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CHARACTERIZATION OF A CLASS OF INFINITE MATRICES WITH APPLICATIONS

P.N. NATARAJAN

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. The class (l_α, l_α) of infininite matrices transforming sequences over K in l_α to sequences in l_α is characterized. Further a Mercerian theorem is proved in the context of the Banach algebra (l_α, l_α), $\alpha \geq 1$ and finally a Steinhaus type result is proved for the space l_α. In the case of $\mathbb{R}$ or $\mathbb{C}$, on the other hand, the best known result so far seems to be a characterization of positive matrix transformations of the class (l_α, l_β), $\infty > \alpha \geq \beta > 1$.

1. Introduction.

K denotes a complete, non-trivially valued field, that is $K = \mathbb{R}$ (the field of real numbers) or $\mathbb{C}$ (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field.

If X, Y are sequence spaces with elements whose entries are in K and if $A = (a_{nk})$, $a_{nk} \in K$, $n,k = 0,1,2,\ldots$ is an infinite matrix, we

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write $A \in (X,Y)$ if $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$ is defined, $n = 0,1,2,\ldots$ and the sequence $Ax = \{(Ax)_n\} \in Y$, for every $x = \{x_k\} \in X$. Ax is called the A transform of x.

The main result of this paper is the characterization of infinite matrices belonging to (l_α, l_α), $\alpha > 0$, where

$$l_\alpha = \{x = \{x_k\},\ x_k \in K,\ k = 0,1,2,\ldots,\ \sum_{k=0}^{\infty} |x_k|^\alpha < \infty\} ,$$

where K is a complete, non-trivially valued, non-archimedean field. Because of the fact that there is, as such, no classical analogue for this result, this result is interesting. When $K = \mathbb{R}$ or $\mathbb{C}$, a complete characterization of the class (l_α, l_β) of infinite matrices, $\alpha, \beta \geq 2$, does not seem to be available in the literature. Even a recent result [5] in this direction characterizes only non-negative matrices in (l_α, l_β), $\alpha \geq \beta > 1$. When $K = \mathbb{R}$ or $\mathbb{C}$, a known simple sufficient condition ([6], p.174, Theorem 9) for an infinite matrix A to belong to (l_α, l_α) is

$$A \in (l_\infty, l_\infty) \cap (l_1, l_1).$$

Sufficient conditions or necessary conditions for $A \in (l_\alpha, l_\beta)$ when $K = \mathbb{R}$ or $\mathbb{C}$ are available in literature (see for example [11]). Necessary and sufficient conditions for $A \in (l_1, l_1)$ are due to Mears [7] (for alternate proofs, see Knopp and Lorentz [4], Fridy [2]).

From the characterization mentioned at the outset, it is then deduced that the Cauchy product of two sequences in l_α, $\alpha > 0$, is again in l_α. This result fails to hold for $\alpha > 1$ when the field is $\mathbb{R}$ or $\mathbb{C}$. In Section 3 we obtain a Mercerian theorem by considering the structure of the space (l_α, l_α), $\alpha \geq 1$, of matrices. In Section 4 we study certain Steinhaus type theorems involving the space l_α.

2. Characterization of matrices in (l_α, l_α), $\alpha > 0$.

THEOREM 2.1. *If $A = (a_{nk})$, $a_{nk} \in K$, $n,k = 0,1,2,\ldots$ where K is a complete, non-trivially valued, non-archimedean field,*

$A \in (l_\alpha, l_\alpha)$, $\alpha > 0$ if and only if

(2.1) $$\sup_{k\geq 0} \sum_{n=0}^{\infty} |a_{nk}|^\alpha < \infty .$$

Proof. Since $|.|$ is a non-archimedean valuation, we first observe that

(2.2) $$\big||a|^\alpha - |b|^\alpha\big| \leq |a+b|^\alpha \leq |a|^\alpha + |b|^\alpha , \quad \alpha > 0 .$$

(Sufficiency). If $x = \{x_k\} \in l_\alpha$, $\sum_{k=0}^{\infty} a_{nk}x_k$ converges, $n = 0,1,2,\ldots,$ since $x_k \longrightarrow 0$, $k \longrightarrow \infty$ and $\sup_{n,k} |a_{nk}| < \infty$ by (2.1). Also,

$$\sum_{n=0}^{\infty} |(Ax)_n|^\alpha \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{nk}|^\alpha \, |x_k|^\alpha$$
$$\leq \Big(\sum_{k=0}^{\infty} |x_k|^\alpha\Big)\Big(\sup_{k\geq 0} \sum_{n=0}^{\infty} |a_{nk}|^\alpha\Big)$$
$$< \infty ,$$

so that $\{(Ax)_n\} \in l_\alpha$.

(Necessity). Suppose $A \in (l_a, l_a)$. We first note that

$\sup_{k\geq 0} |a_{nk}|^\alpha = B_n < \infty$, $n = 0,1,2,\ldots$. For, if for some m ,

$\sup_{k\geq 0} |a_{mk}|^\alpha = \infty$, then, we can choose a strictly increasing sequence

$\{k(i)\}$ of positive integers such that $|a_{m,k(i)}|^\alpha > i^2$, $i = 1,2,\ldots$.

If the sequence $\{x_k\}$ is defined by

$$x_k = \begin{cases} \dfrac{1}{a_{m,k(i)}} , & k = k(i) \\ 0 , & k \neq k(i) \end{cases} \quad , \; i = 1,2,\ldots,$$

$\{x_k\} \in l_\alpha$, for , $\sum_{k=0}^{\infty} |x_k|^\alpha = \sum_{i=1}^{\infty} |x_{k(i)}|^\alpha < \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$,

while $a_{m,k(i)}\, x_{k(i)} = 1 \not\longrightarrow 0$, $i \longrightarrow \infty$ which is contradiction.

Since $(Ax)_n = a_{nk}$ for the sequence $x = \{x_n\}$, $x_n = 0$, $n \neq k$, $x_k = 1$ and $\{(Ax)_n\} \in l_\alpha$,

$$\mu_k = \sum_{n=0}^{\infty} |a_{nk}|^\alpha < \infty , \; k = 0,1,2,\ldots .$$

Suppose $\{\mu_k\}$ is unbounded. Choose a positive integer $k(1)$ such that

$$\mu_{k(1)} > 3 .$$

Then choose a positive integer $n(1)$ such that

$$\sum_{n=n(1)+1}^{\infty} |a_{n,k(1)}|^\alpha < 1 ,$$

so that

$$\sum_{n=0}^{n(1)} |a_{n,k(1)}|^\alpha > 2 .$$

More generally, given the positive integers $k(j)$, $n(j)$, $j \leq m-1$, choose positive integers $k(m)$, $n(m)$ such that $k(m) > k(m-1)$, $n(m) > n(m-1)$,

$$\sum_{n=n(m-2)+1}^{n(m-1)} \sum_{k=k(m)}^{\infty} B_n k^{-2} < 1 ,$$

$$\mu_{k(m)} > 2 \sum_{n=0}^{n(m-1)} B_n + \rho^{-\alpha} m^2 \left\{2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)}\right\}$$

and

$$\sum_{n=n(m)+1}^{\infty} |a_{n,k(m)}|^\alpha < \sum_{n=0}^{n(m-1)} B_n ,$$

where, since K is non-trivially valued, there exists $\pi \in K$ such that $0 < \rho = |\pi| < 1$.

$$\sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\alpha = \mu_{k(m)} - \sum_{n=0}^{n(m-1)} |a_{n,k(m)}|^\alpha - \sum_{n=n(m)+1}^{\infty} |a_{n,k(m)}|^\alpha$$

$$> 2 \sum_{n=0}^{n(m-1)} B_n + \rho^{-\alpha} m^2 \left\{2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)}\right\} - \sum_{n=0}^{n(m-1)} B_n - \sum_{n=0}^{n(m-1)} B_n$$

$$= \rho^{-\alpha} m^2 \left\{2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)}\right\} .$$

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For every $i = 1,2,\ldots,$ there exists a non-negative integer $\lambda(i)$ such that

$$\rho^{\lambda(i)+1} \le i^{-2/\alpha} < \rho^{\lambda(i)} .$$

Define the sequence $x = \{x_k\}$ as follows.

$$\left.\begin{aligned} x_k &= \pi^{\lambda(i)+1} , & k &= k(i) \\ &= 0 , & k &\ne k(i) \end{aligned}\right\}, \quad i = 1,2,\ldots .$$

$$\{x_k\} \in l_\alpha , \text{ for } , \sum_{k=0}^{\infty} |x_k|^\alpha = \sum_{i=1}^{\infty} |x_{k(i)}|^\alpha \le \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty .$$

However, using (2.2),

$$\sum_{n=n(m-1)+1}^{n(m)} |(Ax)_n|^\alpha \ge \Sigma_1 - \Sigma_2 - \Sigma_3 ,$$

where

$$\Sigma_1 = \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\alpha \ |x_{k(m)}|^\alpha ,$$

$$\Sigma_2 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^\alpha \ |x_{k(i)}|^\alpha ,$$

$$\Sigma_3 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=m+1}^{\infty} |a_{n,k(i)}|^\alpha \ |x_{k(i)}|^\alpha .$$

Now

$$\Sigma_1 = \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\alpha \ \rho^{(\lambda(m)+1)\alpha}$$

$$\ge \rho^\alpha \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\alpha \ m^{-2}$$

(2.3) $$> 2 + \sum_{i=1}^{m-1} i^{-2} \ \mu_{k(i)} ,$$

$$\Sigma_2 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^\alpha \ \rho^{(\lambda(i)+1)\alpha}$$

$$\le \sum_{i=1}^{m-1} i^{-2} \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(i)}|^\alpha$$

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$$(2.4) \qquad \leq \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} ,$$

$$\Sigma_3 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=m+1}^{\infty} |a_{n,k(i)}|^{\alpha} \rho^{(\lambda(i)+1)\alpha}$$

$$\leq \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=k(m+1)}^{\infty} B_n i^{-2}$$

$$(2.5) \qquad < 1 .$$

From (2.3) to (2.5), we have,

$$\sum_{n=n(m-1)+1}^{n(m)} |(Ax_n)|^{\alpha} > 1 , \quad m = 2,3,\ldots .$$

This shows that $\{(Ax)_n\} \notin l_\alpha$ while $\{x_k\} \in l_\alpha$, a contradiction. Thus condition (2.1) is also necessary. The proof of the theorem is now complete.

The Cauchy product of two series $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ is $\sum_{k=0}^{\infty} c_k$ where

$$(2.6) \qquad c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0 .$$

In the context of the sequence space l_α , we have the following theorem.

THEOREM 2.2. *If* $\{a_k\}$, $\{b_k\} \in l_\alpha$, *so does their Cauchy product* $\{c_k\}$.

Proof. Consider the matrix

$$(2.7) \qquad A = \begin{bmatrix} a_0 & 0 & 0 & 0 & \ldots \\ a_1 & a_0 & 0 & 0 & \ldots \\ a_2 & a_1 & a_0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} .$$

Noting that the A-transform of $\{b_k\}$ is $\{c_k\}$, since $\{a_k\} \in l_\alpha$, $A \in (l_\alpha, l_\alpha)$ so that $\{c_k\} \in l_\alpha$, since $\{b_k\} \in l_\alpha$.

Remark 2.1. (i) It is easy to establish Theorem 2.2, when $\alpha = 1$, virtually by following the steps in the well-known Cauchy theorem on multiplication of series; Theorem 2.2 could be proved in the same way using (2.2).

(ii) Theorem 2.2 could be formally stated in the following form: If a sequence $\{a_k\}$ is given, then $\{c_k\} \in l_\alpha$ for every sequence $\{b_k\} \in l_\alpha$ if and only if $\{a_k\} \in l_\alpha$ where c_k is defined by (2.6).

(iii) Theorem 2.2 is not true when $K = \mathbb{R}$ or $\mathbb{C}$ and $\alpha > 1$ as illustrated by the following example. Let

$$a_k = b_k = \frac{1}{(k+1)^{\frac{1}{2}(1+\frac{1}{\alpha})}}, \quad \alpha > 1 .$$

Then $\{a_k\}, \{b_k\} \in l_\alpha$ while $\{c_k\} \notin l_\alpha$.

3. A Mercerian theorem.

We now set out to study the structure of (l_α, l_α), $\alpha \geq 1$, with a view to obtain a Mercerian theorem analogous to the one obtained earlier by Rangachari and Srinivasan [9]. (l_α, l_α), $\alpha \geq 1$, is a Banach algebra under the norm

$$(3.1) \qquad ||A|| = \sup_{k\geq 0} \left(\sum_{n=0}^{\infty} |a_{nk}|^\alpha\right)^{\frac{1}{\alpha}}, \quad A = (a_{nk}) \in (l_\alpha, l_\alpha) ,$$

with the usual matrix addition, multiplication and elementwise scalar multiplication. First we note that if $A = (a_{nk})$, $B = (b_{nk}) \in (l_\alpha, l_\alpha)$, $(AB)_{nk}$ is defined, for, $(AB)_{nk} = \sum_{i=0}^{\infty} a_{ni} b_{ik}$ converges, since $b_{ik} \longrightarrow 0$, $i \longrightarrow \infty$ and $\sup_{n,i} |a_{ni}| < \infty$. We next show that (l_α, l_α) is closed with respect to multiplication. For, using (2.2),

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$$\sum_{n=0}^{\infty} |(AB)_{nk}|^{\alpha} = \sum_{n=0}^{\infty} |\sum_{i=0}^{\infty} a_{ni}b_{ik}|^{\alpha}$$

$$\leq \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} |a_{ni}|^{\alpha} |b_{ik}|^{\alpha}$$

$$= \sum_{i=0}^{\infty} |b_{ik}|^{\alpha} (\sum_{n=0}^{\infty} |a_{ni}|^{\alpha})$$

$$\leq ||A||^{\alpha} ||B||^{\alpha}, \quad k = 0,1,2,\ldots .$$

Thus $AB \in (l_{\alpha}, l_{\alpha})$ and $||AB|| \leq ||A|| \, ||B||$. The associative law follows, for, if $A = (a_{nk})$, $B = (b_{nk})$, $C = (c_{nk}) \in (l_{\alpha}, l_{\alpha})$,

$$\{(AB)C\}_{nk} = \sum_{i=0}^{\infty} (AB)_{ni} c_{ik}$$

$$= \sum_{i=0}^{\infty} c_{ik} (\sum_{j=0}^{\infty} a_{nj} b_{ji})$$

$$= \sum_{j=0}^{\infty} a_{nj} (\sum_{i=0}^{\infty} b_{ji} c_{ik})$$

$$= \sum_{j=0}^{\infty} a_{nj} (BC)_{jk}$$

$$= \{A(BC)\}_{nk} .$$

It remains to prove that (l_{α}, l_{α}) is complete under the norm defined by (3.1). To see this, let $\{A^{(n)}\}$ be a Cauchy sequence in (l_{α}, l_{α}) where $A^{(n)} = (a_{ij}^{(n)})$, $i,j = 0,1,2,\ldots$. Since $\{A^n\}$ is Cauchy, for any $\varepsilon > 0$, there exists a positive integer n_o such that for $m,n \geq n_o$,

$$||A^{(m)} - A^{(n)}|| < \varepsilon .$$

That is $$\sup_{j\geq 0} \sum_{i=0}^{\infty} |a_{ij}^{(m)} - a_{ij}^{(n)}|^{\alpha} < \varepsilon^{\alpha}$$

Thus for all $i,j = 0,1,2,\ldots,$

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$$|a_{ij}^{(m)} - a_{ij}^{(n)}| < \varepsilon \ , \ m,n \geq n_o \ .$$

Hence $a_{ij}^{(n)} \longrightarrow a_{ij}$, $n \longrightarrow \infty$, $i,j = 0,1,2,\ldots$, since K is complete. Consider the matrix $A = (a_{ij})$, $i,j = 0,1,2,\ldots$. For all $j = 0,1,2,\ldots$,

$$\sum_{i=0}^{k} |a_{ij}^{(n)}|^{\alpha} \leq ||A^{(n)}||^{\alpha} \leq M^{\alpha} \ , \ n,k = 0,1,2,\ldots,$$

where $M = \sup_{n\geq 0} ||A^{(n)}||$. Allowing $n \longrightarrow \infty$, we have,

$$\sum_{i=0}^{k} |a_{ij}|^{\alpha} \leq M^{\alpha} \ , \ j,k = 0,1,2,\ldots \ .$$

Allowing $k \longrightarrow \infty$,

$$\sum_{i=0}^{\infty} |a_{ij}|^{\alpha} \leq M^{\alpha} \ , \ j = 0,1,2,\ldots,$$

which shows that $A \in (l_{\alpha}, l_{\alpha})$. Again for all $j,k = 0,1,2,\ldots$,

$$\sum_{i=0}^{k} |a_{ij}^{(m)} - a_{ij}^{(n)}|^{\alpha} < \varepsilon^{\alpha} \ , \ m,n \geq n_o \ .$$

For $n \geq n_o$, allowing $m \longrightarrow \infty$, for all $j,k = 0,1,2,\ldots$,

$$\sum_{i=0}^{k} |a_{ij} - a_{ij}^{(n)}|^{\alpha} \leq \varepsilon^{\alpha} \ .$$

Now, allowing $k \longrightarrow \infty$, we have, for all $j = 0,1,2,\ldots$,

$$\sum_{i=0}^{\infty} |a_{ij} - a_{ij}^{(n)}|^{\alpha} \leq \varepsilon^{\alpha} \ , \ n \geq n_o \ .$$

That is, $$\sup_{j\geq 0} \left(\sum_{i=0}^{\infty} |a_{ij} - a_{ij}^{(n)}|^{\alpha} \right)^{1/\alpha} \leq \varepsilon \ , \ n \geq n_o \ .$$

That is, $$||A^{(n)} - A|| \leq \varepsilon \ , \quad n \geq n_o \ ,$$

which shows that $A^{(n)} \longrightarrow A$, $n \longrightarrow \infty$.

The Mercerian theorem mentioned is the following.

THEOREM 3.1. *When* $K = Q_p$, *the* p-*adic field for a prime* p , *if* $y_n = x_n + \lambda p^n (x_o + x_1 + \ldots + x_n)$ *and* $\{y_n\} \in l_{\alpha}$, *then* $\{x_n\} \in l_{\alpha}$ *if*

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$|\lambda|_p < (1-\rho^\alpha)^{1/\alpha}$ *where* $\rho = |p|_p < 1$.

Proof. Since (l_α, l_α) is a Banach algebra, if $\lambda \in Q_p$ is such that $|\lambda|_p < \frac{1}{||A||}$, $A \in (l_\alpha, l_\alpha)$, then $I - \lambda A$, where I is the identity matrix, has an inverse in (l_α, l_α). The matrix of transformation is $I + \lambda A$ where

$$A \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ p & p & 0 & 0 & \cdots \\ p^2 & p^2 & p^2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

We note that $A \in (l_\alpha, l_\alpha)$ with $||A|| = \frac{1}{(1-\rho^\alpha)^{1/\alpha}}$. Then $I + \lambda A$ has an inverse in (l_α, l_α) if $|\lambda|_p < (1-\rho^\alpha)^{1/\alpha}$. Since $y = (I + \lambda A)x$ where $y = \{y_k\}$, $x = \{x_k\}$ and lower triangular matrices are associative, it follows that $(I + \lambda A)^{-1} y = x$. Since $y \in l_\alpha$ and $(I + \lambda A)^{-1} \in (l_\alpha, l_\alpha)$, it follows that $x \in l_\alpha$.

The proof of the theorem is now complete.

4. A Steinhaus type theorem for l_α.

Theorem 4.2 to follow is a Steinhaus type result proved using the characterization of (l_α, l_α) in Theorem 2.1. We write $A \in (l_\alpha, l_\alpha; P)$ if $A \in (l_\alpha, l_\alpha)$ and $\sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k$, $x = \{x_k\} \in l_\alpha$; $A \in (l_\alpha, l_\alpha; P)'$ if $A \in (l_\alpha, l_\alpha; P)$ and $a_{nk} \longrightarrow 0$, $k \longrightarrow \infty$, $n = 0,1,2,\ldots$. It is easy to check that $A \in (l_\alpha, l_\alpha; P)$ if and only if $A \in (l_\alpha, l_\alpha)$ and $\sum_{n=0}^{\infty} a_{nk} = 1$, $k = 0,1,2,\ldots$.

THEOREM 4.1. *If* $A \in (l_\alpha, l_\alpha)$ *such that* $a_{nk} \longrightarrow 0$, $k \longrightarrow \infty$,

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$n = 0,1,2,\ldots$ and $\overline{\lim_{k\to\infty}} \sum_{n=0}^{\infty} |a_{nk}|^{\alpha} > 0$, then there exists a sequence $x = \{x_k\} \in l_\beta$, $\beta > \alpha$, $Ax = \{(Ax)_n\} \notin l_\alpha$.

Proof. By hypothesis, for some $\varepsilon > 0$, there exists a subsequence $\{k(i)\}$ of positive integers such that

$$\sum_{n=0}^{\infty} |a_{n,k(i)}|^{\alpha} \geq 2\varepsilon\ ,\ i = 1,2,\ldots\ .$$

In particular,

$$\sum_{n=0}^{\infty} |a_{n,k(1)}|^{\alpha} \geq 2\varepsilon\ .$$

Choose a positive integer $n(1)$ such that

$$\sum_{n=n(1)+1}^{\infty} |a_{n,k(1)}|^{\alpha} < \min\ (2^{-1}, \tfrac{\varepsilon}{2})\ ,$$

so that

$$\sum_{n=0}^{n(1)} |a_{n,k(1)}|^{\alpha} > \varepsilon\ .$$

More generally, having chosen the positive integers $k(j)$, $n(j)$, $j \leq m-1$, choose a positive integer $k(m)$ such that $k(m) > k(m-1)$,

$$\sum_{n=0}^{\infty} |a_{n,k(m)}|^{\alpha} \geq 2\varepsilon\ ,$$

$$\sum_{n=0}^{n(m-1)} |a_{n,k(m)}|^{\alpha} < \min\ (2^{-m}\ , \tfrac{\varepsilon}{2})\ ,$$

and then choose a positive integer $n(m)$ such that $n(m) > n(m-1)$,

$$\sum_{n=n(m)+1}^{\infty} |a_{n,k(m)}|^{\alpha} < \min\ (2^{-m}, \tfrac{\varepsilon}{2})\ ,$$

so that

$$\sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^{\alpha} > 2\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \varepsilon.$$

Since K is non-trivially valued, there exists $\pi \in K$ such that $0 < \rho = |\pi| < 1$. For each $i = 1,2,\ldots,$ choose a non-negative integer $\lambda(i)$ such that

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$$\rho^{\lambda(i)+1} \le \frac{1}{i^{1/\alpha}} < \rho^{\lambda(i)} .$$

If the sequence $x = \{x_k\}$ is defined by

$$\left.\begin{aligned} x_k &= \pi^{\lambda(i)} , \quad k = k(i) \\ &= 0 \qquad , \quad k \neq k(i) \end{aligned}\right| , \quad i = 1,2,\ldots,$$

$\{x_k\} \in l_\beta \setminus l_\alpha$, for,

$$\sum_{k=0}^{\infty} |x_k|^\beta = \sum_{i=1}^{\infty} |x_{k(i)}|^\beta = \sum_{i=1}^{\infty} \rho^{\beta\lambda(i)}$$

$$\le \frac{1}{\rho^\beta} \sum_{i=1}^{\infty} \frac{1}{i^{\beta/\alpha}} < \infty ,$$

since $\beta > \alpha$, while,

$$\sum_{k=0}^{\infty} |x_k|^\alpha = \sum_{i=1}^{\infty} |x_{k(i)}|^\alpha = \sum_{i=1}^{\infty} \rho^{\alpha\lambda(i)} > \sum_{i=1}^{\infty} \frac{1}{i} = \infty .$$

Defining $n_o = 0$,

$$\sum_{n=0}^{n(N)} |(Ax)_n|^\alpha \ge \sum_{m=1}^{N} \sum_{n=n(m-1)+1}^{n(m)} \left| \sum_{i=1}^{\infty} a_{n,k(i)} x_{k(i)} \right|^\alpha$$

$$= \sum_{m=1}^{N} \sum_{n=n(m-1)+1}^{n(m)} \left| \sum_{i=1}^{\infty} a_{n,k(i)} \pi^{\lambda(i)} \right|^\alpha$$

$$\ge \sum_{m=1}^{N} \sum_{n=n(m-1)+1}^{n(m)} \left\{ |a_{n,k(m)}|^\alpha \rho^{\alpha\lambda(m)} - \sum_{i\neq m} |a_{n,k(i)}|^\alpha \rho^{\alpha\lambda(i)} \right\}$$

(using (2.2))

$$\ge \sum_{m=1}^{N} \sum_{n=n(m-1)+1}^{n(m)} \left\{ |a_{n,k(m)}|^\alpha m^{-1} - \frac{1}{\rho^\alpha} \sum_{i\neq m} |a_{n,k(i)}|^\alpha \right\}$$

$$> \sum_{m=1}^{N} \epsilon\, m^{-1} - \frac{1}{\rho^\alpha} \sum_{m=1}^{N} \sum_{n=n(m-1)+1}^{n(m)} \sum_{i\neq m} |a_{n,k(i)}|^\alpha . \tag{4.1}$$

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We note that

$$\sum_{m=1}^{\infty} \sum_{n=n(m-1)+1}^{n(m)} \sum_{i<m} |a_{n,k(i)}|^{\alpha} = \sum_{m=1}^{\infty} \sum_{n=n(m)+1}^{\infty} |a_{n,k(m)}|^{\alpha}$$

(4.2) $$\leq \sum_{m=1}^{\infty} 2^{-m} .$$

Similarly it can be shown that

(4.3) $$\sum_{m=1}^{\infty} \sum_{n=n(m-1)+1}^{n(m)} \sum_{i>m} |a_{n,k(i)}|^{\alpha} < \sum_{m=1}^{\infty} 2^{-(m+1)} .$$

Thus it follows from (4.1) to (4.3) that

$$\sum_{n=0}^{n(N)} |(Ax)_n|^{\alpha} > \varepsilon \sum_{m=1}^{N} \frac{1}{m} - \frac{3}{2} .$$

Since $\sum_{m=1}^{\infty} \frac{1}{m} = \infty$, it follows that $\{(Ax)_n\} \notin l_{\alpha}$.

THEOREM 4.2. $(l_{\alpha},l_{\alpha};P)' \cap (l_{\beta},l_{\alpha}) = \emptyset$, $\beta > \alpha$.

Proof. Suppose $A \in (l_{\alpha},l_{\alpha};P)' \cap (l_{\beta},l_{\alpha})$, $\beta > \alpha$. Then $\sum_{n=0}^{\infty} |a_{nk}|^{\alpha} \geq |\sum_{n=0}^{\infty} a_{nk}|^{\alpha} = 1$, $k = 0,1,2,\ldots$ so that $\overline{\lim_{k\to\infty}} \sum_{n=0}^{\infty} |a_{nk}|^{\alpha} \geq 1.$ By Theorem 4.1, there exists $x = \{x_k\} \in l_{\beta}$ such that $\{(Ax)_n\} \notin l_{\alpha}$, a contradiction.

Remark 4.1. When $K = \mathbb{R}$ or $\mathbb{C}$, it was proved by Fridy [3] that $(l_1,l_1;P) \cap (l_{\alpha},l_1) = \emptyset$, $\alpha > 1$. This result, as such, fails to hold when K is a complete, non-trivially valued, non-archimedean field, as the following example shows. Let $K = Q_3$ and $A = (a_{nk})$ where

$$a_{nk} = \frac{1}{4} (\frac{3}{4})^n , n,k = 0,1,2,\ldots .$$

$\sup_{k\geq 0} \sum_{n=0}^{\infty} |a_{nk}|_3 = \frac{1}{1-\rho} < \infty$ where $\rho = |3|_3$ and $\sum_{n=0}^{\infty} a_{nk} = 1$, $k = 0,1,2,\ldots,$ so that $A \in (l_1,l_1;P)$; but, for $\alpha > 1$, if $x = \{x_k\} \in l_{\alpha}$,

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$$\sum_{n=0}^{\infty} |(Ax)_n|_3 = \sum_{n=0}^{\infty} |\sum_{k=0}^{\infty} a_{nk}x_k|_3$$

$$= \sum_{n=0}^{\infty} |\tfrac{1}{4} (\tfrac{3}{4})^n|_3 \; |\sum_{k=0}^{\infty} x_k|_3$$

$$= |\sum_{k=0}^{\infty} x_k|_3 \frac{1}{1-\rho} < \infty \text{ (since } \{x_k\} \in l_\alpha,$$

$x_k \longrightarrow 0$, $k \longrightarrow \infty$ and so $\sum_{k=0}^{\infty} x_k$ converges as $K = Q_3$ is complete).

Thus $A \in (l_\alpha, l_1)$ also.

Remark 4.2. $(l_\alpha, l_\alpha; P) \cap (l_\infty, l_\alpha) = \emptyset$ where l_∞ is the space of all bounded sequences with entries in K. For, if $A \in (l_\alpha, l_\alpha; P) \cap (l_\infty, l_\alpha)$, then $A \in (l_\alpha, l_\alpha; P)' \cap (l_\beta, l_\alpha)$, $\beta > \alpha$, a contradiction.

Remark 4.3. Virtually by following the proof of Theorem 4.1, we can show that given any matrix $A \in (l_\alpha, l_\alpha; P)$, there exists a sequence of 0's and 1's whose A-transform is not in l_α. This is analogous to Schur's version of the well-known Steinhaus theorem for regular matrices (see [10], when $K = \mathbb{R}$ or $\mathbb{C}$ and [8] when K is a complete, non-trivially valued, non-archimedean field).

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Department of Mathematics,
Vivekananda College,
Madras - 600 004
India.