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# Separation of Variables for $U_q(\mathfrak{sl}_{n+1})^+$

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*Abstract.* Let $U_q(\mathfrak{sl}_{n+1})^+$ be the positive part of the quantized enveloping algebra $U_q(\mathfrak{sl}_{n+1})$. Using results of Alev–Dumas and Caldero related to the center of $U_q(\mathfrak{sl}_{n+1})^+$, we show that this algebra is free over its center. This is reminiscent of Kostant's separation of variables for the enveloping algebra $U(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$, and also of an analogous result of Joseph–Letzter for the quantum algebra $\check{U}_q(\mathfrak{g})$. Of greater importance to its representation theory is the fact that $U_q(\mathfrak{sl}_{n+1})^+$ is free over a larger polynomial subalgebra $N$ in $n$ variables. Induction from $N$ to $U_q(\mathfrak{sl}_{n+1})^+$ provides infinite-dimensional modules with good properties, including a grading that is inherited by submodules.

## 1 Introduction

We work over a field $\mathbb{K}$ of characteristic 0 and assume $q \in \mathbb{K}^\times$ is not a root of unity. In this paper we show that the algebra $U_q(\mathfrak{sl}_{n+1})^+$, the quantized version of the enveloping algebra of the nilpotent Lie algebra of strictly upper triangular $(n+1) \times (n+1)$ matrices, is free when viewed as a module over its center. This has consequences for the representation theory of $U_q(\mathfrak{sl}_{n+1})^+$, one of which being the existence of simple modules with arbitrary central character. In fact, we show first that $U_q(\mathfrak{sl}_{n+1})^+$ is free over a polynomial subalgebra $N$ in variables $\Delta_1, \dots, \Delta_n$ that commute with the Chevalley generators $e_1, \dots, e_n$ up to a power of the parameter $q$.

Our motivation is the study of infinite-dimensional $U_q(\mathfrak{sl}_{n+1})^+$-modules. We use the latter result to construct modules by inducing from one-dimensional $N$-modules. Given an $N$-character $\chi \in \widehat{N} = \mathrm{Alg}(N, \mathbb{K})$ with corresponding simple module $V_\chi = \mathbb{K}v_\chi$, the induced $U_q(\mathfrak{sl}_{n+1})^+$-module $M_\chi = U_q(\mathfrak{sl}_{n+1})^+ \otimes_N V_\chi$ has a *weight space* decomposition with respect to $N$,

$$M_\chi = \bigoplus_{\eta \in \widehat{N}} M_\chi^{(\eta)},$$

where $M_\chi^{(\eta)} = \{m \in M_\chi \mid x.m = \eta(x)m \text{ for all } x \in N\}$, and it is easy to see that every subquotient of $M_\chi$ inherits this grading.

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For the case $n = 2$, the algebra $U_q(\mathfrak{sl}_3)^+$ is isomorphic to the down-up algebra $A(q+q^{-1},-1,0)$ with generators $d$, $u$ and defining relations

$$d^2u - (q+q^{-1})dud + ud^2 = 0,$$
$$du^2 - (q+q^{-1})udu + u^2d = 0.$$

In this case, the polynomial algebra $N$ is just $\mathbb{K}[du, ud]$, and the modules we discuss are universal amongst cyclic weight modules for the down-up algebra $A(q+q^{-1},-1,0)$. The case $n = 3$ is more intricate, but we obtain two distinct two-parameter families of representations.

We begin with the basic definitions, including the description of a PBW (Poincaré–Birkhoff–Witt) basis and a filtration for which the associated graded algebra is a *quantum affine space*. After briefly reviewing results of Caldero [5, 6] and of Alev–Dumas [1] on the center $Z$ of $U_q(\mathfrak{sl}_{n+1})^+$, we show that $U_q(\mathfrak{sl}_{n+1})^+$ is free over $N$ and also over $Z$, by working in the graded algebra first. We can then exploit this result to develop the representation theory of $U_q(\mathfrak{sl}_{n+1})^+$.

The techniques of [7] can be used instead to show the freeness of $U_q(\mathfrak{sl}_{n+1})^+$ over its center. Our approach is perhaps more pedestrian. But the same methods as we use here apply to the enveloping algebra of the Lie algebra $\mathfrak{sl}_{n+1}^+$, using Dixmier's description of the center in [9]. We therefore see that $U(\mathfrak{sl}_{n+1}^+)$ is also free over its center, a result that suggests that the class of algebras for which the separation of variables is true goes well beyond the universal enveloping algebras of the finite-dimensional complex semisimple Lie algebras and their quantum analogues. Further evidence of this comes from the theory of down-up algebras, which are known to behave similarly to enveloping algebras. In [2], the authors prove separation and annihilation theorems for the down-up algebra $A(\alpha,\beta,\gamma)$ for all choices of parameters $\alpha,\beta,\gamma$. See also the remarks at the end of Section 5.

## 2 Definitions and Notation

**2.1** Let $\mathbb{K}$ be a field of characteristic 0 and assume $q \in \mathbb{K}^\times$ is not a root of unity. The algebra we are concerned with is the unital, associative $\mathbb{K}$-algebra having generators $e_1, \dots, e_n$, which satisfy the relations

$$e_ie_j - e_je_i = 0 \qquad \text{if } |i-j| \neq 1, \tag{1}$$

$$e_i^2e_j - (q+q^{-1})e_ie_je_i + e_je_i^2 = 0 \qquad \text{if } |i-j| = 1. \tag{2}$$

We will denote this algebra by $U_q(\mathfrak{sl}_{n+1})^+$; it is the positive part of the quantized enveloping algebra $U_q(\mathfrak{sl}_{n+1})$ with respect to the usual triangular decomposition (see [8, 10, 12, 13], for example).

**2.2** Let $\mathfrak{sl}_{n+1}$ be the Lie algebra of traceless $(n+1)\times(n+1)$ matrices over the complex field $\mathbb{C}$; $R$ the set of roots with respect to a Cartan subalgebra $\mathfrak{h}$; $\alpha_1, \dots, \alpha_n$ a base

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of $R$; $\varpi_1, \dots, \varpi_n$ the fundamental weights; $Q = \bigoplus_{k=1}^n \mathbb{Z}\alpha_k$ the root lattice; $Q^+ = \bigoplus_{k=1}^n \mathbb{N}\alpha_k$ the positive root lattice; $P = \bigoplus_{k=1}^n \mathbb{Z}\varpi_k$ the weight lattice; and $R^+ = R \cap Q^+$ the set of positive roots. There is a nondegenerate bilinear form on $Q \times Q$ given by $(\alpha_i, \alpha_j) = 2\delta_{i,j} - \delta_{i,j\pm 1}$ for all $i, j = 1, \dots, n$.

The algebra $U_q(\mathfrak{sl}_{n+1})^+$ can be graded by the positive root lattice $Q^+$ by assigning to $e_i$ the degree $\alpha_i$, as the defining relations are homogeneous. We use the terminology *weight* instead of degree for this gradation and write $\mathrm{wt}(u) = \beta$ if $u \in U_q(\mathfrak{sl}_{n+1})^+$ has weight $\beta \in Q^+$.

## 3 PBW Basis and a Filtration

Many authors have studied PBW-bases of $U_q(\mathfrak{sl}_{n+1})^+$ (*e.g.*, [16–19]); here we follow Ringel [17]. The filtration in 3.2 below is similar to the one in [8] and yields the same graded algebra.

**3.1** For each $1 \le i < j \le n+1$, we can define weight elements $X_{ij}$ recursively by setting $X_{i,i+1} = e_i$ for all $i \in \{1, \dots, n\}$ and $X_{ij} = X_{ik}X_{kj} - q^{-1}X_{kj}X_{ik}$ for $1 \le i < k < j \le n+1$. It can be shown that this definition does not depend on $k$ (see [17, App. 2]). These elements correspond bijectively to the positive roots of $\mathfrak{sl}_{n+1}$, as $\mathrm{wt}(X_{ij}) = \alpha_i + \dots + \alpha_{j-1}$ for all $i < j$. The set $\{X_{ij}\}_{1 \le i < j \le n+1}$ can be linearly ordered using the rule

$$X_{ij} < X_{kl} \quad \Leftrightarrow \quad (k < i) \quad \text{or} \quad (k = i \quad \text{and} \quad l < j).$$

We use the alternative notation $X_k$ for the $k$-th element in this increasing chain, so that $\{X_{ij}\}_{1 \le i < j \le n+1} = \{X_k\}_{1 \le k \le m}$, where $m = |R^+| = \frac{1}{2}n(n+1)$.

Let $\mathbf{b} \in \mathbb{N}^m$ and write $X^{\mathbf{b}} := X_1^{b_1} \cdots X_m^{b_m}$. By [17, Thm. 2], the monomials $X^{\mathbf{b}}$ ($\mathbf{b} \in \mathbb{N}^m$) form a basis of $U_q(\mathfrak{sl}_{n+1})^+$. Furthermore, for all $i < j$ we have

$$X_j X_i = q^{(\mathrm{wt}(X_i), \mathrm{wt}(X_j))} X_i X_j + \sum c_{a_{i+1}, \dots, a_{j-1}} X_{i+1}^{a_{i+1}} \cdots X_{j-1}^{a_{j-1}}, \tag{3}$$

where $c_{a_{i+1}, \dots, a_{j-1}} \in \mathbb{K}$, and the sum is over all sequences $(a_{i+1}, \dots, a_{j-1})$ of natural numbers such that the homogeneity of (3) is preserved.

**3.2** We order $\mathbb{N}^m$ by setting $\mathbf{b} < \mathbf{c} \Leftrightarrow$ there is $l \in \{1, \dots, m\}$ such that $b_l < c_l$ and $b_t = c_t$ for all $t > l$. Naturally, $\mathbf{b} \le \mathbf{c}$ means $\mathbf{b} < \mathbf{c}$ or $\mathbf{b} = \mathbf{c}$. This is easily seen to be a well-ordered relation on $\mathbb{N}^m$. Define

$$U_q^+(\mathbf{a}) = \bigoplus_{\mathbf{b} \le \mathbf{a}} \mathbb{K}X^{\mathbf{b}} \quad \text{and} \quad U_q^+(< \mathbf{a}) = \bigcup_{\mathbf{b} < \mathbf{a}} U_q^+(\mathbf{b}).$$

The family $\{U_q^+(\mathbf{a})\}_{\mathbf{a} \in \mathbb{N}^m}$ is an increasing filtration of $U_q(\mathfrak{sl}_{n+1})^+$ by $\mathbb{N}^m$ with respect to the order defined above. In particular, $U_q^+(\mathbf{b}) \subseteq U_q^+(\mathbf{a})$ if $\mathbf{b} \le \mathbf{a}$,

$$\bigcup_{\mathbf{a} \in \mathbb{N}^m} U_q^+(\mathbf{a}) = U_q(\mathfrak{sl}_{n+1})^+ \quad \text{and} \quad U_q^+(\mathbf{a}) \cdot U_q^+(\mathbf{b}) \subseteq U_q^+(\mathbf{a} + \mathbf{b}).$$

The latter property is essentially a consequence of (3).

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**3.3** By 3.2 we can define the associated graded algebra as

$$S \stackrel{\text{def}}{=} gr\left(U_q(\mathfrak{sl}_{n+1})^+\right) = \bigoplus_{\mathbf{a}\in\mathbb{N}^m} U_q^+(\mathbf{a})/U_q^+(<\mathbf{a}), \quad \left(U_q^+(<\mathbf{0}) = (0)\right),$$

where multiplication is defined by linearity in the following way: Given $u \in U_q^+(\mathbf{a}) \setminus U_q^+(<\mathbf{a})$, we say $u$ has degree $\mathbf{a}$ (by convention, $\deg(0) = (-\infty,\dots,-\infty)$). Write $gr(u) = u + U_q^+(<\mathbf{a})$. If $v \in U_q^+(\mathbf{b}) \setminus U_q^+(<\mathbf{b})$, then

$$gr(u)\cdot gr(v) = uv + U_q^+(<(\mathbf{a}+\mathbf{b})).$$

This is well defined by 3.2, and we have the relations

$$gr(X_j)gr(X_i) = q^{(\mathrm{wt}(X_i),\mathrm{wt}(X_j))}gr(X_i)gr(X_j) \quad \text{if } i<j.$$

Therefore $\deg(uv) = \deg(u) + \deg(v)$, and the associated graded algebra $S$ is an integral domain. Also, $gr(u)gr(v) = gr(uv)$. In fact, $S$ is the *quantum affine space* given by generators $\theta_1,\dots,\theta_m$ and relations $\theta_j\theta_i = t_{ij}\theta_i\theta_j$, where $\theta_i = gr(X_i)$, and

$$t_{ij} = \begin{cases} q^{(\mathrm{wt}(X_i),\mathrm{wt}(X_j))} & \text{if } i<j,\\ 1 & \text{if } i=j,\\ t_{ji}^{-1} & \text{if } j<i. \end{cases} \tag{4}$$

## 4 Central and $q$-Central Elements of $U_q(\mathfrak{sl}_{n+1})^+$

Alev and Dumas [1] as well as Caldero [4,5] have determined the center of $U_q(\mathfrak{sl}_{n+1})^+$. According to their work, there exist algebraically independent elements $\Delta_1,\dots,\Delta_n$ of $U_q(\mathfrak{sl}_{n+1})^+$ that commute with the generators $e_1,\dots,e_n$ up to a power of $q$. They generate a (commutative) polynomial subalgebra that contains the center. We summarize results of [5] regarding the $\Delta_i$, and then determine $gr(\Delta_i)$ $(1 \le i \le n)$ explicitly in the graded algebra $S$ of 3.3.

**4.1** Consider the matrix

$$\mathcal{X} = \begin{pmatrix} \xi & X_{1,2} & X_{1,3} & \cdots & X_{1,n+1}\\ & \xi & X_{2,3} & \cdots & X_{2,n+1}\\ & & \ddots & \vdots & \vdots\\ & 0 & & \xi & X_{n,n+1}\\ & & & & \xi \end{pmatrix}$$

with $\xi = q(q-q^{-1})^{-1}$. For every $i = 1,\dots,n$, define $\Delta_i = \mathrm{Det}_q(\mathcal{X}_i)$, where $\mathcal{X}_i$ is the $i\times i$ matrix obtained from the top $i$ rows and rightmost $i$ columns of $\mathcal{X}$, and $\mathrm{Det}_q$ is a *quantum determinant* that associates to any matrix $M = (m_{kl})_{1\le k,l\le p}$ with entries in a $\mathbb{K}$-algebra $C$ the element

$$\mathrm{Det}_q M = \sum_{\sigma\in\Sigma_p} (-q^{-1})^{l(\sigma)} m_{\sigma(p),p}\cdots m_{\sigma(1),1}, \tag{5}$$

$l(\sigma)$ being the length of the permutation $\sigma$ in the symmetric group $\Sigma_p$.

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**4.2** Let $\check{U}_q^0$ be the group algebra of the weight lattice $P$. Then $\check{U}_q^0$ is the algebra of Laurent polynomials $\mathbb{K}[K_{\varpi_1}^{\pm}, \dots, K_{\varpi_n}^{\pm}]$, where each $K_{\varpi_i}$ corresponds to the fundamental weight $\varpi_i$. The "positive Borel" $\check{U}_q(\mathfrak{sl}_{n+1})^{\geqslant 0}$ is defined so that $U_q(\mathfrak{sl}_{n+1})^+$ and $\check{U}_q^0$ are subalgebras and $\check{U}_q(\mathfrak{sl}_{n+1})^{\geqslant 0} \simeq U_q(\mathfrak{sl}_{n+1})^+ \otimes_{\mathbb{K}} \check{U}_q^0$ as a vector space, with the additional relations:

$$K_{\varpi_i} e_j K_{\varpi_i}^{-1} = q^{\delta_{ij}} e_j, \text{ for all } 1 \leq i, j \leq n. \tag{6}$$

There exists a Hopf algebra structure on $\check{U}_q(\mathfrak{sl}_{n+1})^{\geqslant 0}$, endowing this algebra with a (left) adjoint action denoted by ad. For each $1 \leq i \leq n$ let $L_q(\varpi_i)$ be the finite-dimensional simple module of highest weight $\varpi_i$ for the quantized enveloping algebra $U_q(\mathfrak{sl}_{n+1})$ (see [12], for example). The submodule ad $U_q(\mathfrak{sl}_{n+1})^+(K_{\varpi_i}^{-2})$ of $\check{U}_q(\mathfrak{sl}_{n+1})^{\geqslant 0}$ is isomorphic to $L_q(\varpi_i)$ as a $U_q(\mathfrak{sl}_{n+1})^+$-module [5, 6, 14], and the element $e_{s(\varpi_i)} \in U_q(\mathfrak{sl}_{n+1})^+$ is defined in [5,6] so that $K_{\varpi_i}^{-2} e_{s(\varpi_i)}$ corresponds to a highest weight vector of $L_q(\varpi_i)$ under that isomorphism. In other words, ad $e_j(K_{\varpi_i}^{-2} e_{s(\varpi_i)}) = 0$ for all $1 \leq i, j \leq n$.

**4.3** The following theorem describes the center of $U_q(\mathfrak{sl}_{n+1})^+$ and the nature of the $\Delta_i$, $1 \leq i \leq n$. Part (c) is the quantum analogue of [9, Thm. 1].

***Theorem 1*** ( [5,6]) *For $1 \leq i, j \leq n$, the following hold:*

(a) $e_i \Delta_j = q^{\delta_{ij} - \delta_{i,n+1-j}} \Delta_j e_i$.

(b) *The subalgebra $N$ of $U_q(\mathfrak{sl}_{n+1})^+$ generated by $\Delta_1, \dots, \Delta_n$ is a polynomial algebra $\mathbb{K}[\Delta_1, \dots, \Delta_n]$ in $n$ variables.*

(c) *The center $Z$ of $U_q(\mathfrak{sl}_{n+1})^+$ is the polynomial algebra in the variables $\{\Delta_k \Delta_{n+1-k} \mid 1 \leq k \leq n/2\}$ if $n$ is even and $\{\Delta_k \Delta_{n+1-k} \mid 1 \leq k \leq (n-1)/2\} \cup \{\Delta_{(n+1)/2}\}$ if $n$ is odd.*

**Proof** Let $\zeta \colon U_q(\mathfrak{sl}_{n+1})^+ \to U_q(\mathfrak{sl}_{n+1})^+$ be the antiautomorphism with $\zeta(e_i) = e_i$ for all $i$. Using [5, Thm. 4.1], it is not hard to see that $e_{s(\varpi_i)} = \zeta(\Delta_i)$ for all $1 \leq i \leq n$. Then, part (a) follows from the proof of [5, Thm. 3.2], part (c) from [5, Thm. 4.1] and part (b) from [5, Prop. 3.2] and [6, Rem. 2.2]. ■

In the case of the algebra $U_q(\mathfrak{sl}_3)^+$, for example, $\Delta_1 = X_{1,3} = e_1 e_2 - q^{-1} e_2 e_1$ and $\Delta_2 = X_{2,3} X_{1,2} - q^{-1} X_{13} \xi = \xi(e_2 e_1 - q^{-1} e_1 e_2)$. Hence the center of $U_q(\mathfrak{sl}_3)^+$ is the polynomial subalgebra $\mathbb{K}[z]$, where $z = \Delta_1 \Delta_2$.

The $\Delta_i$ are said to be $q$-central, because they commute with the Chevalley generators of $U_q(\mathfrak{sl}_{n+1})^+$, up to a power of $q$. The set of $q$-central elements is a proper subset of $N$ which is closed under multiplication, but is not a subspace. For example, $\Delta_1 + \Delta_n$ is not $q$-central. See [6, Thm. 2.2] for details.

**4.4** It is easy to see that the term of highest order of $\Delta_i$, $1 \leq i \leq n$, when expressed in terms of the PBW-basis of 3.1 is obtained by taking the identity permutation in (5).

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Therefore,

$$\Delta_i = X_{i,n+1}X_{i-1,n}\cdots X_{2,n+3-i}X_{1,n+2-i} + \text{(lower order terms)}$$

and consequently, in $S = gr\left(U_q(\mathfrak{sl}_{n+1})^+\right)$,

$$gr(\Delta_i) = gr(X_{i,n+1})\,gr(X_{i-1,n})\cdots gr(X_{2,n+3-i})\,gr(X_{1,n+2-i}). \tag{7}$$

Hence, each of the elements $gr(X_{i,j})$, $1 \le i < j \le n+1$, occurs exactly once in precisely one of the monomials $gr(\Delta_k)$, $1 \le k \le n$.

## 5 $U_q(\mathfrak{sl}_{n+1})^+$ as a Module Over Its Center

Recall that the algebraically independent elements $\Delta_1, \dots, \Delta_n$ generate a polynomial algebra denoted by $N$. We show that $U_q(\mathfrak{sl}_{n+1})^+$ is free as a module over $N$, acting via (right or left) multiplication, and as a consequence, we see that it is also free over its center, $Z$. When we write $A \cong_{\mathbb{K}} B \otimes_{\mathbb{K}} C$ for a $\mathbb{K}$-algebra $A$, we mean that $B$ and $C$ are subspaces of $A$ and that the map $\mathfrak{m}\colon B \otimes_{\mathbb{K}} C \to A$ that sends $b \otimes c$ to $bc$ is a vector space isomorphism.

**5.1** Let $T = (t_{ij})_{1 \le i,j \le r}$ be a matrix with nonzero scalar entries satisfying $t_{ii} = 1$ and $t_{ij} = t_{ji}^{-1}$ for all $i, j$. The *quantum affine space* associated with $T$ is the unital, associative $\mathbb{K}$-algebra with generators $z_1, \dots, z_r$, and relations $z_j z_i = t_{ij} z_i z_j$ for all $i, j$. We denote it by $\mathbb{K}_T[z_1, \dots, z_r]$. The subalgebra generated by the monomial $z_1 \cdots z_r$ is a polynomial algebra in one variable that we naturally denote by $\mathbb{K}[z_1 \cdots z_r]$. The following technical lemma is straightforward to prove:

***Lemma 1*** *$\mathbb{K}_T[z_1, \dots, z_r]$ is free over $\mathbb{K}[z_1 \cdots z_r]$ (acting by multiplication). Indeed, there is a set of linearly independent monomials $B_r \subseteq \mathbb{K}_T[z_1, \dots, z_r]$ such that if $H_r$ is the vector space spanned by $B_r$, then*

$$\mathbb{K}_T[z_1, \dots, z_r] \cong_{\mathbb{K}} H_r \otimes_{\mathbb{K}} \mathbb{K}[z_1 \cdots z_r].$$

*The set $B_r$ can be defined recursively (and independently of $T$) by*

$$B_r = B_{r-1} \cdot \left(\{(z_1 \cdots z_{r-1})^a \mid a \in \mathbb{N}\} \dot\cup \{z_r^c \mid c \in \mathbb{N} \setminus \{0\}\}\right), \quad B_1 = \{1\}.$$

**5.2** Let $S$ be the graded algebra introduced in 3.3. As noted earlier, it is the quantum affine space $\mathbb{K}_T[\theta_1, \dots, \theta_m]$ where $\theta_i = gr(X_i)$ and $t_{ij}$ is given by (4). As in 3.1, we also use the notation $\theta_{ij} = gr(X_{ij})$. For each $1 \le i \le n$, let $S_i$ be the subalgebra of $S$ generated by $\{\theta_{k,k+n+1-i} \mid 1 \le k \le i\}$. Set

$$y_i := gr(\Delta_i) = \theta_{i,n+1} \cdots \theta_{1,n+2-i} \in S_i$$

and $J_i = \mathbb{K}[y_i] \subseteq S_i$. Denote by $J$ the subalgebra of $S$ generated by $y_1, \dots, y_n$. Since $y_i = gr(\Delta_i)$ for all $1 \le i \le n$, by (7), we conclude that the $y_j$ commute with

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each other, and hence that $J$ is the polynomial algebra in the variables $y_1, \dots, y_n$. Therefore,

$$S \cong_{\mathbb{K}} S_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} S_n, \quad \text{and} \quad J \cong_{\mathbb{K}} J_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} J_n.$$

It is clear that $S_i$ is the quantum affine space $\mathbb{K}_{T_i}[\theta_{1,n+2-i}, \dots, \theta_{i,n+1}]$, $T_i$ being obtained from $T$ in the obvious way. Thus $S_i \cong_{\mathbb{K}} H_i \otimes_{\mathbb{K}} J_i$ by Lemma 1, where $H_i$ is the linear span of the monomial basis given in this lemma. Since the spaces $H_j$ are homogeneous, (in the sense that they have a basis consisting of certain monomials in the variables $\theta_k$) it follows that $J_i \otimes_{\mathbb{K}} H_j \cong_{\mathbb{K}} H_j \otimes_{\mathbb{K}} J_i$ for all $i, j$ and so

$$\begin{aligned} S &\cong_{\mathbb{K}} (H_1 \otimes_{\mathbb{K}} J_1) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} (H_n \otimes_{\mathbb{K}} J_n) \\ &\cong_{\mathbb{K}} (H_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} H_n) \otimes_{\mathbb{K}} (J_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} J_n) \\ &\cong_{\mathbb{K}} H \otimes_{\mathbb{K}} J \end{aligned} \tag{8}$$

with $H = H_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} H_n$. This shows that $S$ is free over $J$: if $\mathcal{B}$ is a $\mathbb{K}$-basis for $H$, then $S \cong \bigoplus_{b \in \mathcal{B}} bJ$ as (right) $J$-modules.

**5.3** Consider the linear isomorphism $\beta \colon U_q(\mathfrak{sl}_{n+1})^+ \to S$ defined by

$$\sum_{\mathbf{a} \in \mathbb{N}^m} c_{\mathbf{a}} X^{\mathbf{a}} \mapsto \sum_{\mathbf{a} \in \mathbb{N}^m} c_{\mathbf{a}} \theta^{\mathbf{a}},$$

and let $\mathcal{K} = \beta^{-1}(H)$.

***Proposition 1*** *$U_q(\mathfrak{sl}_{n+1})^+$ is free over the polynomial algebra $N$. Specifically,*

$$U_q(\mathfrak{sl}_{n+1})^+ \cong_{\mathbb{K}} \mathcal{K} \otimes_{\mathbb{K}} N.$$

**Proof** Let $\psi : \mathcal{K} \otimes_{\mathbb{K}} N \longrightarrow U_q^+$ be the multiplication map.

*$\psi$ is surjective*: We will show that $X^{\mathbf{a}} \in \operatorname{Im} \psi$ by induction on $\mathbf{a} \in \mathbb{N}^m$. If $\mathbf{a} = (0, \dots, 0)$, then $1 = X^{\mathbf{a}} \in \psi(\mathcal{K} \otimes_{\mathbb{K}} N)$, as $1 \in \mathcal{K}$. Suppose the result is true for all $\mathbf{d} < \mathbf{a}$. By (8), $gr(X^{\mathbf{a}}) = \theta^{\mathbf{a}} = \sum_{i=1}^{k} h_i p_i$ with $h_i \in H$ and $p_i = p_i(y_1, \dots, y_n) \in J$. It can be assumed that the $h_i$ are monomials in the $\theta_j$, and the $p_i$ are monomials in the $y_j$ (and hence in the $\theta_i$ also) up to a nonzero scalar multiple. Since $\theta^{\mathbf{a}}$ is itself a monomial, we can further assume $k = 1$ and $\theta^{\mathbf{a}} = hp$, say $h = \theta^{\mathbf{b}}$ and $p = \lambda y^{\mathbf{c}}$. Notice that $X^{\mathbf{b}} = \beta^{-1}(h) \in \mathcal{K}$ and $gr\, \psi(X^{\mathbf{b}} \otimes \lambda \Delta^{\mathbf{c}}) = gr(X^{\mathbf{a}})$. Therefore $X^{\mathbf{a}} - \psi(X^{\mathbf{b}} \otimes \lambda \Delta^{\mathbf{c}}) \in U_q^+(\mathbf{d})$ for some $\mathbf{d} < \mathbf{a}$, and the induction hypothesis implies that $X^{\mathbf{a}} \in \operatorname{Im} \psi$.

*$\psi$ is injective*: Suppose $\beta^{-1}(h_1)p_1 + \cdots + \beta^{-1}(h_k)p_k = 0$ with $h_i \in H$ and $p_i \in N$. We can assume the $h_i$ are (distinct) monomials in the $\theta_j$ and that the elements $\beta^{-1}(h_i)p_i$ all have the same degree, say $\mathbf{d} \in \mathbb{N}^m$. Then we have

$$\begin{aligned} 0 &= gr\left(\beta^{-1}(h_1)p_1 + \cdots + \beta^{-1}(h_k)p_k\right) \\ &= h_1 gr(p_1) + \cdots + h_k gr(p_k). \end{aligned} \tag{9}$$

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Since the $h_i \in H$ are linearly independent over $\mathbb{K}$ and $gr(p_i) \in J$, equations (8) and (9) force $gr(p_i) = 0$ for all $1 \le i \le k$, and hence $p_1 = \cdots = p_k = 0$.

Therefore $\psi$ is a linear isomorphism and the proposition is proved. ■

This brings us to an analogue of Kostant's separation of variables [15] (see also [14] for a version for $\check{U}_q(\mathfrak{g})$, $\mathfrak{g}$ semisimple). Since the center $Z$ of $U_q(\mathfrak{sl}_{n+1})^+$ is a polynomial algebra in the variables $\Delta_1\Delta_n$, $\Delta_2\Delta_{n-1}, \ldots$ (see Theorem 1(c)), we see that $N$ is free over $Z$. Combining this with Proposition 1 yields the following separation theorem for $U_q(\mathfrak{sl}_{n+1})^+$:

***Theorem 2*** *$U_q(\mathfrak{sl}_{n+1})^+$ is free over its center.*

***Remarks*** 1. Recently, Futorny and Ovsienko [11] have proved a similar result for what they call *special PBW algebras* over algebraically closed fields of characteristic 0. These are algebras $R$ with a PBW-type basis and with an increasing filtration over $\mathbb{N}$, such that the associated graded algebra is a (commutative) polynomial ring. Their hypothesis is that there are mutually commuting *regular* elements $x_1, \ldots, x_t$, that generate a polynomial subalgebra $\Gamma \subseteq R$. They prove that $R$ is free as a left or right $\Gamma$-module. A major difference between their work and ours is that our associated graded algebra is not commutative, and $\mathbb{K}$ is not assumed to be algebraically closed. Consequently, the algebraic geometry methods of [11] do not apply here.

2. $U_q(\mathfrak{sl}_{n+1})^+$ is not finite over $Z$, as the proof shows and as is also apparent from the fact that there are infinite-dimensional simple modules.

## 6 Applications to Representations

**6.1** As before, $Z$ denotes the center and $N = \mathbb{K}[\Delta_1, \ldots, \Delta_n]$. If $\mathbb{K}$ is algebraically closed, the irreducible $N$-modules are parametrized by the characters of $N$, *i.e.,* algebra homomorphisms in $\mathrm{Alg}(N, \mathbb{K})$, which in turn can be identified with the elements of $\mathbb{K}^n$. Following this idea, we think of $\chi = (\chi_1, \ldots, \chi_n) \in \mathbb{K}^n$ as the character $N \longrightarrow \mathbb{K}$, $\Delta_i \mapsto \chi_i$.

Let $V_\chi = \mathbb{K}v_\chi$ be the simple $N$-module corresponding to $\chi$, and define the induced $U_q(\mathfrak{sl}_{n+1})^+$-module $M_\chi = U_q(\mathfrak{sl}_{n+1})^+ \otimes_N V_\chi$. By Proposition 1,

$$M_\chi = \mathcal{K} \otimes_{\mathbb{K}} V_\chi = \bigoplus_{\eta \in \mathbb{K}^n} M_\chi^{(\eta)}$$

as vector spaces, where each $M_\chi^{(\eta)}$ is a semisimple $N$-module with simple summands isomorphic to $V_\eta$. The space $M_\chi^{(\chi)}$ is nonzero and generates $M_\chi$ as a $U_q(\mathfrak{sl}_{n+1})^+$-module. Any maximal submodule of $M_\chi$ inherits this grading by $\mathbb{K}^n$, and the corresponding factor module is an irreducible $U_q(\mathfrak{sl}_{n+1})^+$-module, which is semisimple as an $N$-module and has a common eigenvector for $N$ with eigenvalue $\chi$.

Thus, we see that any character $\chi$ of $N$ can be "lifted" to a simple $U_q(\mathfrak{sl}_{n+1})^+$-module $L = \bigoplus_{\eta \in \mathbb{K}^n} L^{(\eta)}$, with $L^{(\chi)} \neq (0)$ and $L^{(\eta)}$ a direct sum of copies of the simple $N$-module $V_\eta$, for all $\eta \in \mathbb{K}^n$. An analogous statement is true if we use $Z$ instead of $N$, but in such a case, $L = L^{(\theta)}$ for $\theta$ a given character of $Z$.

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**6.2** Throughout this subsection we consider the algebra $U_q(\mathfrak{sl}_3)^+$, so that $n = 2$. We will construct a family of modules for $U_q(\mathfrak{sl}_3)^+$, each universal with respect to the property that they are generated by a common eigenvector for the $q$-central elements $\Delta_1$ and $\Delta_2$ with a given eigenvalue. These turn out to be closely related to the weight modules for the down-up algebra $A(q+q^{-1}, -1, 0)$, defined in [3].

The generators of the PBW basis of $U_q(\mathfrak{sl}_3)^+$ described in 3.1 are:

$$X_1 = e_2, \quad X_2 = e_1e_2 - q^{-1}e_2e_1, \quad X_3 = e_1,$$

and the $q$-central elements $\Delta_1$ and $\Delta_2$ can be taken to be

$$\Delta_1 = X_2 \quad \text{and} \quad \Delta_2 = e_1e_2 - qe_2e_1.$$

A basis for $U_q(\mathfrak{sl}_3)^+$ over $\mathbb{K}[\Delta_1, \Delta_2]$ is $B = \{X_1^a \mid a \geq 1\} \cup \{X_3^b \mid b \geq 0\}$. Let $(\alpha, \beta) \in \mathbb{K}^2$ be a character of $\mathbb{K}[\Delta_1, \Delta_2]$. The induced module $M_{(\alpha,\beta)} = U_q(\mathfrak{sl}_3)^+ \otimes_N V_{(\alpha,\beta)}$ has a $\mathbb{K}$-basis indexed by $B$. Computing in $U_q(\mathfrak{sl}_3)^+$, we see that $M_{(\alpha,\beta)}$ is the $U_q(\mathfrak{sl}_3)^+$-module $\mathbb{K}[x^{\pm 1}]$ with action:

$$e_1.x^a = \begin{cases} {}_\alpha[a]_\beta\, x^{a-1} & \text{if } a \geq 1, \\ x^{a-1} & \text{if } a \leq 0, \end{cases}$$

$$e_2.x^a = \begin{cases} x^{a+1} & \text{if } a \geq 0, \\ {}_\alpha[a+1]_\beta\, x^{a+1} & \text{if } a \leq -1, \end{cases}$$

where we have identified $x^a$ with $X_1^a$ if $a \geq 1$ and with $X_3^{-a}$ if $a \leq 0$. The quantity ${}_\lambda[k]_\mu$, with $\lambda, \mu \in \mathbb{K}$ and $k \in \mathbb{Z}$ is given by

$$(10) \qquad {}_\lambda[k]_\mu = \frac{\lambda q^k - \mu q^{-k}}{q - q^{-1}}.$$

In the particular case where $\lambda = 1 = \mu$ we recover the $q$-integer $[k] = {}_1[k]_1$.

Notice that

$$\Delta_1.x^a = q^a \alpha x^a \quad \text{and} \quad \Delta_2.x^a = q^{-a}\beta x^a, \text{ for all } a \in \mathbb{Z}$$

and hence, if $(\alpha, \beta) \neq (0, 0)$, this module is graded by $\mathbb{Z}$, with $\deg x^a = a$. Every submodule inherits this grading. This implies that $M_{(\alpha,\beta)}$ has a unique maximal submodule when $(\alpha, \beta) \neq (0, 0)$, as the graded components have dimension 1. Let us examine this in more detail. We have two cases:

(A) $\alpha\beta^{-1} = q^{-2m}$, for some $m \in \mathbb{Z}$. Then ${}_\alpha[a]_\beta = 0 \Leftrightarrow a = m$. The unique maximal submodule is $\operatorname{span}_{\mathbb{K}}\{x^r \mid r \geq m\}$ in case $m \geq 1$, or $\operatorname{span}_{\mathbb{K}}\{x^r \mid r \leq m-1\}$ in case $m \leq 0$;

(B) If we are not in the situation of case (A), then (0) is the unique maximal submodule, and $M_{(\alpha,\beta)}$ is simple.

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If $(\alpha, \beta) = (0, 0)$, there is no longer a unique maximal submodule. For example, if $\gamma \in \mathbb{K}^\times$ then the following are all maximal submodules of $M_{(0,0)}$ of codimension 1:

$$U_q(\mathfrak{sl}_3)^+(x - \gamma 1), \quad U_q(\mathfrak{sl}_3)^+(\gamma 1 - x^{-1}), \quad U_q(\mathfrak{sl}_3)^+(x, x^{-1}).$$

In fact, if the field $\mathbb{K}$ is algebraically closed, then as $\gamma$ runs through all nonzero scalars, these are all its maximal submodules, and the corresponding simple quotients account for all isomorphism classes of finite-dimensional simple $U_q(\mathfrak{sl}_3)^+$-modules. There is a nonzero vector $v_0$ such that the simple quotient is isomorphic to $\mathbb{K}v_0$ with action given by $e_1.v_0 = 0$, $e_2.v_0 = \gamma v_0$; $e_1.v_0 = \gamma v_0$, $e_2.v_0 = 0$; or $e_1.v_0 = 0 = e_2.v_0$, respectively.

The class of modules $M_{(\alpha,\beta)}$ is, by construction, universal in the sense that if $V$ is any $U_q(\mathfrak{sl}_3)^+$-module generated by an element $v_0 \in V$ with $\Delta_1.v_0 = \alpha v_0$ and $\Delta_2.v_0 = \beta v_0$, then $V$ is a homomorphic image of $M_{(\alpha,\beta)}$.

We are now ready to make the connection with the down-up algebra $A = A(q + q^{-1}, -1, 0)$. The reader is referred to [3] for all the definitions concerning this algebra, which we shall not review here. After identifying $d$ and $u$ in $A$ with $e_1$ and $e_2$ in $U_q(\mathfrak{sl}_3)^+$ respectively, we see that these algebras coincide.

According to [3], a *weight module* for $A$ is one for which the operators $du$ and $ud$ are simultaneously diagonalizable. Since a common eigenvector for $du$ and $ud$ is also a common eigenvector for $du - q^{-1}ud$ and $du - qud$, and vice versa, it follows that such modules are the ones having a basis of common eigenvectors for $\Delta_1$ and $\Delta_2$. Furthermore, as $\Delta_1$ and $\Delta_2$ are $q$-central, it suffices that the module be generated by such eigenvectors in order for it to be a weight module. Given the universal property of the modules $M_{(\alpha,\beta)}$, we see that any cyclic weight module is a homomorphic image of $M_{(\alpha,\beta)}$, for some $(\alpha, \beta) \in \mathbb{K}^2$. In particular, the following proposition is easy to prove:

***Proposition 2*** *Let $\kappa, \lambda \in \mathbb{K}$ and define the highest weight module $V(\lambda)$, lowest weight module $W(\kappa)$ and doubly infinite module $V(\kappa, \lambda)$ as in* [3]*. Then,*

(a) $\mathrm{Span}_{\mathbb{K}}\{x^i \mid i \leq -1\}$ *is a submodule of $M_{(\lambda,\lambda)}$, and the corresponding factor module is isomorphic to $V(\lambda)$;*
(b) $\mathrm{Span}_{\mathbb{K}}\{x^i \mid i \geq 1\}$ *is a submodule of $M_{(-q^{-1}\kappa,-q\kappa)}$, and the corresponding factor module is isomorphic to $W(\kappa)$;*
(c) *If $(\kappa, \lambda) = (0, 0)$, then $V(0, 0)$ is not a Noetherian module, and therefore is not isomorphic to a subquotient of $M_{(\alpha,\beta)}$, for any $(\alpha, \beta) \in \mathbb{K}^2$;*
(d) *If $\lambda - q\kappa = q^{2(m+1)}(\lambda - q^{-1}\kappa)$, for some $m \in \mathbb{Z}$, then $V(\kappa, \lambda)$ is isomorphic to $M_{(\alpha,\beta)}$, where $\alpha = -q^{-1}(\lambda[m] - \kappa[m-1])$ and $\beta = q^2\alpha$;*
(e) *If $(\kappa, \lambda)$ satisfies neither of the conditions from* (c) *or* (d)*, then $V(\kappa, \lambda)$ is isomorphic to $M_{(\lambda - q^{-1}\kappa, \lambda - q\kappa)}$, and is therefore simple.*

**6.3** Now we want to study the next simplest case, $U_q(\mathfrak{sl}_4)^+$. A PBW basis is given by:

$$X_1 = e_3, \quad X_2 = e_2e_3 - q^{-1}e_3e_2, \quad X_3 = e_2,$$

$$X_4 = e_1X_2 - q^{-1}X_2e_1, \quad X_5 = e_1e_2 - q^{-1}e_2e_1, \quad \text{and} \quad X_6 = e_1,$$

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and we can take

$$\Delta_1 = X_4, \quad \Delta_2 = X_2X_5 - q^{-1}X_3X_4,$$
$$\Delta_3 = q^{-2}\big((q-q^{-1})^2X_1X_3X_6 - (q-q^{-1})X_1X_5 - (q-q^{-1})X_2X_6 + X_4\big),$$
$$z_1 = \Delta_1\Delta_3, \quad z_2 = \Delta_2.$$

The center is $Z = \mathbb{K}[z_1, z_2]$, and a basis for $U_q(\mathfrak{sl}_4)^+$ over $N = \mathbb{K}[\Delta_1, \Delta_2, \Delta_3]$ is

$$\{X_1^aX_2^bX_3^c \mid (a,b,c) \in \mathbb{N}^3\} \cup \{X_1^aX_3^bX_5^c \mid (a,b,c) \in \mathbb{N}^3\}$$
$$\cup \{X_1^aX_2^bX_6^c \mid (a,b,c) \in \mathbb{N}^3\} \cup \{X_1^aX_5^bX_6^c \mid (a,b,c) \in \mathbb{N}^3\}$$
$$\cup \{X_2^aX_3^bX_6^c \mid (a,b,c) \in \mathbb{N}^3\} \cup \{X_3^aX_5^bX_6^c \mid (a,b,c) \in \mathbb{N}^3\},$$

which is already a considerably large set. Instead of inducing modules from $N$, we would like to find a bigger subalgebra to induce from, so that $U_q(\mathfrak{sl}_4)^+$ is still free over this larger subalgebra, but with a free basis that is somewhat easier to manage.

Since $X_1^aX_6^b$, $(a,b) \in \mathbb{N}^2$, are among the basis elements listed above, it is clear that the $q$-commuting elements $X_1, X_6, \Delta_1$ and $\Delta_2$ are algebraically independent, hence generate the quantum affine subalgebra

$$\Gamma = \mathbb{K}[X_1, X_6, \Delta_1, \Delta_2],$$

with relations $X_1X_6 = X_6X_1$, $X_1\Delta_2 = \Delta_2X_1$, $X_1\Delta_1 = q^{-1}\Delta_1X_1$, $X_6\Delta_2 = \Delta_2X_6$, $X_6\Delta_1 = q\Delta_1X_6$, $\Delta_1\Delta_2 = \Delta_2\Delta_1$. It is easily seen by our discussion in Section 5 that $U_q(\mathfrak{sl}_4)^+$ is free over $\Gamma$, with basis $B = \{X_2^aX_3^b \mid (a,b) \in \mathbb{N}^2\} \cup \{X_3^aX_5^b \mid (a,b) \in \mathbb{N}^2\}$. Given $(\alpha, \beta) \in \mathbb{K}^2$, there is a $\Gamma$-character determined by

$$X_1 \mapsto 0, \quad X_6 \mapsto 0, \quad \Delta_1 \mapsto \alpha, \quad \Delta_2 \mapsto \beta.$$

Let $V_{(\alpha,\beta)}$ be the corresponding one-dimensional module, and set

$$M_{(\alpha,\beta)} = U_q(\mathfrak{sl}_4)^+ \otimes_\Gamma V_{(\alpha,\beta)}.$$

This is a cyclic $U_q(\mathfrak{sl}_4)^+$-module with a $\mathbb{K}$-basis indexed by $B$, and if we make the identifications

$$X_2^bX_3^a \leftrightarrow x^ay^b, \quad X_3^aX_5^c \leftrightarrow x^ay^{-c}, \quad a,b,c \in \mathbb{N},$$

we see that this corresponds to the $U_q(\mathfrak{sl}_4)^+$-module $\mathbb{K}[x, y^{\pm 1}]$, with action given by:

$$e_1.x^ay^b = \begin{cases} \alpha q^{-a}[a+b]x^ay^{b-1} + \beta[a]q^{-a-b+1}x^{a-1}y^{b-1} & \text{if } b \geq 1, \\ [a]x^{a-1}y^{b-1} & \text{if } b \leq 0, \end{cases}$$

$$e_2.x^ay^b = \begin{cases} q^bx^{a+1}y^b & \text{if } b \geq 0, \\ x^{a+1}y^b & \text{if } b < 0, \end{cases}$$

$$e_3.x^ay^b = \begin{cases} -q^{a-b}[a]x^{a-1}y^{b+1} & \text{if } b \geq 0 \\ \alpha q[b-a]x^ay^{b+1} - \beta q[a]x^{a-1}y^{b+1} & \text{if } b \leq -1. \end{cases}$$

Similarly, we could have used the $\Gamma$-character determined by

$$X_1 \mapsto \alpha, \quad X_6 \mapsto \beta, \quad \Delta_1 \mapsto 0, \quad \Delta_2 \mapsto \gamma,$$

for $(\alpha, \beta, \gamma) \in \mathbb{K}^3$, and the result would have been a $U_q(\mathfrak{sl}_4)^+$-module $P_{(\alpha,\beta,\gamma)}$, isomorphic to $\mathbb{K}[x, y^{\pm 1}]$ with action:

$$e_1.x^a y^b = \begin{cases} \beta q^{-a-b} x^a y^b + \gamma[a] q^{-a-b+1} x^{a-1} y^{b-1} & \text{if } b \geq 1, \\ \beta q^{-a-b} x^a y^b + [a] x^{a-1} y^{b-1} & \text{if } b \leq 0, \end{cases}$$

$$e_2.x^a y^b = \begin{cases} q^b x^{a+1} y^b & \text{if } b \geq 0, \\ x^{a+1} y^b & \text{if } b < 0, \end{cases}$$

$$e_3.x^a y^b = \begin{cases} \alpha q^{a-b} x^a y^b - q^{a-b}[a] x^{a-1} y^{b+1} & \text{if } b \geq 0, \\ \alpha q^{a-b} x^a y^b - \gamma q[a] x^{a-1} y^{b+1} & \text{if } b \leq -1. \end{cases}$$

Let us look at the module $M_{(\alpha,\beta)}$ more carefully. We have,

$$\Delta_1.x^a y^b = q^b \alpha x^a y^b, \quad \Delta_2.x^a y^b = \beta x^a y^b, \quad \text{and} \quad \Delta_3.x^a y^b = q^{-b} \alpha x^a y^b,$$

for all $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. Assume $\alpha \neq 0$. Then there is a natural $\mathbb{Z}$-grading on $M_{(\alpha,\beta)}$ given by setting $\deg(x^a y^b) = b$ for all $b \in \mathbb{Z}$. It has the additional property that any submodule of $M_{(\alpha,\beta)}$ inherits this grading. Note that the homogeneous subspace of degree $k$ is $\mathbb{K}[x]y^k$. We will show now under the assumption $\alpha \neq 0$ that $M_{(\alpha,\beta)}$ is simple. Let $W$ be a nonzero submodule, and take a nonzero homogeneous element of $W$, say $p$, which we can write as

$$p = (a_0 + a_1 x + \cdots + a_l x^l) y^b = a_0 y^b + a_1 x y^b + \cdots + a_l x^l y^b,$$

where $a_i \in \mathbb{K}$, $a_l \neq 0$, $l \geq 0$, and $b = \deg\ p$.

***Case 1*** $b \geq 0$. Since

$$e_3^l.p = (-1)^l q^{-l(b-1)} [l]!\, a_l y^{b+l},$$

we see that $y^{b+l} \in W$, and hence so is

$$e_1^{b+l}.y^{b+l} = \alpha^{b+l} [b+l]!\, 1.$$

It follows that $1 \in W$ and so $W = M_{(\alpha,\beta)}$, as 1 generates $M_{(\alpha,\beta)}$.

***Case 2*** $b < 0$. As in the previous case, one sees from the following computations that $1 \in W$ and $W = M_{(\alpha,\beta)}$:

$$e_1^l.p = a_l [l]!\, y^{b-l},$$

$$e_3^{l-b}.y^{b-l} = (-q\alpha)^{l-b} [l-b]!\, 1.$$

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So $M_{(\alpha,\beta)}$ is indeed simple for all pairs $(\alpha, \beta) \in \mathbb{K}^\times \times \mathbb{K}$. The center $Z$ of $U_q(\mathfrak{sl}_4)^+$ acts via

$$z_1.m = \alpha^2 m, \tag{11}$$

$$z_2.m = \beta m, \quad \text{for all } m \in M_{(\alpha,\beta)}, \tag{12}$$

where $z_1, z_2$ are as in 6.3. The above equations show that if the modules $M_{(\alpha,\beta)}$ and $M_{(\alpha',\beta')}$ are isomorphic, then $\alpha^2 = (\alpha')^2$ and $\beta = \beta'$, as their central characters should be the same. Furthermore, the eigenvalues of the operator $\Delta_1$ on each module must coincide and hence $\alpha' = q^b\alpha$ for some $b \in \mathbb{Z}$, which forces $\alpha = \alpha'$, as $\alpha^2 = (\alpha')^2$. Therefore the modules $M_{(\alpha,\beta)}$ are pairwise non-isomorphic, and simple if $\alpha \neq 0$. A similar argument shows that $M_{(\alpha,\beta)}$ is not isomorphic to the module $P_{(\gamma,\delta,\epsilon)}$ defined earlier or to any of its simple quotients if $\alpha \neq 0$, as the central element $z_1$ annihilates $P_{(\gamma,\delta,\epsilon)}$.

***Remark*** The subalgebra of $U_q(\mathfrak{sl}_4)^+$ generated by the elements $X_1, X_6, \Delta_1, \Delta_2$, and $\Delta_3$ is isomorphic to *quantum affine* 5-*space*, but $U_q(\mathfrak{sl}_4)^+$ is no longer free over it, and in fact if we try to induce one-dimensional modules for this algebra up to $U_q(\mathfrak{sl}_4)^+$, then corresponding to any character with $\Delta_1 \mapsto \lambda$, $\Delta_3 \mapsto \mu$, and $\lambda \neq \mu$, we obtain just the zero $U_q(\mathfrak{sl}_4)^+$-module.

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