HILBERT SPACES OF GENERALIZED FUNCTIONS EXTENDING L^2, (II)

J. B. MILLER

(received 9 November 1961)

1. Introduction

The present note continues the discussion, begun in the first paper with the above title, of classes of spaces $\mathscr{H}$ which are extensions of $L^2(0, \infty)$ and whose elements, which we call 'sequence-functions', exhibit some of the properties of distributions. The previous paper defined the spaces, and described how they can be used to extend the domain of definition of Watson transforms. Further related applications to transform theory are described in [2] and [3]. In this paper I pursue the analogy between these sequence-functions and other types of generalized functions further by discussing their local behaviour, the existence of ordinary and convolution-type products, and of derivatives and integrals.

Ordinary and convolution products are dealt with in §§ 3, 4 and 5. In each case it proves necessary to introduce new function spaces, to which at least one factor of a product shall belong, and each of these paragraphs is devoted in part to a description of the new spaces.

Turning to the definition of derivatives and integrals of sequence-functions, we find that expressions of the form $x^\alpha F^{(\alpha)}(x)$ and $x^{-\alpha} F^{(-\alpha)}(x)$ arise more naturally than $F^{(\alpha)}(x)$ and $F^{(-\alpha)}(x)$. We therefore first find conditions for the existence of these 'affixed' derivatives and integrals, in § 6, and then attempt in § 7 to detach the factors $x^{\pm\alpha}$ by using the ordinary product of § 3 to define $x^{-\alpha} x^\alpha F^{(\alpha)}(x)$ and $x^\alpha x^{-\alpha} F^{(-\alpha)}(x)$.

This method defines sequence-functions which are interpretable as derivatives and integrals only over intervals excluding neighbourhoods of 0 and ∞, and suggest the necessity of some treatment of the local behaviour of sequence-functions. Accordingly, a discussion is included of sequence-functions which belong to L^2 on open sets; the paper begins with this.

The reader is assumed to be acquainted with the notation and contents of the first half of the first paper, [1]; references to equations, theorems etc. in [1] will be shown by a suffix 1. I must again thank Professor E. R. Love for his interest; some sections of this paper, particularly §§ 2 and 3, were suggested by his published papers $[4]_1$, and unpublished work to which he gave me access. The paper contains some modifications suggested by the referee.

2. Locally numerical sequence-functions

If a sequence-function belongs to L^2 it has almost everywhere a numerical representation. A less restricted member of $\mathscr{H}_{-\lambda}$ may be numerical on a subset E of $(0, \infty)$, i.e. may be equivalent to a function of $L^2(E)$, without belonging to $L^2(0, \infty)$. We show that this property of being 'locally numerical' may be defined precisely, at least when E is an open set.

DEFINITION. *A sequence-function* F *of* $\mathscr{H}_{-\lambda}$ *is numerical on* E *if it possesses at least one Cauchy sequence which converges in* $L^2(E)$. *The mean limit in* $L^2(E)$ *of this sequence determines the local value of* F *in* E.

That the local value so defined is unique is shown by

THEOREM 1. *If* F *of* $\mathscr{H}_{-\lambda}$ *is numerical in* E, *an open set in* $(0, \infty)$, *and* $\lambda > \frac{1}{2}$, *then all Cauchy sequences for* F *which converge in* $L^2(E)$ *have the same limit.*

We require a preliminary result.

LEMMA 1. *Let* (h_n), $h_n \in L^2$, *be a null sequence in* $\mathscr{G}_{-\lambda}$, *i.e.*

$$||h_n||_{-\lambda} \to 0,$$

and suppose that $h_n(x) = h(x)$, *independent of* n, *for almost all* x *in the interval* (a, b). *Then* $h(x) \equiv 0$.

PROOF OF THE LEMMA. Since $||h_n||_{-\lambda} \to 0$ as $n \to \infty$, we have

$$\int_a^b \left| x^{-\lambda} \int_0^a (x-t)^{\lambda-1} h_n(t)dt + x^{-\lambda} \int_a^x (x-t)^{\lambda-1} h(t)dt \right|^2 dx \to 0. \tag{2.1}$$

If $a = 0$, it follows immediately that $h^{(-\lambda)}(x) \equiv 0$ and hence that $h(x) \equiv 0$ in (a, b). Suppose $a > 0$. Consider first the case when λ is a positive integer, and write $H_n(x)$, $H(x)$ respectively for the two expressions within the modulus signs in (2.1). Clearly $H_n(x)$ is a polynomial in x^{-1} of degree λ with zero constant term, and therefore converges in $L^2(a, b)$ to a similar polynomial; thus $H(x)$ has this form also. Therefore

$$\int_a^x (x-t)^{\lambda-1} h(t)dt = c_0 + c_1 x + \cdots + c_{\lambda-1} x^{\lambda-1};$$

the λ-th derivatives of both sides are equal almost everywhere, and so $h(x) \equiv 0$. Now suppose that λ is non-integral, $l < \lambda < l+1$ for some integer l. Then $||h_n||_{-\lambda} \to 0$ implies $||h_n||_{-(l+1)} \to 0$ by Lemma 2_1: the previous case gives $h = 0$.

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

PROOF OF THE THEOREM. Let (f_n), (g_n) be two equivalent sequences for F which converge to f, g respectively in $L^2(a, b)$. Let (f'_n) be another sequence given by

$$f'_n(x) = f(x) \quad (a < x < b), \quad f_n(x) \ \text{(otherwise)}.$$

We show first that $(f'_n) \sim (f_n)$. Now

$$\begin{aligned}||f'_n - f_n||^2_{-\lambda} &= \int_a^b \left| \frac{x^{-\lambda}}{\Gamma(\lambda)} \int_a^x (x-t)^{\lambda-1}\{f(t)-f_n(t)\}dt \right|^2 dx \\ &\quad + \int_b^\infty \left| \frac{x^{-\lambda}}{\Gamma(\lambda)} \int_a^b (x-t)^{\lambda-1}\{f(t)-f_n(t)\}dt \right|^2 dx \\ &= I_1 + I_2, \text{ say.}\end{aligned}$$

If $\lambda > \frac{1}{2}$,

$$\begin{aligned}\Gamma(\lambda)I_1^{\frac{1}{2}} &\leqq \int_a^b |f(t)-f_n(t)|dt \left\{\int_t^b x^{-2\lambda}(x-t)^{2\lambda-2}dx\right\}^{\frac{1}{2}} \\ &\leqq \left[\int_a^b |f(t)-f_n(t)|^2 dt\right]^{\frac{1}{2}} \cdot \left[\int_a^b dt \int_t^b x^{-2\lambda}(x-t)^{2\lambda-2}dx\right]^{\frac{1}{2}},\end{aligned}$$

and the second factor is finite if $0 < a \leqq b < \infty$. Thus under these conditions $I_1 \to 0$; and similarly $I_2 \to 0$; (f'_n) defines F. Let (g'_n) be obtained similarly from (g_n), so that

$$(g'_n) \sim (g_n) \sim (f_n) \sim (f'_n)$$

in $\mathscr{H}_{-\lambda}$. Then $h_n = f'_n - g'_n$ satisfies the conditions of the lemma, and $h = f - g = 0$. Hence the theorem holds when E is a finite interval not having 0 as an endpoint, and the extension to a countable union of open such intervals and so to any open set is immediate.

We note the corollary: A sequence-function is numerical on an open set if E and only if it has a sequence which repeats on E, as a function of $L^2(E)$.

3. Ordinary products

We look for a product $F\phi$ in $\mathscr{H}$ which will reduce to the ordinary product of two functions if $F \in L^2$. The evidence of other theories of generalized functions shows that such a product cannot exist between two arbitrary sequence-functions; rather, the further removed F is from L^2, the greater the restriction necessary upon ϕ. In the present case it turns out that the restrictions take forms involving the norm

$$||\phi||_0 = \sqrt{\int_0^\infty |\phi(t)|^2 t^{-1} dt}.$$

We shall denote by $\mathscr{P}$ the new linear spaces of measurable functions which this norm determines.

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

DEFINITION. I. *$\mathscr{P}_0$ is the space of measurable functions ϕ for which $\|\phi\|_0 < \infty$.*

II. *ϕ belongs to $\mathscr{P}_\lambda$ if there exists some function $\phi^{(\lambda)}$ related to ϕ as in* $(2.1)_1$ *(the Lebesgue integral there given existing for almost all $x > 0$), and if $t^\lambda \phi^{(\lambda)}(t) \in \mathscr{P}_0$. We write $\|\phi\|_\lambda = \|t^\lambda \phi^{(\lambda)}(t)\|_0$.*

III. *ϕ belongs to $\mathscr{P}_{-\lambda}$ if the Lebesgue integral* $(2.2)_1$ *defining $\phi^{(-\lambda)}(x)$ exists for almost all $x > 0$, and $x^{-\lambda}\phi^{(-\lambda)}(x) \in \mathscr{P}_0$. We write $\|\phi\|_{-\lambda} = \|x^{-\lambda}\phi^{(-\lambda)}(x)\|_0$.*

IV. *Spaces $\mathscr{P}_{[\lambda]}$ and $\mathscr{P}_{[-\lambda]}$ are defined similarly.*

These spaces in their orderings show less symmetry than spaces $\mathscr{G}$ and $\mathscr{H}$. We find (by judicious use of Schwarz' and Minkowski's inequalities) that, if $0 < \lambda < \mu$,

$$\|\phi\|_\lambda \leqq \frac{\Gamma(\lambda)}{\Gamma(\mu)} \|\phi\|_\mu, \quad \|\phi\|_{[-\mu]} \leqq \frac{\Gamma(\lambda)}{\Gamma(\mu)} \|\phi\|_{[-\lambda]}; \tag{3.1}$$

and if $0 \leqq \lambda < \mu$,

$$\|\phi\|_{[\lambda]} \leqq \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \|\phi\|_{[\mu]}, \quad \|\phi\|_{-\mu} \leqq \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \|\phi\|_{-\lambda}. \tag{3.2}$$

From these can be deduced the existence of two broken systems

$$\begin{aligned} &\mathscr{P}_\mu \subset \mathscr{P}_\lambda; \quad \mathscr{P}_0 \subset \mathscr{P}_{-\lambda} \subset \mathscr{P}_{-\mu} \\ &\mathscr{P}_{[\mu]} \subset \mathscr{P}_{[\lambda]} \subset \mathscr{P}_0; \quad \mathscr{P}_{[-\lambda]} \subset \mathscr{P}_{[-\mu]}. \end{aligned} \qquad (0 < \lambda < \mu) \tag{3.3}$$

We shall prove theorems which determine sufficient conditions for $(f_n\phi)$ to be Cauchy when (f_n) is Cauchy, and thus allow us to define $F\phi$. The proofs depend upon interated ordinary integration by parts, and we are obliged to restrict the parameter λ to take integer values only. There is a distinction between cases where $(f_n\phi)$ is to be Cauchy in the same space $\mathscr{G}_{-\lambda}$ as (f_n), and where (f_n) is restricted to a smaller space than $(f_n\phi)$. We begin with a preliminary lemma.

LEMMA 2. (i) *If $f \in L^2$, then $x^{-\lambda}f^{(-\lambda)}(x) = o(x^{-\frac{1}{2}})$ as $x \to 0$, if $\lambda > \frac{1}{2}$.*
(ii) *If $\phi \in \mathscr{P}_\lambda$, $0 < \alpha < \lambda - \frac{1}{2}$, then as $x \to 0$,*

$$\phi(x) = O(\sqrt{\log(1/x)}), \quad x^\alpha \phi^{(\alpha)}(x) = o(1).$$

(iii) *If $\phi \in \mathscr{P}_{[\lambda]}$, $\lambda > \frac{1}{2}$, then $\phi(x)$ is bounded,*

$$|\phi(x)| \leqq \{2\lambda(2\lambda-1)\}^{-\frac{1}{2}}\Gamma(\lambda)^{-1} \cdot \|\phi\|_{[\lambda]}.$$

The proofs of (i) and (iii) are immediate consequences of Schwarz' inequality. The proof of the second order relation in (ii) is a slight modification of that of $[6]_1$, Theorem 4; the first of (ii) follows from the same proof, by a varied treatment of the term 'J_2' there.

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

LEMMA 3. *Let l, m be integers, $0 < l < m$, and let $f \in L^2$. If $\phi \in \mathscr{P}_0 \cap \mathscr{P}_l$, then*

$$\|f\phi\|_{-m} \leqq \|f\|_{-l} \cdot \{A\|\phi\|_0 + B\|\phi\|_l\}, \tag{3.4}$$

where A, B depend only upon l, m.

If instead $l = m$, $\phi \in \mathscr{P}_m$ and $\phi(x)$ is bounded, then

$$\|f\phi\|_{-m} \leqq \|f\|_{-m} \cdot \{\sup|\phi(x)| + C\|\phi\|_m\}, \tag{3.5}$$

where C depends only on m.

PROOF. Assume $\phi \in \mathscr{P}_0 \cap \mathscr{P}_l$, and write $\Phi(t) = \phi(t)(x-t)^{m-1}$, so that

$$\Gamma(m)(f\phi)^{(-m)}(x) = \int_0^x f(t)\Phi(t)dt, \tag{3.6}$$

and integrate by parts l times, so as to obtain $f^{(-l)}(t)$ in the integrand. The integrated terms which arise form a sum of expressions

$$\left[f^{(-r)}(t)\frac{d^s}{dt^s}\phi(t)(x-t)^{m-r+s}\right]_0^x \qquad (s = 0, 1, \cdots, r-1;\ r = 1, 2, \cdots, l).$$

Lemma 2 shows that the terms for $t = 0$ vanish; clearly those for $t = x$ also vanish if $l < m$, but if $l = m$ there remains one non-zero term $f^{(-l)}(x)\phi(x)$. If we write $\Phi^{\langle r\rangle}(t)$ for the ordinary r-th derivative of Φ with respect to t, supposing x fixed, we get from (3.6):

$$(f\phi)^{(-m)}(x) = \Gamma(m)^{-1}(-1)^l \int_0^x f^{(-l)}(t)\Phi^{\langle l\rangle}(t)dt \qquad (\text{if } l < m) \tag{3.7}$$

or

$$= f^{(-m)}(x)\phi(x) + \Gamma(m)^{-1}(-1)^m \int_0^x f^{(-m)}(t)\Phi^{\langle m\rangle}(t)dt \qquad (\text{if } l = m). \tag{3.8}$$

Suppose $0 < l < m$; from (3.7),

$$\begin{aligned} \Gamma(m)\|f\phi\|_{-m} &= \left\{\int_0^\infty \left|x^{-m}\int_0^x f^{(-l)}(t)\Phi^{\langle l\rangle}(t)dt\right|^2 dx\right\}^{\frac{1}{2}} \\ &\leqq \int_0^\infty |f^{(-l)}(t)|dt \left\{\int_t^\infty |x^{-m}\Phi^{\langle l\rangle}(t)|^2 dx\right\}^{\frac{1}{2}} \\ &\leqq \left[\int_0^\infty |t^{-l}f^{(-l)}(t)|^2 dt\right]^{\frac{1}{2}} \cdot \left[\int_0^\infty t^{2l}dt\int_t^\infty |x^{-m}\Phi^{\langle l\rangle}(t)|^2 dx\right]^{\frac{1}{2}} \\ &= \|f\|_{-l}L(\phi), \text{ say.} \end{aligned} \tag{3.9}$$

Now

$$\begin{aligned} \Phi^{\langle l\rangle}(t) &= \frac{\partial^l}{\partial t^l}\{\phi(t)(x-t)^{m-1}\} \\ &= (-1)^l \sum_{r=0}^{l} a_r (x-t)^{m-l+r-1}\phi^{(r)}(t), \text{ say,} \end{aligned} \tag{3.10}$$

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

since $(-1)^r\phi^{(r)}(t) = \phi^{(r)}(t)$ (cf. § 2_1). Then

$$\left\{\int_t^\infty |x^{-m}\Phi^{(l)}(t)|^2 dx\right\}^{\frac{1}{2}} \leqq \sum_{r=0}^{l} a_r \left\{\int_t^\infty |\phi^{(r)}(t)x^{-m}(x-t)^{m-l+r-1}|^2 dx\right\}^{\frac{1}{2}}$$
$$= \sum_{r=0}^{l} b_r|\phi^{(r)}(t)|t^{-l+r-\frac{1}{2}}, \text{ say;}$$

and therefore, in (3.9),

$$\begin{aligned} L(\phi) &\leqq \left\{\int_0^\infty t^{2l}\left(\sum_{r=0}^{l} b_r|\phi^{(r)}(t)|t^{-l+r-\frac{1}{2}}\right)^2 dt\right\}^{\frac{1}{2}} \\ &\leqq \sum_{r=0}^{l} b_r \left\{\int_0^\infty |t^r\phi^{(r)}(t)|^2 t^{-1} dt\right\}^{\frac{1}{2}} \\ &= \sum_{r=0}^{l} b_r\|\phi\|_r \\ &\leqq b_0\|\phi\|_0 + \|\phi\|_l \cdot \sum_{r=1}^{l} b_r\Gamma(r)\Gamma(l)^{-1}, \end{aligned} \tag{3.11}$$

by (3.1). The first part of the lemma follows from (3.9) and (3.11).

Suppose now that $l = m$. Equation (3.10) is valid again, but now $a_0 = 0$. With this modification the work down to (3.11) holds, with $b_0 = 0$, and (3.8) gives the second part of the lemma, (3.5).

Since $\mathscr{P}_\lambda \neq \mathscr{P}_{[\lambda]}$, a cognate form of Lemma 3 will provide for multiplication by some functions not covered by that lemma.

LEMMA 4. *Let l, m be integers, $0 < l \leqq m$, and let $f \in L^2$. If $\phi \in \mathscr{P}_{[l]}$, then*

$$\|f\phi\|_{-m} \leqq B'\|f\|_{-l} \cdot \|\phi\|_{[l]}, \tag{3.12}$$

where B' depends only on l, m.

The proof follows the general pattern of the previous one, but the details are considerably more awkward. If $l < m$, we find in place of (3.4):

$$\|f\phi\|_{[-m]} \leqq \|f\|_{[-l]} \cdot \{A''\|\phi\|_{[0]} + B''\|\phi\|_{[l]}\},$$

but this time the first term inside the brackets may be incorporated in the second, by (3.2); and $\|f\|_{[-l]} = \|f\|_{-l}$ by $(4.4)_1$, while $\|f\phi\|_{[-m]} = \|f\phi\|_{-m}$ by virtue also of Lemma 2 (iii): this gives (3.12). If $l = m$ we get a cognate form of (3.5), in which again (by Lemma 2 (iii)) the first term may be incorporated in the second, giving (3.12) in this case also.

Returning to products of sequence-functions, we now frame the

DEFINITION. *The product $\mathsf{F}\phi$ of a sequence-function F and a function ϕ exists if all sequences $(f_n\phi)$ for which $(f_n) \sim \mathsf{F}$, $f_n \in L^2$, are Cauchy and equivalent; then $\mathsf{F}\phi$ is the sequence-function defined by this class of Cauchy sequences.*

Here, the space in which the sequences are to be Cauchy and equivalent is purposely left unspecified; it is in that space that $F\phi$ will exist.

It is easily seen that under this definition the distributive and associative laws are valid, in the sense that

$$F(\phi+\psi) = F\phi+F\psi, \qquad (F+G)\phi = F\phi+G\phi \tag{3.13}$$
$$F(\phi\psi) = (F\phi)\psi$$

hold provided the right-hand side is defined, in each case. If $F \in L^2$ i.e. if F possesses a repeating sequence (f), $f \in L^2$, and if $F\phi$ exists in $\mathscr{H}_{-\lambda}$, then $F\phi$ is the principal sequence-function $f\phi$, a function of $\mathscr{G}_{-\lambda}$ (cf. § 6_1).

The following theorem is an immediate consequence of Lemmas 3 and 4.

THEOREM 2. *Let l be a positive integer, and suppose $F \in \mathscr{H}_{-l}$. Then $F\phi$ exists as a member of $\mathscr{H}_{-(l+1)}$ if $\phi \in \mathscr{P}_0 \cap \mathscr{P}_l$, or even as a member of $\mathscr{H}_l$ if $\phi \in \mathscr{P}_l$ and ϕ is bounded, or if $\phi \in \mathscr{P}_{[l]}$.*

4. Resultant products

Here we consider a product of the form

$$(f \cdot g)(x) = \int_0^\infty f(xt)\,\frac{1}{t}\,g\left(\frac{1}{t}\right)dt = \int_0^\infty f(t)\,\frac{1}{t}\,g\left(\frac{x}{t}\right)dt. \tag{4.1}$$

Such products arise naturally in work with Watson transforms, where they are perhaps more appropriate than the usual convolution product of two functions, to which this resultant reduces after simple changes of variables. We say, at first, that $f \cdot g$ exists if the integral is absolutely convergent for almost all positive x. Clearly $f \cdot g$ exists if both f and g belong to L^2.

In order to be able to discuss the existence of the resultant in $\mathscr{H}_{-\lambda}$, we find it necessary to introduce new spaces $\mathscr{A}$ and $\mathscr{B}$, derived from the norm

$$|f|_0 = \int_0^\infty |f(t)|t^{-\frac{1}{2}}dt \tag{4.2}$$

in the same way as $\mathscr{G}$ and $\mathscr{H}$ spaces are derived from the norm $\|\cdot\|_0$ and $\mathscr{P}$ spaces from the norm $\|\cdot\|_0$.

DEFINITION. I. *M is the space of measurable functions f for which $|f|_0 < \infty$.*

II. *f belongs to $\mathscr{A}_\lambda (\lambda > 0)$ if there exists some function $f^{(\lambda)}$ related to f as in* $(2.1)_1$*(the Lebesgue integral existing for almost all $x > 0$), for which $t^\lambda f^{(\lambda)}(t) \in M$; and $\mathscr{A}_0 = M$. We write $|f|_\lambda = |t^\lambda f^{(\lambda)}(t)|_0$.*

III. *f belongs to $\mathscr{A}_{-\lambda} (\lambda > 0)$ whenever $f^{(-\lambda)}(x)$ exists almost everywhere (as a Lebesgue integral, defined by* $(2.2)_1$*), and $x^{-\lambda} f^{(-\lambda)}(x) \in M$; $\mathscr{A}_{-0} = \mathscr{A}_0 = M$. We write $|f|_{-\lambda} = |x^{-\lambda} f^{(-\lambda)}(x)|_0$.*

IV. *Spaces* $\mathscr{A}_{[\lambda]}$, $\mathscr{A}_{[-\lambda]}$ *are defined analogously.*

V. $\mathscr{B}_{-\lambda}$, $\mathscr{B}_{[-\lambda]}$ *are the spaces* $\mathscr{A}_{-\lambda}$, $\mathscr{A}_{[-\lambda]}$ *respectively, completed by the adjunction of limits in the respective norms* (*cf.* § 6_1).

It will be seen that $\mathscr{P}_0$ is related to M as $\mathscr{G}_0(=L^2)$ is to L. Before applying these spaces to the discussion of the resultant, we list in summary form some of their properties.

1°. The spaces L^2 and M are distinct, and their intersection is dense in each of them.

2°. For $f \in L^2$, $0 \leqq \lambda < \mu$, we have (cf. Lemma 2_1)

$$|f|_\lambda \leqq \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}|f|_\mu, \qquad |f|_{-\mu} \leqq \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}|f|_{-\lambda}. \tag{4.3}$$

Also, if $0 \leqq \lambda < \mu - \frac{1}{2}$,

$$\|f\|_\lambda < K(\lambda, \mu)|f|_\mu, \qquad \|f\|_{-\mu} < K(\lambda, \mu)|f|_{-\lambda} \tag{4.4}$$

where

$$K(\lambda, \mu) = \frac{1}{\Gamma(\mu-\lambda)}\left(\frac{\Gamma(2\lambda+1)\Gamma(2\mu-2\lambda-1)}{\Gamma(2\mu)}\right)^{\frac{1}{2}}.$$

The same inequalities hold for norms associated with [ ] forms. [The proofs use Schwarz' and Minkowski's inequalities. To prove the second of (4.4) for example, we write

$$\phi(x, t) = x^{-\mu}(x-t)^{\mu-\lambda-1}f^{(-\lambda)}(t) \qquad (0 < t < x), \qquad 0\ (t > x),$$

so that, by $(3.1)_1$,

$$\begin{aligned}\|f\|_{-\mu} &= \left\{\int_0^\infty |x^{-\mu}(f^{(-\lambda)})^{(-(\mu-\lambda))}(x)|^2 dx\right\}^{\frac{1}{2}} \\ &= \Gamma(\mu-\lambda)^{-1}\left\{\int_0^\infty dx\left|\int_0^\infty \phi(x, t)dt\right|^2\right\}^{\frac{1}{2}} \\ &\leqq \Gamma(\mu-\lambda)^{-1}\int_0^\infty dt\left\{\int_0^\infty |\phi(x, t)|^2 dx\right\}^{\frac{1}{2}} = K(\lambda, \mu)|f|_{-\lambda}.]\end{aligned}$$

3°. We deduce from the inequalities of 2° that

$$\mathscr{A}_\mu \subset \mathscr{A}_\lambda \subset M \subset \mathscr{A}_{-\lambda} \subset \mathscr{A}_{-\mu} \qquad (0 < \lambda < \mu) \tag{4.5}$$

and

$$\mathscr{A}_\mu \subset \mathscr{G}_\lambda, \qquad \mathscr{A}_{-\lambda} \subset \mathscr{G}_{-\mu} \qquad (0 \leqq \lambda < \mu - \tfrac{1}{2}). \tag{4.6}$$

[The examples c, d of $(5.3)_1$ show that the inclusions are proper.] On the other hand, no $\mathscr{G}$ space is wholly contained in any $\mathscr{A}$ space. [It can be verified that if

$$t^\lambda f^{(\lambda)}(t) = t^{-\frac{1}{2}}(\log t)^{-1}\ (1 < 1+\varepsilon \leqq t),\ 0\ \text{(otherwise)},$$

then $f \in \mathscr{G}_\lambda$ but $f \notin \mathscr{A}_{-\mu}$ for all μ.]

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

4°. The formula $(5.4)_1$ for fractional integration by parts holds if $\lambda > \frac{1}{2}$ and either $f \in L^2$, $g \in \mathscr{A}_\lambda$ or $f \in M$, $g \in \mathscr{G}_\lambda$.

5°. The spaces $\mathscr{A}_{-\lambda}(\lambda > 0)$ are incomplete, so that $\mathscr{A}_{-\lambda} \subset \mathscr{B}_{-\lambda} \subset \mathscr{H}_{-\mu}$ if $0 < \lambda < \mu - \frac{1}{2}$. [The sequence (η_n), where $\eta_n(x, \xi) = n$ $(\xi < x < \xi + 1/n)$, 0 (otherwise), is Cauchy in $\mathscr{A}_{-\lambda}$, with no limit in the space.]

6°. M is dense in $\mathscr{B}_{-\lambda}$ [This follows from 1°, 5°, and the fact that L^2 is dense in $\mathscr{H}_{-\lambda}$ (Theorem 3_1).]

We return now to the resultant (4.1). The linear space M with this form of product becomes a commutative Banach algebra, without identity (since M does not contain a delta function). Thus

7°. If $f, g \in M$, then $f \cdot g \in M$ and

$$|f \cdot g|_0 \leqq |f|_0 \cdot |g|_0. \tag{4.7}$$

We also have

8°. If $f \in L^2$ and $g \in M$, then $f \cdot g \in L^2$ and

$$\|f \cdot g\|_0 \leqq \|f\|_0 \cdot |g|_0. \tag{4.8}$$

9°. If $f, g \in L^2$, then $(f \cdot g)(x)$ exists for all $x > 0$.

It will be convenient to introduce the operator $\sharp$, defined by

$$f^\sharp(x) = \frac{1}{x} f\left(\frac{1}{x}\right).$$

Real M is an algebra with involution $\sharp$. The operation also maps L^2 onto L^2, $\mathscr{G}_\lambda$ onto $\mathscr{G}_{[\lambda]}$, $\mathscr{A}_\lambda$ onto $\mathscr{A}_{[\lambda]}$.

We seek to extend the definition of $\cdot$ to the sequence-function spaces $\mathscr{H}$ and $\mathscr{B}$, and to this end we first construct inequalities for norms of the product. These are set out in Lemma 6 below, and the existence of the extended product is then described in Theorem 3. As a preliminary, we prove

LEMMA 5. *The formula*

$$x^{-\lambda}(f \cdot g)^{(-\lambda)}(x) = \left(x^{-\lambda} f^{(-\lambda)}(x)\right) \cdot g \tag{4.9}$$

is valid for all positive x in each of the following cases:

(a) $f \in L^2$, $g \in L^2$, $\lambda > 0$,

(b) $f \in L^2$, $g \in M$, $\lambda > \frac{1}{2}$,

(c) $f \in M$, $g \in L^2$, $\lambda > \frac{1}{2}$;

and for almost all positive x if $\lambda > 0$ in (b) or (c), or if

(d) $f \in M$, $g \in M$, $\lambda > 0$.

PROOF. Equation (4.9) in cases (a) to (c) is a consequence of Tonelli's and Fubini's theorems, by which

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

$$(4.10)\qquad \Gamma(\lambda)x^{-\lambda}(f\cdot g)^{(-\lambda)}(x) = x^{-\lambda}\int_0^\infty g\,(u)du\int_0^x (x-t)^{\lambda-1}f(ut)dt;$$

the repeated integral is shown to be absolutely convergent by judicious use of Schwarz' and Minkowski's inequalities appropriate to each case; we omit the details. Case (d) is slightly more involved: to show that the repeated integral (4.10) converges absolutely, suppose f and g non-negative, and consider the integral of (4.10),

$$(4.11)\qquad \Gamma(\lambda)\int_0^\infty x^{-\frac{1}{2}}dx\int_0^\infty g^{\#}(u)(xu)^{-\lambda}f^{(-\lambda)}(xu)du.$$

Invert the order of integration; the repeated integral

$$\begin{aligned}
&= \int_0^\infty g^{\#}(u)du\int_0^\infty x^{-\lambda-\frac{1}{2}}u^{-\lambda}dx\int_0^{xu}(xu-t)^{\lambda-1}f(t)dt\\
&= \int_0^\infty g^{\#}(u)u^{-\frac{1}{2}}du\int_0^\infty y^{-\lambda-\frac{1}{2}}dy\int_0^y (y-t)^{\lambda-1}f(t)dt \qquad (xu=y)\\
&= |g\,|_0\cdot\int_0^\infty f(t)dt\int_t^\infty y^{-\lambda-\frac{1}{2}}(y-t)^{\lambda-1}dy\\
&= |g|_0\cdot|f|_0\cdot\int_1^\infty w^{-\lambda-\frac{1}{2}}(w-1)^{\lambda-1}dw \qquad (y=wt).
\end{aligned}$$

Since this is finite, applications of Tonelli's and Fubini's theorems justify the inversions of order of integration used, and show that (4.11) converges; Tonelli's theorem shows that, in the general case (d), the integral in (4.10) converges absolutely for almost all positive x; finally, both theorems then justify the inversion involved in (4.10).

Cases (b) and (c) for $\lambda > 0$ are proved similarly. For example, to show the convergence of (4.10) in case (b), we note that, if $0 < a < b < \infty$,

$$\begin{aligned}
&\int_a^b dx\int_0^\infty |g^{\#}(u)|du\,(xu)^{-\lambda}\int_0^{xu}(xu-t)^{\lambda-1}|f(t)|dt\\
&\quad= \int_0^\infty |g^{\#}(u)|du\int_a^b dx\int_0^1 (1-v)^{\lambda-1}|f(xuv)|dv\\
&\quad\leqq \cdots\left(\int_a^b 1^2dx\right)^{\frac{1}{2}}\left(\int_a^b\left\{\int_0^1(1-v)^{\lambda-1}|f(xuv)|dv\right\}^2dx\right)^{\frac{1}{2}}\\
&\quad\leqq \cdots\;\cdots\int_0^1(1-v)^{\lambda-1}dv\left(\int_a^b |f(xuv)|^2dx\right)^{\frac{1}{2}}\\
&\quad\leqq \int_0^\infty |g\,(u)|u^{-\frac{1}{2}}du\cdot\sqrt{(b-a)}\cdot\int_0^1(1-v)^{\lambda-1}v^{-\frac{1}{2}}dv\cdot\left(\int_0^\infty |f(y)|^2dy\right)^{\frac{1}{2}},
\end{aligned}$$

which is finite if $g\in M$, $f\in L^2$, $\lambda>0$; the argument proceeds as before.

LEMMA 6. (i) *If* $f\in L^2$, $g\in M$, $\lambda\geqq 0$ *and* $\alpha\geqq 0$, *then*

$$(4.12)\qquad ||f\cdot g||_{-(\lambda+\alpha)}\leqq c_1||f||_{-\lambda}\cdot|g|_{-\alpha}.$$

(ii) *If* f, $g\in M$, $\lambda\geqq 0$ *and* $\alpha\geqq 0$, *then*

$$|f \cdot g|_{-(\lambda+\alpha)} \leqq c_2 |f|_{-\lambda} \cdot |g|_{-\alpha}. \tag{4.13}$$

Here c_1 and c_2 are constants depending only on λ and α.

PROOF. By two successive applications of Lemma 5, first for the case (c) and then for the case (b), we get

$$\begin{aligned} x^{-\lambda}\{x^{-\alpha}(f \cdot g)^{(-\alpha)}(x)\}^{(-\lambda)} &= x^{-\lambda}\{f \cdot (x^{-\alpha}g^{(-\alpha)}(x))\}^{(-\lambda)} \\ &= (x^{-\lambda}f^{(-\lambda)}(x)) \cdot (x^{-\alpha}g^{(-\alpha)}(x)). \end{aligned}$$

This is valid for almost all $x > 0$ under the conditions in (i) (we use 3° above). Then by (4.8),

$$||x^{-\alpha}(f \cdot g)^{(-\alpha)}(x)||_{-\lambda} \leqq ||f||_{-\lambda} \cdot |g|_{-\alpha}.$$

From 8° above we have $f \cdot g \in L^2$; an application of Lemma 7 (ii), from § 6 below, gives (4.12).

In the same way, Lemma 5 (d) and Lemma 7 (iii) lead to (4.13).

The criterion for the existence of $F \cdot g$ will be the same as for $F\phi$; that is, for the definition of $F \cdot g$ we repeat exactly the definition of $F\phi$ in § 3, with only the formal changes from $F\phi$ to $F \cdot g$ and $(f_n\phi)$ to $(f_n \cdot g)$. Then the remarks that follow that definition will also apply for the resultant, with the same kinds of amendments.

But now we shall more generally define also the resultant of two sequence-functions: $F \cdot G$ will be said to exist if $(f_n \cdot g_n)$ is Cauchy for any pair of sequences $(f_n) \sim F$, $(g_n) \sim G$ and if all such sequences are equivalent, and then $F \cdot G$ is the equivalence class so determined. Here the F, G may be limits in either $\mathscr{H}$ or $\mathscr{B}$ spaces, and the sequences are from L^2 or M, as the case may be. (Here we invoke § 4, 6°). We have:

THEOREM 3. *Let $\lambda \geqq 0$, $\alpha \geqq 0$. If $F \in \mathscr{H}_{-\lambda}$ and $G \in \mathscr{B}_{-\alpha}$, then $F \cdot G$ exists as an element of $\mathscr{H}_{-(\lambda+\alpha)}$. If instead $F \in \mathscr{B}_{-\lambda}$ and $G \in \mathscr{B}_{-\alpha}$, then $F \cdot G$ exists as an element of $\mathscr{B}_{-(\lambda+\alpha)}$.*

PROOF. For the first part, let $(f_n) \sim F$ in $\mathscr{H}_{-\lambda}$ and $(g_n) \sim G$ in $\mathscr{B}_{-\lambda}$. Then by (4.12),

$$||f_n \cdot g_n - f_m \cdot g_m||_{-(\lambda+\alpha)} \leqq c_1\{||f_n - f_m||_{-\lambda} \cdot |g_n|_{-\alpha} + ||f_m||_{-\lambda} \cdot |g_n - g_m|_{-\alpha}\},$$

and hence $(f_n \cdot g_n)$ is Cauchy in $\mathscr{H}_{-(\lambda+\alpha)}$. The non-dependence upon the particular sequences follows likewise from (4.12). The second part of the theorem is proved similarly.

The theorem shows that the $\mathscr{B}$ spaces are perhaps more appropriate vehicles for the resultant product than the $\mathscr{H}$ spaces: no sufficient condition for $F \cdot G \in \mathscr{H}_{-\lambda}$ in terms of $\mathscr{H}$ spaces alone seems to exist. We have from 5° that $\mathscr{B}_{-\lambda} \subset \mathscr{H}_{-\mu}$ if $\lambda < \mu - \frac{1}{2}$, but no reverse inclusion is possible.

5. Convolution products

A theory like that of § 4 can be constructed for the product

$$(f * g)(x) = \int_0^x f(t)g(x-t)dt.$$

The auxiliary spaces are $\mathscr{J}$, $\mathscr{K}$ (say), derived from the norm $|\cdot|_0$ of $L = L(0, \infty)$, so that, for example, we write $f \in \mathscr{J}_{-\lambda}$ if $|f|_{-\lambda} = |t^{-\lambda} f^{(-\lambda)}(t)|_0 < \infty$, and $\mathscr{K}_{-\lambda}$ is the space $\mathscr{J}_{-\lambda}$, completed. These spaces form a broken system:

$$\mathscr{J}_\mu \subset \mathscr{J}_\lambda \subset L; \qquad \mathscr{J}_{-\lambda} \subset \mathscr{J}_{-\mu} \qquad (0 < \lambda < \mu).$$

In place of (4.9) we can first show that $(f * g)^{(-\lambda)}(x) = (f * g^{(-\lambda)})(x)$ if $f, g \in L^2(0, x)$ and $\lambda > 0$; and then we can use the inequality

$$||x^{-(\lambda+\alpha)}(h * k)(x)||_0 \leqq K_1(\lambda, \alpha)||x^{-\lambda}h(x)||_0 \cdot |x^{-\alpha}k(x)|_0 \qquad (\alpha \geqq 0, \lambda \geqq 0)$$

to deduce that

$$||f * g||_{-(\lambda+\alpha)} \leqq K_1(\lambda, \alpha)||f||_{-\lambda} \cdot |g|_{-\alpha}.$$

Thus $F * G$ exists in $\mathscr{H}_{-(\lambda+\alpha)}$ if $F \in \mathscr{H}_{-\lambda}$, $G \in \mathscr{K}_{-\alpha}$.

6. Affixed integrals and derivatives

We have already in § 7_1 defined $F^{(-\lambda)}$, when $F \in \mathscr{H}_{-\lambda}$, indirectly as the coefficient of $x^{-\lambda}$ in the L^2 function $x^{-\lambda}F^{(-\lambda)}(x)$. We now consider the existence of expressions of the form $x^{-\alpha}F^{(-\alpha)}(x)$ for general $\alpha > 0$, which we shall call 'affixed integrals'.

LEMMA 7. *If $f \in L^2$, $\lambda \geqq 0$, $\alpha \geqq 0$, then*
(i) $x^{-\alpha}(x^{-\lambda}f^{(-\lambda)}(x))^{(-\alpha)} = x^{-\lambda}(x^{-\alpha}f^{(-\alpha)}(x))^{(-\lambda)}$;
(ii) *there exist constants c_1, c_2 depending only on λ, α such that*

$$0 < c_1 \leqq \frac{||x^{-\alpha}f^{(-\alpha)}(x)||_{-\lambda}}{||f||_{-(\lambda+\alpha)}} \leqq c_2 < \infty.$$

If instead $f \in M$, $\lambda \geqq 0$, $\alpha \geqq 0$, then
(iii) *the inequalities of* (ii) *hold with $|\cdot|$ in place of $||\cdot||$ (and possibly with different constants, c_3, c_4 say).*

PROOF. To prove (i) it is sufficient to observe that either side of the equation has Mellin transform

$$\mathfrak{F}(s)\frac{\Gamma(1-s)}{\Gamma(\lambda+1-s)}\frac{\Gamma(1-s)}{\Gamma(\alpha+1-s)},$$

by $(4.3)_1$. Likewise for (ii) we have

$$\|x^{-\alpha}f^{(-\alpha)}(x)\|_{-\lambda}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\mathfrak{F}(s)\Gamma(1-s)\Gamma(1-s)}{\Gamma(\lambda+1-s)\Gamma(\alpha+1-s)}\right|^2 dt \qquad (s = \tfrac{1}{2}+it)$$
$$\leqq \frac{c_2^2}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\mathfrak{F}(s)\Gamma(1-s)}{\Gamma(\lambda+\alpha+1-s)}\right|^2 dt = c_2^2\|f\|_{-(\lambda+\alpha)}^2,$$

since

$$c_1^{-1} \leqq |\Gamma(1-s)\Gamma(\lambda+\alpha+1-s)/\Gamma(\lambda+1-s)\Gamma(\alpha+1-s)| \leqq c_2$$

when $\Re(s) = \frac{1}{2}$, for some positive constants c_1, c_2. The other inequality is proved similarly.

Mellin transform theory is no longer adequate for the proof of (iii), for which we work differently. By $(3.1)_1$,

$$|f|_{-(\lambda+\alpha)} = \int_0^{\infty}\left|\frac{x^{-(\lambda+\alpha)}}{\Gamma(\lambda)}\int_0^x (x-t)^{\lambda-1}t^{-\alpha}f^{(-\alpha)}(t)t^{\alpha}dt\right| x^{-\frac{1}{2}}dx.$$

Let t^{α} in the integrand be expanded as a power series in $(x-t)/x$,

$$t^{\alpha} = x^{\alpha}\sum_{r=0}^{\infty}\binom{\alpha}{r}(-1)^r\left(\frac{x-t}{x}\right)^r;$$

term-by-term integration is permissible because the series terminates or is at least boundedly convergent. We get

$$|f|_{-(\lambda+\alpha)} = \int_0^{\infty}\left|\frac{1}{\Gamma(\lambda)}\sum_{r=0}^{\infty}\binom{\alpha}{r}(-1)^r x^{-(\lambda+r)}\int_0^x (x-t)^{\lambda+r-1}t^{-\alpha}f^{(-\alpha)}(t)dt\right| x^{-\frac{1}{2}}dx$$
$$\leqq \sum_{r=0}^{\infty}\left|\binom{\alpha}{r}\right|\frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}|x^{-\alpha}f^{(-\alpha)}(x)|_{-(\lambda+r)}$$
$$\leqq \sum_{r=0}^{\infty}\left|\binom{\alpha}{r}\right|\frac{\Gamma(\lambda+r)\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\lambda+r+\frac{1}{2})}|x^{-\alpha}f^{(-\alpha)}(x)|_{-\lambda},$$

by (4.3). The series converges; thus for a positive constant c_3, say, we have

$$|f|_{-(\lambda+\alpha)} \leqq c_3^{-1}|x^{-\alpha}f^{(-\alpha)}(x)|_{-\lambda},$$

which is one of the inequalities of (iii). The other can be proved in a similar fashion. (Similar methods can be used to prove (ii) also.)

If $x^{-\alpha}F^{(-\alpha)}(x)$ is defined as the equivalence class of sequences $(x^{-\alpha}f_n^{(-\alpha)}(x))$ where $(f_n) \sim F$, $f_n \in L^2$, the lemma gives

THEOREM 4. *Every sequence-function* F *of* $\mathscr{H}_{-\lambda}$ *has affixed integrals of all positive orders, and* $x^{-\alpha}F^{(-\alpha)}(x)$ *belongs to* $\mathscr{H}_{-(\lambda-\alpha)}$ *if* $\alpha \leqq \lambda$.

If in fact $F \in L^2$, so that $(f) \sim F$ for some $f \in L^2$, then $x^{-\alpha}F^{(-\alpha)}(x)$ so defined is equal to $x^{-\alpha}$ multiplied by the fractional integral $f^{(-\alpha)}(x)$. Moreover, if $F \in \mathscr{H}_{-\kappa}$ and $\alpha < \kappa < \lambda$, then $(x^{-\alpha}f_n^{(-\alpha)}(x))$ defines the same sequence-

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

function in both $\mathscr{H}_{-(\kappa-\alpha)}$ and $\mathscr{H}_{-(\lambda-\alpha)}$, so that the definition is consistent within the class of spaces $\mathscr{H}$.

It is easily seen that the statements of the above lemma and theorem remain valid if everywhere the [ ] forms are read in place of the ( ) forms.

Affixed derivatives can be defined in an analogous manner.

LEMMA 8. *If* $f \in \mathscr{G}_\alpha$, $\lambda \geqq 0$, $\alpha \geqq 0$, *then*

(i) $f(x) = x^{-\alpha}(x^\alpha f^{(\alpha)}(x))^{[-\alpha]}$;

(ii) *there exist constants* c_1', c_2' *depending only on* λ, α *such that*

$$0 < c_1' \leqq \frac{||x^\alpha f^{(\alpha)}(x)||_{[-(\lambda+\alpha)]}}{||f||_{[-\lambda]}} \leqq c_2' < \infty.$$

The identity in (i) is implicit in the formula defining $f^{[-\alpha]}$ and in the definitions of § 2_1. Inequalities (ii) can be proved by using Mellin transform theory as in Lemma 7, (ii). We note that, by $(4.4)_1$, the norms appearing in the lemma may be replaced by the equivalent norms without square brackets.

Define $x^\alpha F^{(\alpha)}(x)$ when $F \in \mathscr{H}_{-\lambda}$ as the class of sequences $(x^\alpha f_n^{(\alpha)}(x))$ where (f_n) is any sequence of $\mathscr{G}_\alpha$ functions for F (recalling that $\mathscr{G}_\alpha$ is dense in $\mathscr{H}_{-\lambda}$). We have

THEOREM 5. *Every sequence-function* f *of* L^2 *has an affixed derivative* $x^\alpha f^{(\alpha)}(x)$ *in* $\mathscr{H}_{-\alpha}$, *if* $\alpha > 0$. *More generally, if* $F \in \mathscr{H}_{-\lambda}$, *the affixed derivative* $x^\alpha F^{(\alpha)}(x)$ *is defined in* $\mathscr{H}_{-(\lambda+\alpha)}$ *if* $\lambda \geqq 0$, $\alpha \geqq 0$.

As in the previous case, the definition of $x^\alpha F^{(\alpha)}(x)$ is an extension of its usual meaning for the case $F \in \mathscr{G}_\alpha$, and is consistent within the class of spaces $\mathscr{H}$.

Lemma 8 (i) may be interpreted as a statement on the integrability of derivatives. More generally, if $F \in \mathscr{H}_{-\lambda}$, so that $x^\alpha F^{(\alpha)}(x)$ belongs to $\mathscr{H}_{-(\lambda+\alpha)}$ and $x^{-\alpha}(x^\alpha F^{(\alpha)}(x))^{[-\alpha]}$ belongs to $\mathscr{H}_{[-\lambda]} = \mathscr{H}_{-\lambda}$, Lemma 8 (i) shows that $x^{-\alpha}(x^\alpha F^{(\alpha)}(x))^{[-\alpha]}$ equals F. Likewise $x^\alpha(x^{-\alpha}F^{[-\alpha]}(x))^{(\alpha)} = F$.

As an example on the use of these derivatives, one may prove the formula

$$\int_0^\infty g(x)x^\alpha F^{(\alpha)}(x)dx = \int_0^\infty x^\alpha g^{[\alpha]}(x)F(x)dx, \tag{7.1}$$

valid if $g \in \mathscr{G}_\lambda$, $F \in \mathscr{H}_{-(\lambda-\alpha)}$, $0 < \alpha < \lambda$, and where the integrals have the meaning given in § 8_1. (The proof uses Lemma 5_1.) The particular case $F = \Delta_\xi$ (§ 10_1) and $\lambda - \alpha \geqq 1$ gives

$$\int_0^\infty g(x)x^\alpha \Delta_\xi^{(\alpha)}(x)dx = \xi^\alpha g^{[\alpha]}(\xi). \tag{7.2}$$

https://doi.org/10.1017/S1446788700028299 Published online by Cambridge University Press

7. Local integrals and derivatives

Powers $\phi(t) = t^\beta$ as they stand do not satisfy the requirements on ϕ of Theorem 2. However, functions which behave like them on subsets of $(0, \infty)$ are easily constructed. Thus, for positive integral l and positive numbers β, A, let ϕ be determined by

$$t^l\phi^{(l)}(t) = 0 \quad (0 < t < A), \qquad \Gamma(\beta+l)\Gamma(\beta)^{-1}t^{-\beta} \quad (t > A).$$

It is easily shown that ϕ is then the continuous function

$$x^{-\beta}_{(A,\infty)} = \begin{cases} \text{a polynomial in } x \text{ of degree } l-1 & (0 \leqq x \leqq A), \\ x^{-\beta} & (x \geqq A), \end{cases}$$

and this belongs to $\mathscr{P}_l$ and is bounded.

Again, let ψ be determined by

$$t^l\psi^{[l]}(t) = \Gamma(\beta+l+1)\Gamma(\beta+1)^{-1}t^\beta \quad (0 < t < A), \quad 0 \ (t > A),$$

so that ψ is the continuous function

$$x^\beta_{(0,A)} = \begin{cases} x^\beta & (0 \leqq x \leqq A), \\ \text{a polynomial in } x^{-1} \text{ of degree } l, \text{ with zero constant term} & (x \geqq A); \end{cases}$$

and this belongs to $\mathscr{P}_{[l]}$.

We can use these functions to detach the factors $x^{\pm\alpha}$ in the affixed integrals and derivatives of § 6, on subsets of $(0, \infty)$. Thus, let $F \in \mathscr{H}_{-(l-\alpha)}$, so that by Theorems 4 and 2, the sequence-function

$$F^{(\alpha)}(x)_{(A,\infty)} = x^{-\alpha}_{(A,\infty)} x^\alpha F^{(\alpha)}(x)$$

exists in $\mathscr{H}_{-l}$. If in fact $F \in \mathscr{G}_a$ (a integral), then $x^a F^{(a)}(x) \in L^2$ and $F^{(a)}(x)_{(A,\infty)}$ is a function also in L^2 which equals $(-1)^a(d/dx)^a F(x)$ for almost all $x > A$, in the sense of § 2. Again, let $F \in \mathscr{H}_{-(l+\alpha)}$, so that $x^{-\alpha}F^{(-\alpha)}(x) \in \mathscr{H}_{-l}$; the sequence-function

$$F^{(-\alpha)}(x)_{(0,A)} = x^\alpha_{(0,A)} x^{-\alpha} F^{(-\alpha)}(x)$$

exists in $\mathscr{H}_{-l}$; and it equals the α-order integral of F for almost all x in $0 < x < A$ in the sense of § 2 if, for example, $F \in \mathscr{G}_{-\alpha}$.

References

[1] Miller, J. B., Hilbert spaces of generalized functions extending L^2 (I), *J. Aust. Math. Soc.* **1** (1960), 281–298.

To the references given at the end of [1] may be added:

[2] Miller, J. B., A class of series involving integral kernels, I and II, *Math. Zeitschr.* **76** (1961), 257–269 and **79** (1962), 227–236.

[3] Miller, J. B., On the solution of certain integral equations by generalized functions, *Proc. Camb. Phil. Soc.* **57** (1961), 767–777.

[4] Miller, J. B., Normed spaces of generalized functions, *Compositio Mathematica* (to appear).

Australian National University,
Canberra, A.C.T.