BOOK REVIEWS

HARDY, G. H., *Collected Papers*, Vol. III (Clarendon Press: Oxford University Press, 1969), 748 pp., 105s.

The collected papers of G. H. Hardy are being published in seven volumes, edited by a committee appointed by the London Mathematical Society. All the joint papers will be included and indeed, of the fifty-seven papers in the present volume, thirty-six were written in collaboration with J. E. Littlewood and five with W. W. Rogosinski. The papers have been divided into sections by subject matter, and this third volume contains two sections, of papers on trigonometric series and papers whose principal results lie in the field of mean values of power series. These have had far reaching consequences in other fields, such as functional analysis and function theory, and the volume will have a fascination for anyone interested in analysis.

Naturally the division of papers into sections is somewhat arbitrary. Thus the second section of the present volume includes both of the famous Hardy and Littlewood papers on fractional integrals from *Mathematische Zeitschrift* (1928 and 1932), although the first is concerned only with real functions. This section begins with a paper of 1915 in which Hardy showed that the mean value

$$\mu_p(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \quad (p>0)$$

of an analytic function $f(z)$ behaves like the maximum modulus

$$\mu(r) = \max_{0 \leqq \theta \leqq 2\pi} |f(re^{i\theta})|,$$

for example in that $\log \mu_p(r)$ is a convex function of $\log r$. This is followed by a brief note announcing the results of the first *Zeitschrift* paper. The second *Zeitschrift* paper is on fractional integrals and derivatives of functions of what F. Riesz called the Hardy class H^p, functions for which the mean $\mu_p(r)$ is bounded in the unit disc. There are also joint papers on the mean values of a function and its real and imaginary parts (1931) and on Cesàro means of functions of H^p (1934), and one of 1937 whose main result is that, under suitable conditions if $\Sigma a_n z^n$ is in H^p and if the mean of the function $\Sigma n b_n z^{n-1}$ satisfies $\mu_s(r) \leqq K(1-r)^{-s}$, then $\Sigma a_n b_n z^n$ is in H^λ, where

$$\lambda^{-1} = p^{-1}+s^{-1}-1.$$

In the final paper (1941), Hardy and Littlewood give a unified account of the main results of their work in this field. (The paper of 1930, in which Hardy and Littlewood proved the maximal theorem which is basic in the subject, has already appeared with other papers on inequalities in Volume II of the collected works.)

The papers on trigonometric series have been divided into five groups. The first group contains eleven papers on the convergence, and the second group nineteen on the summability, of a Fourier series or its conjugate. Most of the second group are joint papers with Littlewood on Cesàro summability, but there are also a 1931 paper on summability by logarithmic means, two of Hardy's last papers, written with Rogosinski, on Riemann summability, and two Hardy and Littlewood papers (1935 and 1936) on strong Cesàro summability. As a small indication of the constant interest which the volume holds for any analyst, one might remark that this paper of

1936, under the innocent title " Some more theorems concerning Fourier series and Fourier power series ", contains the theorem that, if $x^{\alpha}U(x) \in L^{p}(0, \infty)$ where

$$-1-p^{-1}<\alpha<1-p^{-1},$$

and if $U(x)$ is an even function, then its Hilbert transform $V(x)$ is odd and

$$\int_{0}^{\infty} (x^{\alpha} \mid V(x)|)^{p} dx \leqq K(p, \alpha) \int_{0}^{\infty} (x^{\alpha} \mid U(x)|)^{p} dx,$$

a deep result on Hilbert transforms which one would hardly expect to find in a paper on strong summability of Fourier series.

Many papers in the first group are closely related to Hardy and Littlewood's work on summability, since the convergence criteria often consist of a condition on the function which ensures summability of its Fourier series, together with one on the coefficients which then ensures convergence, or have been developed from this simple situation, for example by strengthening one condition and relaxing the other. This group also includes papers on absolute convergence, Lebesgue's constants and Gibbs phenomenon.

The third group contains seven papers on topics connected with the Young-Hausdorff inequalities, and then come five papers on special trigonometric series. In a paper on series with decreasing coefficients of 1928, for example, Hardy proves that $a_n \sim n^{-\alpha}$ implies $\Sigma a_n \cos n\theta \sim \sin \frac{1}{2}\alpha\pi\Gamma(1-\alpha)\theta^{\alpha-1}$ as $\theta \to 0$ $(0<\alpha<1)$, and the corresponding result for sine series. A paper of 1931 gives the Tauberian converses of these results, and two papers written with Rogosinski (1943 and 1945) are on similar topics. The other paper in this group (1941) is on the orthogonal functions

$$(-1)^{n}\pi^{-1}(x-n)^{-1} \sin \pi x$$

of Whittaker's cardinal series.

The final group is of seven miscellaneous papers. These include one in which Hardy evaluates the sum

$$\sum_{0}^{\infty} (-t)^{n}(n!)^{2}e^{-x}L_{n}(x)L_{n}(y),$$

where L_n is a Laguerre polynomial, one in which he proves that, within certain limits, a function orthogonal with respect to its own zeros must be a Bessel function, one by Hardy and Littlewood on the analogue for conjugate functions of Fourier's double integral formula, and one in which these authors extend Parseval's theorem, showing for example that if f, g have complex Fourier coefficients a_n, b_n and are in L^p, L^q respectively, then the series

$$\sum_{n \neq 0} \mid n \mid^{-\mu} e^{-\frac{1}{2}\mu\pi i \operatorname{sgn} n} a_n b_n,$$

where $\mu = p^{-1}+q^{-1}-1$, (i) converges absolutely if $1<p \leqq q \leqq 2$, (ii) converges if $1<p<2<q$, and that in each case the number μ is best possible.

The papers have been reproduced photographically, so that the original pagination is preserved for reference, and there is a list of misprints at the end of each paper. The present volume has been edited by Professor Rogosinski, who had drafted introductions to each of the groups of papers on trigonometric series before his death, and by Professor T. M. Flett, who completed these drafts and added an introduction to the second section. Professor Flett has also added illuminating comments which trace subsequent developments in the topics covered by individual papers. These editorial additions enhance the value of the collection considerably, and are thoroughly worthy of the papers which they accompany. It is a tremendously worthwhile project to make Hardy's papers available as a collection, for this will be a source of stimulation even to those of us who are already familiar with parts of his work, and mathematicians everywhere owe a great debt to the London Mathematical Society and its editorial committee.

PHILIP HEYWOOD

HARDY, G. H.. *Collected Papers*, including joint papers with J. E. Littlewood and others. Vol. IV. Edited by a Committee appointed by the London Mathematical Society. (Clarendon Press; Oxford University Press, 1969), 722 pp, 120s.

For anyone to attempt a critical review of a volume of Hardy's papers would be very difficult; for one of his former pupils to do so would be impertinent. All I can do is to state briefly what this volume contains.

There are 54 papers, of which only five are joint, written between 1902 and 1946; and they are divided into two sections.

The first section of 42 papers is on Special Functions. § I(*a*) contains 12 papers on the zeros and asymptotic properties of certain special integral functions. At the beginning of the century, the theory of integral functions was an active field of research. Hardy's objective was to illustrate the general theory and to try to find new results by the consideration of particular examples. He starts by considering the zeros of large modulus of such functions as $\sin x - x$ and $\dfrac{1}{\Gamma(x+1)} - c$. He then goes on to integral functions defined by Taylor Series. An interesting example is

$$\sum_{0}^{\infty} \frac{z^{n^3}}{(n^3)!}$$

When $|z| = N^3$ where N is large, this function is dominated by its Nth term, from which it follows that there are N^3 zeros in $|z| < N^3$. In the annulus

$$N^3 < |z| < (N+1)^3,$$

the function is dominated by the sum of its Nth and $(N+1)$th terms. And the $3N^2+3N+1$ zeros in the annulus are given approximately by equating to zero the sum of those terms. They lie approximately at the vertices of a regular polygon midway between the bounding circles. The result is generalised by replacing n^3 by a function $\phi(n)$ whose increase is regular and sufficiently rapid.

§ I(*b*), consisting of five papers, deals with the singularities on its circle of convergence $|z| = 1$ of a function defixed by a Taylor series $\Sigma a_n z^n$ where

$$a_n = \int_0^1 \left(\log \frac{1}{u}\right)^{\alpha-1} (1-u)^{\beta-1} u^{\gamma-1+n} \phi(u) du.$$

The discussion involves replacing this integral by an integral round a loop in the complex u-plane. Various examples, such as the hypergeometric series, are worked out in detail. He also discusses functions of two variables; the problem is to determine the behaviour of a series $\Sigma a_{mn} c_{mn} x^m y^n$ from a knowledge of the behaviour of a " base series " $\Sigma c_{mn} x^m y^n$ of simple type.

The three papers of § I(*c*) are essentially a supplement to Hardy's 1910 Cambridge Tract *Orders of Infinity* and applications of the Infinitärcalcül of du Bois-Reymond to oscillating Dirichlet integrals.

§ I(*d*) is a miscellaneous collection of 21 papers on Special Functions, the Gamma and Zeta Functions and their generalisations, Airy's Integral, the Modular Functions,