Proceedings of the Edinburgh Mathematical Society (1996) 39, 409–423 ©

ON RHODES EXPANSIONS OF BANDS

by H. SEZINANDO*

(Received 18th May 1994)

We present a direct computation of the Rhodes expansions of the free objects in the varieties of bands, based in the manipulation of the invariants i_n introduced by Gerhard and Petrich [4] in the study of bands.

1991 *Mathematics subject classification:* primary 20M10, secondary 20M17.

1. Introduction

In [8] Reilly studied the Rhodes expansions of completely regular semigroups, using techniques deriving from the work of Polák [5, 6, 7]. Here we give an alternative and quite different computation of the Rhodes expansion of S in the special case where S is a free object in a variety of bands. This is based on manipulation of the invariants i_n, introduced by Gerhard and Petrich [4] and on results obtained by the author [10] concerning the cardinalities of the free objects in varieties of bands. The work is part of the author's Ph.D. thesis [9].

2. Preliminaries

We record here the notation to be used throughout the paper and we state some theorems to be used later.

X: a fixed countably infinite set. Elements of X are called variables.

A: a set, called an alphabet. Elements of A are called letters.

A^*: the free monoid on A. Elements of A^* are called words. They are finite strings of elements of A written as $a_1 \cdots a_n$, where $a_1, \dots a_n \in A$. The product is concatenation. The identity of A^* is denoted by 1. It is thought of as the empty string.

$c(w)$: the content of $w \in A^*$ is the set of letters occurring w. By definition, $c(1) = \phi$.

$\bar{w}$: the dual of w is the word obtained from w by reversing the order of the variables. That is, if $w = a_1 \cdots a_n$ with $a_1 \cdots a_n \in A$, then $\bar{w} = a_n \cdots a_1$.

$\bar{V}$: the dual variety of V.

$F_A(V)$: the free object of the variety V, on the set of generators A.

$[u = v]$: the variety of semigroups defined by the equations $u = v$, $x^2 = x$.

* This work was partially supported by S.A.L. (JNICT)

$[i, i+j]$: the set $\{i, i+1, \ldots i+j\}$, where $i \geq 0, j \geq 1$.
$\mathcal{S}$: the variety of semilattices $[xy = yx]$.
$\mathcal{LB}$: the lattice of varieties of bands.
$\mathcal{LB}_0$: the lattice obtained from $\mathcal{LB}$ by taking away the varieties of rectangular bands.
$\mathcal{L}, \mathcal{J}$: the Green relations.

Given a function $t: A^* \to A^*$, we denote by $\bar{t}$ the function from A^* into A^* defined by $\bar{t}(w) = \overline{t(\bar{w})}$.
Let $w = uxv$, where $c(w) = c(ux)$ and $c(w) \neq c(u)$. Set

$s(w) = u$: the longest left cut of w that contains all but one of the variables of w.
$\sigma(w) = x$: the last variable to occur in w in order from the left.
$e(w) = \bar{s}(w)$: the longest right cut of w that contains all but one of the variables of w.
$\varepsilon(w) = \bar{\sigma}(w)$: the last variable to occur in w in order from the right.

Following Fennemore [3], we define the words R_n, S_n and Q_n for $n \geq 2$, as follows.

$R_2 = R_2(x_1, x_2, x_3) = x_3x_2x_1$,
$R_3 = R_3(x_1, x_2, x_3) = x_1x_2x_3$,
$Q_2 = Q_2(x_1, x_2, x_3) = x_2x_3x_1$,
$Q_3 = Q_3(x_1, x_2, x_3) = x_1x_2x_3x_1x_3$,
$S_2 = S_2(x_1, x_2, x_3) = x_3x_1x_2x_1$,
$S_3 = S_3(x_1, x_2, x_3) = x_1x_2x_3x_1x_3x_2x_3$,
$R_n = R_n(x_1, \ldots, x_n) = R_{n-1}x_n$, for $n = 4, 6 \ldots$
$R_n = R_n(x_1, \ldots, x_n) = x_nR_{n-1}$, for $n = 5, 7 \ldots$
$Q_n = Q_n(x_1, \ldots, x_n) = Q_{n-1}x_nR_n$, for $n = 4, 6 \ldots$
$Q_n = Q_n(x_1, \ldots, x_n) = R_nx_nQ_{n-1}$, for $n = 5, 7 \ldots$
$S_n = S_n(x_1, \ldots, x_n) = S_{n-1}x_nR_n$, for $n = 4, 6 \ldots$
$S_n = S_n(x_1, \ldots, x_n) = R_nx_nS_{n-1}$, for $n = 5, 7 \ldots$

Set $V_1 = \mathcal{S}$, $V_2 = [xyz = xy]$, $V_2' = [\bar{R}_2 = \bar{Q}_2]$, $V_n = [R_n = S_n]$, $V_n' = [R_n = Q_n]$ for $n \geq 3$, n odd, $V_n = [\bar{R}_n = \bar{S}_n]$, $V_n' = [\bar{R}_n = \bar{Q}_n]$ for $n \geq 3$, n even. Let V_∞ denote the variety of all bands. The lattice $\mathcal{LB}_0$ is represented in Figure 1.

We call the varieties V_1, V_∞, V_n and V_n', $n \geq 2$, *left varieties*. The duals of these varieties are called *right varieties*.

Given a variety of bands V, we denote by $V^l\,[V^r]$ the minimum of the set of left varieties [right varieties] containing V.

Let S be a semigroup. We define the relations $\leq_{\mathcal{L}}$ and $<_{\mathcal{L}}$ by

$$a \leq_{\mathcal{L}} b \Leftrightarrow S^1a \subseteq S^1b \quad \text{and}$$
$$a <_{\mathcal{L}} b \Leftrightarrow S^1a \subseteq S^1b \quad \text{but} \quad S^1a \neq S^1b \quad (a, b \in S).$$

Remark 2.1. Let S be a band and let $s, t \in S$ be such that $s \leq_{\mathcal{L}} t$. Then $s\mathcal{L}t$ if and only if $s\mathcal{J}t$. Hence, if $u, v \in S$ are such that $c(u) = c(v)$ and $u \leq_{\mathcal{L}} v$, then $u\mathcal{L}v$.

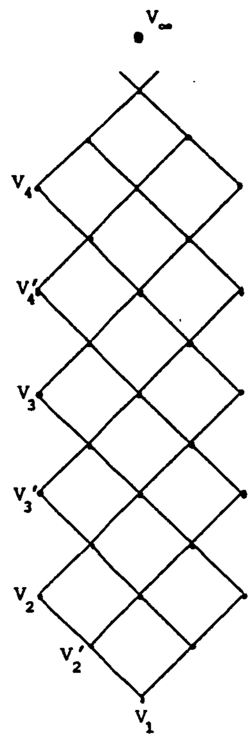


FIGURE 1 The lattice $\mathcal{L}\mathcal{B}_0$.

Theorem 2.2. **[10]** *Let us fix a variety of bands V. Given a finite subset B of A, let $\mathcal{X}_B(V)$ denote the $\mathcal{J}$-class of $F_A(V)$ consisting of the elements of content B. Let $c_n(V)$ denote the cardinality of $\mathcal{X}_B(V)$ when $|B|$ equals n. Then*

$$c_k(V_2) = k!,$$
$$c_k(V_n) = k^2 c_{k-1}(V_n) c_{k-1}(\bar{V}_{n-1}), \quad n \geq 3,$$
$$c_k(V_n \vee \bar{V}_n) = k^2 c_{k-1}^2(V_n), \quad n \geq 2.$$

3. The Rhodes expansion

Let S be a semigroup. A finite sequence $\bar{a} = (a_n, \ldots, a_1)$ of elements of S is an $\mathcal{L}$-chain of

$$a_n \leq_{\mathcal{L}} a_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} a_1$$

and is a *reduced* $\mathcal{L}$-chain if

$$a_n <_{\mathcal{L}} a_{n-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} a_1.$$

The reduction $Red(\bar{a})$ of $\bar{a}$ is the sequence obtained from $\bar{a}$ by successively deleting the right most element of any pair of $\mathcal{L}$-equivalent elements until no such pairs remain.

Definition 3.1. Let S be a semigroup. The *Left Rhodes expansion* $\hat{S}^{\mathcal{L}}$ of S is the set of all reduced $\mathcal{L}$-chains with the multiplication

$$(a_n <_{\mathcal{L}} \dots <_{\mathcal{L}} a_1)(b_m <_{\mathcal{L}} \dots <_{\mathcal{L}} b_1) = Red(a_n b_m \leq_{\mathcal{L}} \dots \leq_{\mathcal{L}} a_1 b_m \leq_{\mathcal{L}} b_m <_{\mathcal{L}} \dots <_{\mathcal{L}} b_1).$$

We define a map $\eta_S : \hat{S}^{\mathcal{L}} \to S$ by

$$\eta_S(a_n <_{\mathcal{L}} \dots <_{\mathcal{L}} a_1) = a_n;$$

η_S is a surjective morphism and is called the *canonical morphism.*

Given a morphism $\varphi : S \to T$, we define a morphism $\hat{\varphi}^{\mathcal{L}} : \hat{S}^{\mathcal{L}} \to \hat{T}^{\mathcal{L}}$ by

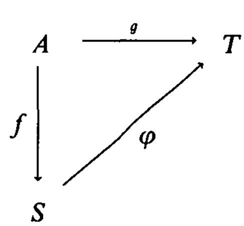
$$\hat{\varphi}^{\mathcal{L}}(a_n <_{\mathcal{L}} \dots <_{\mathcal{L}} a_1) = Red(\varphi(a_n) \leq_{\mathcal{L}} \dots \leq_{\mathcal{L}} \varphi(a_1)).$$

It is easy to check that we get an expansion.

Dually, we can define the *right Rhodes expansion* $\hat{S}^{\mathcal{R}}$ of S.

We refer to **[1]** for a discussion of the basic properties of the left and right Rhodes expansions.

We now consider the category $\mathcal{S}_A$ of semigroups generated by a given set A. Its objects are the pairs (S,f), where S is a semigroup and $f : A \to S$ is a map such that $f(A)$ generates S (in the classical sense). A morphism from (S,f) into (T,g) is a morphism of semigroups $\varphi : S \to T$ such that the following diagram is commutative.



Definition 3.2. Let (S,f) be an object of $\mathcal{S}_A$. The *left Rhodes expansion cutdown to generators* A of S is the pair $(\hat{S}^{\mathcal{L}}_A, \hat{f}^{\mathcal{L}}_A)$, where $\hat{S}^{\mathcal{L}}_A$ is the subsemigroup of $\hat{S}^{\mathcal{L}}$ generated by the set $\{(f(a)) : a \in A\}$ and $\hat{f}^{\mathcal{L}}_A : A \to \hat{S}^{\mathcal{L}}_A$ is defined by

$$\hat{f}^{\mathcal{L}}_A(a) = (f(a)) \qquad (a \in A).$$

The canonical morphism $\eta_{S,A}:\hat{S}^{\mathcal{L}}_A \to S$ is defined simply by

$$\eta_{S,A} = \eta_S|_{\hat{S}^{\mathcal{L}}_A}.$$

Finally, if φ is a morphism from (S,f) into (T,g), we define $\hat{\varphi}^{\mathcal{L}}_A$ from $(\hat{S}^{\mathcal{L}}_A, \hat{f}^{\mathcal{L}}_A)$ into $(\hat{T}^{\mathcal{L}}_A, \hat{g}^{\mathcal{L}}_A)$ by

$$\hat{\varphi}^{\mathcal{L}}_A = \hat{\varphi}^{\mathcal{L}}|_{\hat{S}^{\mathcal{L}}_A}.$$

It is easy to check that we still get an expansion.

We refer again to [1] for a discussion of the basic properties of the left and right Rhodes expansion cutdown to generators.

From now on we work with the left Rhodes expansions cutdown to generators and we omit the word "left".

Theorem 3.3. [Tilson, **11**] *Let S be a semigroup. Then the Rhodes expansion cutdown to A of* $\hat{S}^{\mathcal{L}}_A$ *is isomorphic to* $\hat{S}^{\mathcal{L}}_A$.

Fact 3.4. If (S,f) and (T,g) are objects of S_A, there is at most one morphism from (S,f) to (T,g). Consequently, if $\varphi:(S,f)\to(T,g)$ and $\psi:(T,g)\to(S,f)$ are morphisms, then (S,f) and (T,g) are isomorphic.

Fact 3.5. [**1**] If S is a band, then $\hat{S}^{\mathcal{L}}_A$ is also a band.

Lemma 3.6. *Let S be a semigroup and let* $x, y \in \hat{S}^{\mathcal{L}}$. *Then*

$$xy = y \Leftrightarrow \eta_S(xy) = \eta_S(y).$$

Proof. Let $x=(x_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} x_1)$, $y=(y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)$, $x_1,\ldots,x_n$, $y_1,\ldots y_m \in S$, be such that $\eta_S(xy)=\eta_S(y)$. This means that

$$\eta_S(\mathrm{Red}(x_n y_m \le_{\mathcal{L}} \cdots \le_{\mathcal{L}} x_1 y_m \le_{\mathcal{L}} y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)) = \eta_S(y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)$$

that is,

$$\eta_S(\mathrm{Red}(x_n y_m \le_{\mathcal{L}} \cdots \le_{\mathcal{L}} x_1 y_m \le_{\mathcal{L}} y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)) = y_m.$$

By the definitions of Red and η_S we get

$$\mathrm{Red}(x_n y_m \le_{\mathcal{L}} \cdots \le_{\mathcal{L}} x_1 y_m \le_{\mathcal{L}} y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1) = (y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)$$

that is, $xy = y$.

The converse is trivial.

4. The Rhodes expansions of the free objects

If S is a rectangular band, then $\hat{S}_A^{\mathcal{L}}$ is isomorphic to S. This comes from the fact that all the sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ have length one, since, if $s, t \in S$ and $s \leq_{\mathcal{L}} t$, then $s\mathcal{L}t$.

From now on we work on $\mathcal{LB}_0$.

We start with $V = \mathcal{S}$ and show that if $S = F_A(V)$, then $\hat{S}_A^{\mathcal{L}} \simeq F_A(\bar{V}_2)$. In order to show this, we will show that:

(1) If $S \in \mathcal{S}$, then $\hat{S}^{\mathcal{L}} \in \bar{V}_2$.

(2) If $S = F_A(V)$, then $\hat{S}_A^{\mathcal{L}} = \{(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : k \geq 1,\ s_i \in A^+,\ |c(s_i)| = i,\ i \in [1, k]\}$ and $|\hat{S}_A^{\mathcal{L}}| = |F_A(\bar{V}_2)|$.

Proof of (1). Let $s, t \in \hat{S}^{\mathcal{L}}$, $s = (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$, $t = (i_l <_{\mathcal{L}} \cdots <_{\mathcal{L}} i_1)$, where $k, l \geq 1$, $s_i, i_j \in A^+$, $|c(s_i)| = i$, $|c(i_j)| = j$, $i \in [1, k]$, $j \in [1, l]$. Then

$$\begin{aligned} sts &= \mathrm{Red}(s_k i_l \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1 i_l <_{\mathcal{L}} i_l <_{\mathcal{L}} \cdots <_{\mathcal{L}} i_1) . (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\ &= \mathrm{Red}(s_k i_l s_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1 i_l s_k \leq_{\mathcal{L}} i_l s_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} i_1 s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\ &= \mathrm{Red}(i_l s_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} i_1 s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\ &= ts. \end{aligned}$$

Proof of (2). Let $X = \{(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : k \geq 1, s_i \in A^+, |c(s_i)| = i,\ i \in [1, k]\}$.

Let $s \in \hat{S}_A^{\mathcal{L}}$. If $s = (a)$, $a \in A$, then $s \in X$. Suppose that any product $(a_n) \ldots (a_1)$, $n \geq 1$, $a_i \in A$, $i \in [1, n]$ is in X. Let $a_{n+1} \in A$. Then

$$\begin{aligned} (a_{n+1}) \ldots (a_1) &= (a_{n+1}) [(a_n) \ldots (a_1)] \\ &= (a_{n+1}) (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \end{aligned}$$

by the induction hypothesis, where $s_i \in A^+$, $|c(s_i)| = i, i \in [1, k]$. Hence

$$(a_{n+1}) \ldots (a_1) = \mathrm{Red}(a_{n+1} s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

If $a_{n+1} \notin c(s_k)$, then $a_{n+1} s_k <_{\mathcal{L}} s_k$ and

$$(a_{n+1}) \ldots (a_1) = (a_{n+1} s_k <_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1),$$

with $|c(a_{n+1} s_k)| = k + 1$.

If $a_{n+1} \in c(s_k)$, then $a_{n+1} s_k \mathcal{L} s_k$ and

$$(a_{n+1}) \ldots (a_1) = (a_{n+1} s_k <_{\mathcal{L}} s_{n-1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1),$$

where $|c(a_{n+1} s_k)| = k$.

Thus $\hat{S}_A^{\mathcal{L}} \subseteq X$.

Conversely, let $s = (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in X$ and let $\{a_i\} = c(s_i) \backslash c(s_{i-1})$, $i \in [1, k]$. In S, $s_i = a_i s_{i-1}$, $i \in [1, k]$ (since $c(s_i) = c(a_i s_{i-1})$). It is then a mere routine to check that $s = (a_k) \ldots (a_1)$. Thus, $s \in \hat{S}_A^{\mathcal{L}}$.

Finally, if $|A| = N$, we have

$$\begin{aligned} |\hat{S}_A^{\mathcal{L}}| &= |\{(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : k \geq 1, s_i \in A^+, |c(s_i)| = i, i \in [1,k]\}| \\ &= \sum_{k=1}^{N} |\{(s_k, \ldots, s_1) : s_i \in A^+, c(s_i) \supseteq c(s_{i-1}), |c(s_i)| = i, i \in [1,k]\}| \\ &= \sum_{k=1}^{N} N(N-1)\ldots(N-k+1) \\ &= |F_A(\bar{V}_2)| \quad \text{(See Theorem 2.2.)} \end{aligned}$$

We now state the main theorem. As mentioned in the introduction, this is a special case of a result due to Reilly **[8]**.

Theorem 4.1. *If $V \in \mathcal{LB}_0$ and $S = F_A(V)$, then $\hat{S}_A^{\mathcal{L}} \simeq F_A(V^r)$.*

The next statement shows that it suffices to prove the theorem for the left varieties of $\mathcal{LB}_0$. Moreover, since the remaining cases can be proved similarly, we shall only consider the cases $V = V_2$ or $V = V_k$, k odd, $k \geq 3$.

Proposition 4.2. *Let $S, T, U \in S_A$ and let $\varphi : S \to T$, $\psi : T \to U$ be morphisms. Then*

$$\hat{U}_A^{\mathcal{L}} \simeq S \Rightarrow \hat{T}_A^{\mathcal{L}} \simeq S.$$

Proof. Let $S, T, U \in S_A$ and let $\varphi : S \to T$, $\psi : T \to U$ be morphisms. Suppose that $U_A^{\mathcal{L}} \simeq S$ and let χ be an isomorphism from $\hat{U}_A^{\mathcal{L}}$ into S.

By Theorem 3.3, $\hat{S}_A^{\mathcal{L}} \simeq S$ and by Fact 3.4, the natural morphism $\eta_{S,A} : \hat{S}_A^{\mathcal{L}} \to S$ is an isomorphism. Let $\eta_{S,A}^{-1}$ be its inverse. Let $\hat{\varphi}_A^{\mathcal{L}} : \hat{S}_A^{\mathcal{L}} \to \hat{T}_A^{\mathcal{L}}$ and $\hat{\psi}_A^{\mathcal{L}} : \hat{T}_A^{\mathcal{L}} \to \hat{U}_A^{\mathcal{L}}$ be the morphisms determined by φ and ψ, respectively. Then $\hat{\varphi}_A^{\mathcal{L}} \circ \eta_{S,A}^{-1} \circ \chi \circ \hat{\psi}_A^{\mathcal{L}}$ is a morphism from $\hat{T}_A^{\mathcal{L}}$ into itself and so it is the identity morphism. Hence $\chi \circ \hat{\psi}_A^{\mathcal{L}}$ is an isomorphism from $\hat{T}_A^{\mathcal{L}}$ into S.

We now state a series of lemmas, whose proofs we shall defer until we prove Theorem 4.1.

Lemma 4.3. *If V is a left variety and $S \in V$, then $\hat{S}^{\mathcal{L}} \in V^r$.*

Lemma 4.4. *Let $u, v \in A^+$ be such that $c(u) \supseteq c(v)$, $|c(u)| = |c(v)| + 1$.*
(i) *If $V = V_2$, then $u <_{\mathcal{L}} v$ in $F_A(V)$.*
(ii) *If $V = V_k$ odd, $k \geq 3$, then*

$$i_k(u) \leq_{\mathcal{L}} i_k(v) \Leftrightarrow \bar{i}_{k-1}(e(u))\mathcal{L}\bar{i}_{k-1}(v).$$

Proposition 4.5. *Let $S = F_A(V)$.*

(i) *If* $V = V_2$, *then*

$$\hat{S}_A^{\mathcal{L}} = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, i \in [1, n]\}.$$

(ii) *If* $V = V_k$, *k odd*, $k \geq 3$, *then*

$$\hat{S}_A^{\mathcal{L}} = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, \bar{i}_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{i-1}), i \in [1, n]\}.$$

Lemma 4.6. *Let* $n \geq 1$ *and let* $u \in A^+$ *be such that* $|c(u)| = n$.

(i) $V = V_2$, then

$$|\{(u \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, i \in [1, n-1]\}| = \frac{c_n(\bar{V}_3)}{c_n(V_2)}.$$

(ii) *If* $V = V_k$, *k odd*, $k \geq 3$, *then*

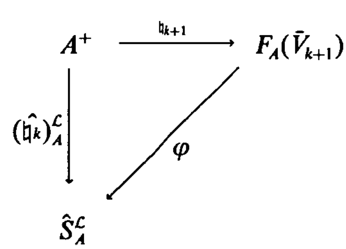
$$\begin{aligned}|\{(u \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, \bar{i}_{k-1}(e(u)) = \bar{i}_{k-1}(s_{n-1}),\\ \bar{i}_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{i-1}), i \in [1, n-1]\}|\\ = \frac{c_n(\bar{V}_{k+1})}{c_n(V_k)}.\end{aligned}$$

Lemma 4.7. *Let* $n \leq 1$ *and let* $A_n \subseteq A$ *be such that* $|A_n| = n$. *Let* $S = F_A(V_k)$, $k = 2$ *or* *k odd*, $k \geq 3$. *Then*

$$|\{s \in \hat{S}_A^{\mathcal{L}} : c(\eta_{S,A}(s)) = A_n\}| = c_n(\bar{V}_{k+1}).$$

Proof of Theorem 4.1. Let $S = F_A(V)$. If $V = V_\infty$, by Fact 3.5, $\hat{S}_A^{\mathcal{L}} \in V_\infty$. Consequently, there is a morphism $\psi : S \to \hat{S}_A^{\mathcal{L}}$. Since $\eta_{S,A}$ is a morphism from $\hat{S}_A^{\mathcal{L}}$ into S, Fact 3.4 yields that $\varphi, \eta_{S,A}$ are isomorphisms and so $\hat{S}_A^{\mathcal{L}} \simeq S$.

Let $S = F_A(V_k)$, $k \geq 2$. By Lemma 4.3 there is a surjective morphism $\varphi : F_A(\bar{V}_{k+1}) \to \hat{S}_A^{\mathcal{L}}$ such that the following diagram is commutative.

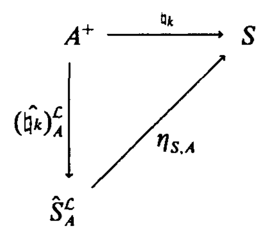


(4.1.1)

where $\natural_k$ denotes the canonical epimorphism from A^+ into $F_A(V_k)$.

https://doi.org/10.1017/S0013091500023208 Published online by Cambridge University Press

Also the following diagram is commutative.



(4.1.2)

We now show that φ is injective. It is enough to show that the restriction of φ to the $\mathcal{J}$-classes of $F_A(\bar{V}_{k+1})$ is injective. Indeed, if t and $t' \in F_A(\bar{V}_{k+1})$ are such that $\varphi(t) = \varphi(t')$, then there are $u, v \in A^+$ such that $t = \natural_{k+1}(u)$, $t' = \natural_{k+1}(v)$ and so we get

$$\begin{aligned}\varphi(t) = \varphi(t') &\Rightarrow \varphi\natural_{k+1}(u) = \varphi\natural_{k+1}(v)\\ &\Rightarrow (\widehat{\natural_k})^{\mathcal{L}}_A(u) = (\widehat{\natural_k})^{\mathcal{L}}_A(v) \quad \text{by (4.1.1)}\\ &\Rightarrow \eta_{S,A}(\widehat{\natural_k})^{\mathcal{L}}_A(u) = \eta_{S,A}(\widehat{\natural_k})^{\mathcal{L}}_A(v)\\ &\Rightarrow \natural_k(u) = \natural_k(v) \quad \text{by (4.1.2)}\\ &\Rightarrow c(u) = c(v)\end{aligned}$$

and therefore t and t' are $\mathcal{J}$-related in $F_A(\bar{V}_{k+1})$.

Let $A_n \subseteq A$. We have

$$\varphi(\{t \in F_A(\bar{V}_{k+1}) : c(t) = A_n\}) = \{\varphi(t) \in \hat{S}^{\mathcal{L}}_A : c(t) = A_n\}$$

But if $t = \natural_{k+1}(u)$, $u \in A^+$ we get

$$\begin{aligned}c(t) &= c(\natural_{k+1}(u))\\ &= c(\natural_k(u))\\ &= c(\eta_{S,A}(\widehat{\natural_k})^{\mathcal{L}}_A(u)) \quad \text{by (4.1.2)}\\ &= c(\eta_{S,A}\varphi\natural_{k+1}(u)) \quad \text{by (4.1.1)}\\ &= c(\eta_{S,A}(\varphi(t))).\end{aligned}$$

Hence

$$\begin{aligned}\varphi(\{t \in F_A(\bar{V}_{k+1}) : c(t) = A_n\}) &= \{\varphi(t) \in \hat{S}^{\mathcal{L}}_A : c(\eta_{S,A}(\varphi(t))) = A_n\}\\ &= \{s \in \hat{S}^{\mathcal{L}}_A : c(\eta_{S,A}(s)) = A_n\}\end{aligned}$$

and the injectivity of φ follows from the fact that

$$\begin{aligned}|\varphi(\{t \in F_A(\bar{V}_{k+1}) : c(t) = A_n\})| \\ &= |\{s \in \hat{S}_A^{\mathcal{L}} : c(\eta_{S,A}(s)) = A_n\}| \\ &= c_n(\bar{V}_{k+1}) \quad \text{by Lemma 4.7} \\ &= |\{t \in F_A(\bar{V}_{k+1}) : c(t) = A_n\}|.\end{aligned}$$

We now prove 4.3–4.7.

Proof of Lemma 4.3. The proof is based on Lemma 3.6. The case of $\mathcal{S}$ will be omitted since it was already treated.

If $S \in V_\infty$ the result is an immediate consequence of Fact 3.5.

Let $S \in V_2$ and let $s_1, s_2, s_3 \in \hat{S}^{\mathcal{L}}$. Then

$$\begin{aligned}\eta_S(s_1s_2s_1s_3s_1s_2s_3) &= \eta_S(s_1)\eta_S(s_2)\eta_S(s_1)\eta_S(s_3)\eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\ &= \eta_S(s_1)\eta_S(s_2)\eta_S(s_3)\eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\ &= \eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\ &= \eta_S(s_1s_2s_3).\end{aligned}$$

Hence Lemma 3.6 yields that $s_1s_2s_3 = s_1s_2s_1s_3s_1s_2s_3$, thus $\hat{S}^{\mathcal{L}} \in \bar{V}_3$, as required.

Now let $S \in V_k$, k odd, $k \geq 3$. By definition $R_{k+1} = R_kx_{k+1}$ and $S_{k+1} = S_kx_{k+1}R_{k+1}$. We notice first that if $S_k = S_k(x_1, \ldots, x_k)$, then

$$\begin{aligned}\eta_S(S_k) &= S_k(\eta_S(x_1), \ldots, \eta_S(x_k)) \\ &= R_k(\eta_S(x_1), \ldots, \eta_S(x_k)) \\ &= \eta_S(R_k).\end{aligned}$$

Hence,

$$\begin{aligned}\eta_S(S_kx_{k+1}R_{k+1}) &= \eta_S(S_k)\eta_S(x_{k+1})\eta_S(R_{k+1}) \\ &= \eta_S(R_k)\eta_S(x_{k+1})\eta_S(R_{k+1}) \\ &= \eta_s(R_kx_{k+1})\eta_S(R_{k+1}) \\ &= \eta_S(R_{k+1})\eta_S(R_{k+1}) \\ &= \eta_S(R_{k+1}).\end{aligned}$$

Thus Lemma 3.6 yields that $S_{k+1} = R_{k+1}$ and so $\hat{S}^{\mathcal{L}} \in \bar{V}_{k+1}$, as required.

Proof of Lemma 4.4. Let $u, v \in A^+$ be such that $c(u) \supseteq c(v)$, $|c(u)| = |c(v)| + 1$.

(i) Let $V = V_2$. Then

$$u \leq_{\mathcal{L}} v \Leftrightarrow i_2(uv) = i_2(u).$$

But $c(u) \supseteq c(v)$ implies $c(uv) = c(u)$, $i_2(uv) = i_2(u)$. Hence $u \leq_{\mathcal{L}} v$. If $v \leq_{\mathcal{L}} u$, we would get $c(vu) = c(v)$ and this contradicts the fact that $c(u) \neq c(v)$. Therefore $u <_{\mathcal{L}} v$.

(ii) Let $V = V_k$, k odd, $k \geq 3$. Then

$$
\begin{aligned}
u \leq_{\mathcal{L}} v &\Leftrightarrow i_k(uv) = i_k(u) \\
&\Leftrightarrow \begin{cases} i_k(s(uv)) = i_k(s(u)) \\ \sigma(uv) = \sigma(u) \\ \varepsilon(uv) = \varepsilon(u) \\ \bar{i}_{k-1}(e(uv)) = \bar{i}_{k-1}(e(u)) \end{cases} \\
&\Leftrightarrow \bar{i}_{k-1}(e(uv)) = \bar{i}_{k-1}(e(u)), \quad \text{since } c(u) \supseteq c(v) \\
&\Leftrightarrow \bar{i}_{k-1}(e(u)v) = \bar{i}_{k-1}(e(u)), \quad \text{since } c(u) \neq c(v) \\
&\Leftrightarrow \bar{i}_{k-1}(e(u))\bar{i}_{k-1}(v) = \bar{i}_{k-1}(e(u)) \\
&\Leftrightarrow \bar{i}_{k-1}(e(u)) \leq_{\mathcal{L}} \bar{i}_{k-1}(v) \\
&\Leftrightarrow \bar{i}_{k-1}(e(u)) \mathcal{L} \bar{i}_{k-1}(v),
\end{aligned}
$$

since $c(e(u)) = c(v)$. (See Remark 2.1.)

Proof of Proposition 4.5. (i) Let $V = V_2$ and let

$$X = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, \ i \in [1, n]\}.$$

The proof of the inclusion $\hat{S}_A^{\mathcal{L}} \subseteq X$ is analogous with the proof made for $V = \mathcal{S}$. We now prove the converse inclusion.

Let $s \in X$.

If $s = (s_1)$, $|c(s_1)| = 1$, then $s_1 \in A$ and $s \in \hat{S}_A^{\mathcal{L}}$.

Let $n \geq 1$ and suppose that all sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ of X are in $\hat{S}_A^{\mathcal{L}}$. Let $s = (s_{n+1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in X$ and let $c(s_{n+1}) \backslash c(s_n) = \{a_{n+1}\}$. It is easy to see that

$$i_2(s_{n+1}) = i_2(s(s_{n+1})\sigma(s_{n+1})a_{n+1}s_n). \tag{4.5.1}$$

Now let $s(s_{n+1}) = b_1 \dots b_r$, $b_i \in A$, $i \in [1, r]$. We will see that

$$s = (b_1) \dots (b_r)(\sigma(s_{n+1}))(a_{n+1})(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

Indeed,

$$
\begin{aligned}
&(b_1) \dots (b_r)(\sigma(s_{n+1}))(a_{n+1})(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\
&\quad = \mathrm{Red}(b_1 \dots b_r\sigma(s_{n+1})s_{n+1}s_n \leq_{\mathcal{L}} b_2 \dots b_r\sigma(s_{n+1})a_{n+1}s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\
&\quad = (b_1 \dots b_r\sigma(s_{n+1})a_{n+1}s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)
\end{aligned}
$$

since $c(c_1 \dots b_r\sigma(s_{n+1})a_{n+1}s_n) = c(a_{n+1}s_n)$. Thus

$$(b_1)\dots(b_r)(\sigma(s_{n+1}))(a_{n+1})(s_n <_{\mathcal{L}} \dots <_{\mathcal{L}} s_1) = (s(s_{n+1})\sigma(s_{n+1})a_{n+1}s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \dots <_{\mathcal{L}} s_1)$$
$$= (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \quad \text{by (4.5.1).}$$

Hence the inclusion hypothesis yields that $s \in \hat{S}_A^{\mathcal{L}}$.

(ii) Let $V = V_k$, k odd, $k \geq 3$. Let

$$Y = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, \bar{i}_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{i-1}), i \in [1, n]\}.$$

We prove first the inclusion $\hat{S}_A^{\mathcal{L}} \subseteq Y$.

If $a \in A$, then $(a) \in Y$. Suppose that $n \geq 1$ and that any product $(a_n)\dots(a_1)$, $a_i \in A$, $i \in [1, n]$, is in Y. Let $s = (a_{n+1})(a_n)\dots(a_1)$, $a_i \in A$, $i \in [1, n+1]$. Then

$$s = (a_{n+1})[(a_n)\dots(a_1)]$$
$$= (a_{n+1})(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$$

by the induction hypothesis, where $k \geq 1$, $s_i \in A^+$, $|c(s_i)| = i$, $\bar{i}_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{i-1})$, $i \in [1, k]$. Hence

$$s = \text{Red}(a_{n+1}s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

If $a_{n+1} \in c(s_k)$, then $a_{n+1}s_k\mathcal{L}s_k$ (see Remark 2.1) and

$$s = (a_{n+1}s_k <_{\mathcal{L}} s_{k-1} <_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$$

where $|c(a_{n+1}s_k)| = k$. Moreover in this case

$$\bar{i}_{k-1}(e(a_{n+1}s_k)) = \bar{i}_{k-1}(e(s_k))$$
$$= \bar{i}_{k-1}(s_{k-1}), \quad \text{by the induction hypothesis}$$

and so $s \in Y$.

If $a_{n+1} \notin c(s_k)$ then $a_{n+1}s_k <_{\mathcal{L}} s_k$ and

$$s = (a_{n+1}s_k <_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$$

where $|c(c_{n+1}s_k)| = k + 1$. Moreover

$$\bar{i}_{k-1}(e(a_{n+1}s_k)) = \bar{i}_{k-1}(s_k)$$

since in this case $\varepsilon(a_{n+1}s_k) = a_{n+1}$ and $e(a_{n+1}s_k) = s_k$.

Hence $\hat{S}_A^{\mathcal{L}} \subseteq Y$.

Conversely, let $s \in Y$. If $s = (s_1)$, with $|c(s_1)| = 1$, then $s_1 \in A$ and $s \in \hat{S}_A^{\mathcal{L}}$. Let $n \geq 1$ and suppose that all sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ of Y are in $\hat{S}_A^{\mathcal{L}}$. Let $s = (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in Y$.

Since $\bar{i}_{k-1}(e(s_{n+1})) = \bar{i}_{k-1}(s_n)$, then $c(e(s_{n+1})) = c(s_n)$ and we deduce that

$$\{\varepsilon(s_{n+1})\} = c(s_{n+1})\backslash c(s_n).$$

Again, it is easy t check that

$$i_k(s_{n+1}) = i_k(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n). \tag{4.5.2}$$

Now, let $s(s_{n+1}) = b_1 \dots b_r$, $b_i \in A$, $i \in [1, r]$. We show that

$$s = (b_1)\dots(b_r)\,(\sigma(s_{n+1}))\,(\varepsilon(s_{n+1}))\,(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

Indeed,

$$\begin{aligned}&(b_1)\dots(b_r)\,(\sigma(s_{n+1}))\,(\varepsilon(s_{n+1}))\,(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\ &\quad= \mathrm{Red}(b_1 \dots b_r\sigma(s_{n+1})\varepsilon(s_{n+1})s_n \le_{\mathcal{L}} b_2 \dots b_r\sigma(s_{n+1})\varepsilon(s_{n+1})s_n \le_{\mathcal{L}} \cdots \le_{\mathcal{L}}\\ &\qquad\sigma(s_{n+1})\varepsilon(s_{n+1})s_n \le_{\mathcal{L}} \varepsilon(s_{n+1})s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\ &\quad= (b_1 \dots b_r\sigma(s_{n+1})\varepsilon(s_{n+1})s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\end{aligned}$$

by Remark 2.1, since

$$c(b_1 \dots b_r\sigma(s_{n+1})\varepsilon(s_{n+1})s_n) = c(\varepsilon(s_{n+1})s_n).$$

Hence

$$\begin{aligned}&(b_1)\dots(b_r)\,(\sigma(s_{n+1}))\,(\varepsilon(s_{n+1}))\,(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\ &\qquad= (s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\ &\qquad= (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \quad \text{by (4.5.2)}\end{aligned}$$

Therefore, by the induction hypothesis, $s \in \hat{S}_A^{\mathcal{L}}$.

Proof of Lemma 4.6. (i) Let $V = V_2$. If $|c(u)| = 1$, then the statement is trivially true, since $|c_1(V)| = 1$, for all $V \in \mathcal{LB}$. Suppose the statement holds for $n \ge 1$ and let $u \in A^+$ be such that $|c(u)| = n + 1$.

Fix $z \in A^+$ such that $|c(z)| = n$. Then we have

$$\begin{aligned}&|(u \le_{\mathcal{L}} s_n \le_{\mathcal{L}} \cdots \le_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, i \in [1, n]\}|\\ &\quad= |\{v \in A^+ : u \le_{\mathcal{L}} v, |c(v)| = n\}|\,.\,|\{(z \le_{\mathcal{L}} s_{n-1} \le_{\mathcal{L}} \cdots \le_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i,\\ &\qquad i \in [1, n-1]\}|\end{aligned}$$

Now

$$|\{v : u \le_{\mathcal{L}} v, |c(v)| = n\}| = (n+1)c_n(V_2)$$

and the induction hypothesis yields that

$$|\{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, i \in [1, n-1]\}| = \frac{c_n(\bar{V}_3)}{c_n(V_2)}.$$

Hence

$$|(u \leq_{\mathcal{L}} s_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, i \in [1, n]\}| = \frac{c_{n+1}(\bar{V}_3)}{c_{n+1}(V_2)}.$$

(ii) Let $V = V_k$, k odd, $k \geq 3$.

If $|c(u)| = 1$, the statement is trivially true. Suppose the statement holds for $n \geq 1$ and let $u \in A^+$ be such that $|c(u)| = n+1$. Fix $z \in A^+$ such that $|c(z)| = n$. Then we have

$$\begin{aligned}
&|(u \leq_{\mathcal{L}} s_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, \bar{i}_{k-1}(e(u)) = \bar{i}_{k-1}(v),\\
&\qquad i_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{i-1}), i \in [1, n]\}|\\
&\quad = |\{v : u \leq_{\mathcal{L}} v, |c(v)| = n, \bar{i}_{k-1}(e(u)) = \bar{i}_{k-1}(v)\}|.\\
&|\{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, \bar{i}_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{k-1}),\\
&\qquad i \in [1, n-1], \bar{i}_{k-1}(e(z)) = \bar{i}_{k-1}(s_{n-1})\}|
\end{aligned}$$

Again, the result follows from the next two computations. (See Theorem 2.2.)

$$\begin{aligned}
&|\{v : u \leq_{\mathcal{L}} v, |c(v)| = n, \bar{i}_{k-1}(e(u)) = \bar{i}_{k-1}(v)\}|\\
&\quad = |\{v : \bar{i}_{k-1}(e(u))\mathcal{L}\bar{i}_{k-1}(v), \bar{i}_{k-1}(e(u)) = \bar{i}_{k-1}(v)\}| \quad \text{by Lemma 4.6}\\
&\quad = |\{v : \bar{i}_{k-1}(e(u)) = \bar{i}_{k-1}(v)\}|\\
&\quad = \frac{c_n(V_k)}{c_n(\bar{V}_{k-1})}
\end{aligned}$$

since there are $c_n(V_k)$ words $i_k(v)$ with $|c(v)| = n$ and there are $c_n(\bar{V}_{k-1})$ words $\bar{i}_{k-1}(w)$ with $|c(w)| = n$.

$$\begin{aligned}
&|\{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, \bar{i}_{k-1}(e(s_i)) = \bar{i}_{k-1}(s_{i-1}),\\
&\qquad i \in [1, n-1], \bar{i}_{k-1}(e(z)) = \bar{i}_{k-1}(s_{n-1})\}|\\
&\quad = \frac{c_n(\bar{V}_{k+1})}{c_n(V_k)},
\end{aligned}$$

by the induction hypothesis.

Proof of Lemma 4.7. The proof results immediately from Lemma 4.6. Indeed, if $s = (s_n <_{\mathcal{L}} s_{n-1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in \hat{S}^{\mathcal{L}}_A$, then $\eta_{S,A}(s) = s_n$ and the number of elements s_n such that $c(s_n) = A_n$ is $c_n(V_k)$ if $V = V_k$.

Acknowledgement. The author wishes to thank her supervisor Professor John M. Howie for his help and support during the preparation of her thesis.

REFERENCES

1. J.-C. Birget and J. Rhodes, Iterations of expansions – unambiguous semigroups, *J. Pure Appl. Algebra* **34** (1984), 1–55.

2. S. Eilenberg, *Automata, languages and machines*, Vol. B Academic Press, New York, 1976.

3. C. Fennemore, All varieties of bands, *Semigroup Forum* **44** (1970), 172–179.

4. J. A. Gerhard and M. Petrich, Varieties of bands revisited, *Proc. London Math. Soc.* (3) **58** (1989), 323–350.

5. L. Polak, On varieties of completely regular semigroups I, *Semigroup Forum* **32** (1985), 97–123.

6. L. Polak, On varieties of completely regular semigroups II, *Semigroup Forum* **36** (1987), 253–284.

7. L. Polak, On varieties of completely regular semigroups III, *Semigroup Forum* **37** (1988), 1–30.

8. R. Reilly, The Rhodes expansion and free objects in varieties of completely regular semigroups, *J. Pure Appl. Algebra* **69** (1990), 89–109.

9. H. Sezinando, Formal languages and idempotent semigroups, Ph.D. thesis, University of St. Andrews, Scotland, 1990.

10. H. Sezinando, The cardinalities of the Green classes of the free objects in varieties of bands, *Semigroup Forum* **44** (1992), 164–198.

11. B. Tilson, Chapters IX and XII in [**2**].

Centro de Álgebra, Universidade de Lisboa
Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex
Portugal
E-mail address: MHELENA@PTMAT.LMC.FC.UL.PT