Determinantal Systems of Apolar Triads in the Twisted Cubic.

By Dr W. L. Marr.

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Introduction.

It has already been shown that nine points can be found on a conic to form six triads all apolar to a given triad (ABC), the hessian lines of these six triads and of ABC being concurrent.

The construction of the corresponding system in the twisted cubic is easy. In the first place, the poles of the six triads $P_1 Q_2 R_3$, etc., will lie on the plane of the triad ABC to which they are all apolar. Secondly, the hessian lines of these triads, which in the conic were concurrent, will now be generating lines of a quadric circumscribing the twisted cubic. Now the pole of a triad is in the osculating plane at any of the points of the triad, hence the osculating plane at P_1 will contain the poles of the triads $P_1 Q_2 R_3$ and $P_1 Q_3 R_2$. But these poles are also the intersections of the hessian lines of $P_1 Q_2 R_3$ and $P_1 Q_3 R_2$ with the plane of ABC, and thus lie on the conic formed by the intersection of ABC and the quadric just mentioned.

1. *The nine point system for the twisted cubic is accordingly determined as follows:*

Draw the osculating planes at P_1, Q_2, R_3 to cut the conic which is the intersection of the plane of the fundamental triad ABC with the quadric which is generated by chords joining the corresponding points of the involution determined by the hessian points of ABC and $P_1 Q_2 R_3$. The six points of intersection of these three osculating planes and the conic will be the six poles of the six triads, and from these six poles the nine points of the determinantal system can be readily found.

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2. Let the twisted cubic be given by

$$x=p^3,\ y=3p^2,\ z=3p,\ t=1.$$

If the parameters of the hessian points of $P_1 Q_2 R_3$ be h_1, h_2, then $h_1 h_2=K$ since the product of the parameters of the hessian points of any triad is constant.

The quadric generated by the six hessian lines of the six triads $P_1 Q_2 R_3$, etc., is

$$3zx-y^2-K(3yt-z^2)=0.$$

The plane of ABC (parameters $-1,\ -\omega_1-\omega^2$) is

$$x+t=0.$$

The osculating plane at P_1 (parameter l_1) is

$$x-l_1 y+l_1^2 z-l_1^3 t=0.$$

The six poles of the six triads are the intersections of

$$x(1+l^3)-yl+zl^2=0 \quad (\text{where } l=l_1,\ l_2,\ l_3)$$

and

$$3zx-y^2+K(3xy+z^2)=0.$$

3. Let the poles of $P_1 Q_2 R_3$, $P_2 Q_3 R_1$ and $P_3 Q_1 R_2$ be U_1, U_2, U_3 and of $P_1 Q_3 R_2$, $Q_2 R_1 P_3$ and $R_3 P_2 Q_1$ be V_1, V_2, V_3.

The line $U_1 V_1$ is $x(1+l_1^3)-yl_1+zl_1^2=0$

$U_1 V_2$ is $x(1+l_2^3)$ etc. $=0$

$U_1 V_3$ is $x(1+l_3^3)$ etc. $=0$

for $U_1 V_1$ is the intersection of the osculating plane at P_1 with ABC.

These lines cut $x=0$ when $y=zl_1$, $y=zl_2$, etc.

Hence the nine lines joining the vertices of $U_1 U_2 U_3$ to the vertices of $V_1 V_2 V_3$ cut the line $x=0$ in a determinantal system of points.

4. *The line $x=0$ is the line joining the hessian poles of $U_1 U_2 U_3$ and $V_1 V_2 V_3$.*

The equation of the conic may be written

$$(y+K z)^2=(Ky+z)(3x\sqrt{1-K^3}+K\sqrt{Ky+z})$$

or

$$Y^2=ZX$$

where $Y=0$ is the line joining the pole F of ABC and the pole M of the triangle which is apolar to ABC, and the parameters of whose hessian points are $\pm i\sqrt{K}$.

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The line joining the hessian points of $P_1 Q_2 R_3$ meets the conic in U_1 whose coordinates are $x = K - (h_1 + h_2)^2$, $y = 3(K^3 - \overline{h_1 + h_2})$, $z = 3(K\overline{h_1 + h_2} - 1)$.

The parameter of U_1 with respect to $Y^2 = ZX$ is $h_1 + h_2$, and the parameters of U_1, U_2, U_3 are the roots of the cubic

$$p_3 - 3Kp + h_1^3 + h_2^3 = 0$$

the hessian line of which is

$$X - \frac{h_1^3 + h_2^3}{K} Y + KZ = 0,$$

and hence the hessian poles of $U_1 U_2 U_3$ and $V_1 V_2 V_3$ lie on $X - KZ = 0$, *i.e.* $x = 0$.

This line $(x = 0)$ is also the double tangent of the quartic (class 3) which is the envelope of $x(1 + l^3) - yl + zl^2 = 0$.

5. *The invariant* Q (*v.* Salmon's Higher Algebra) *of the triads* $U_1 U_2 U_3$ *and* $V_1 V_2 V_3$ *vanishes.*

This readily follows from the cubic equation for the parameters of $U_1 U_2 U_3$, and the corresponding equations for those of $V_1 V_2 V_3$, viz. $p^3 - 3Kp + k_1^3 + k_2^3 = 0$.

It means that the pole of $x = 0$ (the line joining their hessian poles) with respect to $U_1 U_2 U_3$ and $V_1 V_2 V_3$ is the same point F on the conic which, as we have said, is also the pole of ABC in the twisted cubic.

6. *Relation to nodal cubic.*

The coordinates of the line $U_1 V_1$ are $1 + l^3$, $-l$, l^2.

Transforming from line to point let $x = 1 + l^3$, $y = -l$, $z = l^2$.

The quartic (class 3) which is the envelope of UV now becomes a nodal cubic, viz. $y^3 - z^3 - xyz = 0$.

i.e. $(x - 3y + 3z)(x - 3\omega y + 3\omega^2 z)(x - 3\omega^2 y + 3\omega z) = x^3$.

The conic which belongs to the four point family $ABCF$ now becomes a conic touching four lines which are the three tangents at the flexes and the line of the flexes of the nodal cubic.

Let $U_1 V_1$, $U_1 V_2$, $U_1 V_3$ transform into three collinear points X_1, X_2, X_3, $U_2 V_1$, $U_2 V_2$, $U_2 V_3$ into Y_1, Y_2, Y_3 and $U_3 V_1$, etc., into Z_1, etc.

The rows and colums of

$$\begin{matrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{matrix}$$

are six tangents to the four-line conic.

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The theorem may be stated thus:—Given any conic touching the tangents at the three flexes and the line of flexes of a nodal cubic, then if tangents be drawn to the conic from three collinear points on the cubic, the line of which three points touches the conic, and tangents again be drawn to the conic from the points where the previous tangents cut the cubic, there will finally result a hexagon circumscribing the conic with nine of the intersections of the sides of the hexagon lying on the nodal cubic.

7. We have seen that if $U_1 U_2 U_3$ and $V_1 V_2 V_3$ be two triads on a conic for which the Q invariant vanishes, the lines joining the vertices of one triad to the vertices of the other cut the line joining their hessian poles in a determinantal system of nine points.

Dr Milne has generalised this property to any pair of triads on a conic, *i e.* if $U_1 U_2 U_3$ and $V_1 V_2 V_3$ be any pair of triads, whether the Q invariant vanishes or not, the lines $U_1 V_1$, etc., cut the line joining their hessian poles in a determinantal system.

8. *Conversely, if two triads are such that the joins of their vertices cut a line in a determinantal system the two triads are con-conical, and the line is the line joining their hessian poles.*

Let $U_1 U_2 U_3$ be a triad of points not collinear, and $P_1 Q_2 R_3$, etc., a nine point system on a straight line of which H and K are the hessian points of ABC, the triad collinear with $P_1 Q_2$, etc., which is apolar to $(P_1 Q_2 R_3)$, etc.

The point V_1 will be the point of concurrence of $U_1 P_1$, $U_2 Q_2$, $U_3 R_3$, and similarly for V_2 and V_3. We have to prove that $U_1 U_2 U_3$ and $V_1 V_2 V_3$ are con-conical.

Let F, H, K be the triangle of reference, F being the pole of HK with regard to $U_1 U_2 U_3$.

$$U_2 U_3 \equiv x + M_1 y + N_1 z \equiv X = 0$$
$$U_3 U_1 \equiv x + M_2 y + N_2 z \equiv Y = 0$$
$$U_1 U_2 \equiv x + M_3 y + N_3 z \equiv Z = 0.$$

Since F is the pole of $x=0$ (*i e.* HK) with respect to $U_1 U_2 U_3$ it follows that $x + M_1 y + N_1 z + x + M_2 y + N_2 z + x + M_3, \ + N_3 z$ is x. Therefore $M_1 + M_2 + M_3 = 0$, $N_1 + N_2 + N_3 = 0$.

Let the point U_1 be $(p_1 q_1 r_1)$

U_2 be $(p_2 q_2 r_2)$

U_3 be $(p_3 q_3 r_3)$.

Then $p_1 = M_2 N_3 - M_3 N_2 = p_2 = p_3$
$q_1 = N_2 - N_3,\ q_2 = N_3 - N_1,$ etc.
$r_1 = M_3 - M_2,\ r_2 =$ etc.

Let V_1 be $(x_1 y_1 z_1)$ so for V_2 and V_3.
The line $V_1 U_1$ is

$$x(q_1 z_1 - r_1 y_1) + y(r_1 x_1 - p_1 z_1) + z(p_1 y_1 - q_1 x_1) = 0.$$

$V_1 U_1$ meets $x=0$ at the point P_1 which belongs to the determinantal system of collinear points, and has coordinates $(0\, l_1\, 1)$.

Therefore $\dfrac{y}{z} = \dfrac{q_1 x_1 - p_1 y_1}{r_1 x_1 - p_1 z_1} = l_1.$

As $V_1 U_2$ meets $x=0$ at Q_2 with coordinates $(0\, l_2\, 1)$

$$\frac{q_2 x_1 - p_2 y_1}{r_2 x_1 - p_2 z_1} = l_2$$

and from $V_1 U_3$ $\dfrac{q_3 x_1 - p_3 y_1}{r_3 x_1 - p_3 z_1} = l_3.$

Hence

$$x_1(q_1 - r_1 l_1) - p_1 y_1 + l_1 p_1 z_1 = 0$$
$$x_1(q_2 - r_2 l_2) - p_2 y_1 + l_2 p_2 z_1 = 0$$
$$x_1(q_3 - r_3 l_3) - p_3 y_1 + l_3 p_3 z_1 = 0.$$

The condition for concurrence of $U_1 P_1$, $U_2 Q_2$, $U_3 R_3$ is found by eliminating $x\,y\,z$ from these three equations, and is

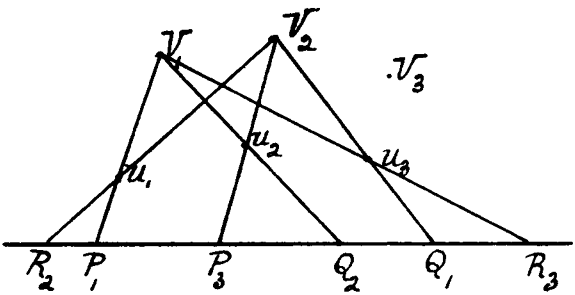
$$(q_1 - r_1 l_1) p_2 p_3 (l_2 - l_3) + (q_2 - r_2 l_2) p_3 p_1 (l_3 - l_1) + \ldots = 0,$$

or since $p_1 = p_2 = p_3$ and $q_1 = N_2 - N_3$, etc.,

$$\Sigma (N_2 - N_3)(l_2 - l_3) + \Sigma (M_2 - M_3)\, l_1 (l_2 - l_3 = 0),$$

or $$N_1 l_1 + N_2 l_2 + N_3 l_3 = M_1 l_2 l_3 + M_2 l_3 l_1 + M_3 l_1 l_2.$$

The line $V_2 U_2$ will meet $x=0$ in P_3 with coordinates $(0\, n_1\, 1)$.
The accompanying diagram will explain this.



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$U_1 V_1$, $U_1 V_2$, $U_1 V_3$ give on $x=0$ $P_1 R_2 Q_3$ corresponding to the $P_1 Q_2 R_3$ of $V_1 U_1$, $V_1 U_2$, $V_1 U_3$.

Thus if $V_2 U_1$ give R_2 $V_2 U_2$ must give P_3.

Where $V_2 U_2$ meets $x=0$

$$\frac{q_2 x_2 - p_2 y_2}{r_2 x_2 - p_2 z_2} = n_1 = \frac{l_1 (l_2 - l_3)}{l_1 - l_2}$$

with corresponding equations resulting from $V_2 U_3$ and $V_2 U_1$.

The condition for concurrence is

$$N_1 l_2 + N_2 l_3 + N_3 l_1 = M_2 l_2 l_3 + M_3 l_3 l_1 + M_1 l_1 l_2.$$

The condition for the concurrence of $V_3 U_3$, $V_3 U_1$, $V_3 U_2$ is in the same way $N_1 l_3 + \ldots = M_1 l_3 l_1 + \ldots$

The following three consistent equations must be satisfied:

$$\begin{aligned} N_1 l_1 + \ldots &= M_1 l_2 l_3 + \ldots \\ N_1 l_2 + \ldots &= M_1 l_1 l_2 + \ldots \\ N_1 l_3 + \ldots &= M_1 l_3 l_1 + \ldots \end{aligned}$$

Square and add

$$\begin{aligned} (N_1^2 + N_2^2 + N_3^2)(l_1^2 + l_2^2 + l_3^2) + 2(N_2 N_3 + \ldots)(l_2 l_3 + \ldots) \\ = (M_1^2 + M_2^2 + M_3^2)(l_2^2 l_3^2 + \ldots) + 2(M_2 M_3 + \ldots)(l_1^2 l_2 l_3 + \ldots). \end{aligned}$$

But $\Sigma 2 N_2 N_3 = -\Sigma N_1^2$ and $\Sigma 2 M_2 M_3 = -\Sigma M_1^2$.

Therefore

$$(N_1^2 + N_2^2 + N_3^2) \Sigma (l_2 - l_3)^2 = (M_1^2 + M_2^2 + M_3^2) \Sigma l_1^2 (l_1 - l_3)^2.$$

But $$\frac{\Sigma l_1^2 (l_2 - l_3)^2}{\Sigma (l_2 - l_3)^2} = K$$

for the term on the left side is the product of the parameters of the hessian points of any triad in the determinantal system, and is constant (K). Hence $N_1^2 + N_2^2 + N_3^2 = K(M_1^2 + M_2^2 + M_3^2)$.

To find the locus of $V_1 V_2 V_3$ we have only to make use of the relation $\Sigma l_1^2 (l_2 - l_3)^2 = K \Sigma (l_2 - l_3)^2$, substituting for $l_1 l_2 l_3$ the expressions in x, y, z, p, q, r already found.

The locus of V thus is

$$\Sigma \left(\frac{q_1 x - p_1 y}{r_1 x - p_1 z}\right)^2 \left(\frac{q_2 x - p_2 y}{r_2 x - p_2 z} - \frac{q_3 x - p_3 y}{r_3 x - p_3 z}\right)^2$$

$$= K \Sigma \left(\frac{q_2 x - p_2 y}{r_2 x - p_2 z} - \frac{q_3 x - p_3 y}{r_3 x - p_3 z}\right)^2$$

or $\Sigma (x + M_1 y + N_1 z)^2 [\{(N_2 - N_3) x - y\}^2 - K\{(M_3 - M_2) x - z\}^2] = 0.$

This quartic passes through F (100) since

$$\Sigma (N_2 - N_3)^2 = K \Sigma (M_3 - M_2)^2.$$

On transferring to $U_1 U_2 U_3$ as triangle of reference the quartic becomes

$$\Sigma X^2 [(N_3 Y - N_2 Z)^2 - K(M_3 Y - M_2 Z)^2] = 0.$$

Use $\Sigma(N_1^2 - K M_1^2) = 0$ and the quartic breaks up into the two conics

$$YZ(N_1^2 - KM_1^2) + ZX(N_2^2 - KM_2^2) + XY(N_3^2 - KM_3^2) = 0,$$

and
$$YZ + ZX + XY = 0.$$

The first conic which alone is related to the nine points passes through F (111), and therefore the hessian pole of $U_1 U_2 U_3$ lies on $X + Y + Z = 0$ or $x = 0$. Hence if two triads by the joining of their vertices give a determinantal system on a line these triads must be con-conical, and the line must contain their hessian poles.

9. The planes $P_1 P_2 P_3$, $Q_1 Q_2 Q_3$, $R_1 R_2 R_3$ in the twisted cubic intersect in a line on the plane ABC and $P_1 Q_1 R_1$, etc., in another line also on ABC. These lines are $Kx + y = 0$ and $Kz - t = 0$, and can be interpreted in terms of the central triad (paragraph 4), the pole of which is M and the parameters of its hessian points $\pm i\sqrt{K}$.

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