp} \psi\} \) is a weakly compact, hence weakly bounded, hence equicontinuous (because \( E \) is barrelled) subset of \( E' \). In view of this we conclude that

\[
\lim_i \phi_i \ast v(x) = \lim_i \langle \phi_i, v_x \rangle = \langle u, v_x \rangle = \langle u_x, v \rangle
\]

uniformly for \( x \in \text{supp} \psi \). It follows that

\[
\int_{\mathbb{R}^n} \langle u_x, v \rangle \psi(-x) \, dx = \lim_i \int_{\mathbb{R}^n} \phi_i \ast v(x) \psi(-x) \, dx = \lim_i \phi_i \ast \psi \ast v(0) = \lim_i \langle \phi_i \ast \psi, v \rangle.
\]

Now we have shown above that the mapping \( w \in E \to w \ast \psi \in E \) is continuous. Therefore

\[
\lim_i \langle \phi_i \ast \psi, v \rangle = \langle u \ast \psi, v \rangle = s \ast \psi(0).
\]

Relations (2.3) and (2.4) together ensure that (2.2) holds; whence we infer that the function \( x \to \langle u_x, v \rangle \) can indeed be identified with a temperate distribution on \( \mathbb{R}^n \). Since \( u \in E \) and \( v \in E' \) were arbitrary, this completes the proof of (a).

The validity of part (b) of the Proposition will be established if we can show that (ii) entails that the mapping \( x \to u^x \) of \( \mathbb{R}^n \) into \( E \) and the mapping \( u \to u^x \) of \( E \) into itself are both continuous (for the given topology on \( E \)); and the truth of this may be verified by using arguments analogous to those which we employed above to establish the continuity of the mappings \( x \to u_x \) of \( \mathbb{R}^n \) into \( E \) and \( u \to u_x \) of \( E \) into itself.

References


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