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PAPER

A MALL geometry of interaction based on indexed linear logic

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Abstract

We construct a geometry of interaction (GoI: dynamic modelling of Gentzen-style cut elimination) for multiplicative-additive linear logic (MALL) by employing Bucciarelli–Ehrhard indexed linear logic MALL(I) to handle the additives. Our construction is an extension to the additives of the Haghverdi–Scott categorical formulation (a multiplicative GoI situation in a traced monoidal category) for Girard's original GoI 1. The indices are shown to serve not only in their original denotational level, but also at a finer grained dynamic level so that the peculiarities of additive cut elimination such as superposition, erasure of subproofs, and additive (co-) contraction can be handled with the explicit use of indices. Proofs are interpreted as indexed subsets in the category Rel, but without the explicit relational composition; instead, execution formulas are run pointwise on the interpretation at each index, with respect to symmetries of cuts, in a traced monoidal category with a reflexive object and a zero morphism. The sets of indices diminish overall when an execution formula is run, corresponding to the additive cut-elimination procedure (erasure), and allowing recovery of the relational composition. The main theorem is the invariance of the execution formulas along cut elimination so that the formulas converge to the denotations of (cut-free) proofs.

Keywords: Geometry of interaction; Multiplicative Additive Linear Logic; Indexed Linear Logic; Cut Elimination; Traced Monoidal Category; Execution Formula

1. Introduction

The indexed multiplicative-additive linear logic MALL(I), introduced by Bucciarelli– Ehrhard (2000), is a conservative extension of Girard's MALL in which all formulas and proofs come equipped with sets of indices. The usual MALL is stipulated to be the restriction to the empty set. The status of the indexed syntactical system is noteworthy as it stems from the denotational semantics of Rel, a simple, yet pivotal categorical model of MALL. With the enabling of an explicit notion of location in linear proof theory, the indices can enumerate the locations of formulas and proofs, corresponding to denotational interpretations of MALL. The notion of location becomes a requirement for the additives, although it is redundant for the multipicatives, for which the singleton $\{*\}$ suffices. To work with *parallelism*, which the additives bring intrinsically, different locations need to be handled rather than the sole location $*$. In the context of parallelism, superpositions are known to typically arise under the syntactic additive &-rule. Indices allow one to deal with superpositions by identifying multiple occurrences of formulas in the different indices and by enlarging (or restricting) the indices.

The original motivation for indexed logic was to provide a bridge between a truth-valued semantics (for provability) for MALL(I) and the denotational semantics of (nonindexed) MALL. By means of this bridge, Bucciarelli–Ehrhard obtained a new kind of denotational completeness theorem in Bucciarelli and Ehrhard (2000) for MALL and later extended it to the exponential in Bucciarelli and Ehrhard (2001).

This paper investigates indexed MALL from the perspective of a dynamic semantics for cut elimination, a topic that – to the best of our knowledge – has remained untouched aside from the precursory work of Duchesne (2009) since the original work of Bucciarelli–Ehrhard (2000). The dynamic semantics is the Girard project of Geometry of Interaction (GoI), whereby cut elimination is modelled, using operator algebras (Girard 1989) and more generally traced monoidal categories (Joyal et al. 1996). The GoI project was successful (Girard 1989; Haghverdi and Scott 2006) for MLL with the exponential, and inspired a new model of computation for β reduction of λ-calculus (Danos and Regnier 1995). This paper aims to initiate an exploration of how to combine the two notions of *location*, which the indexed logic brings, and of *dynamics*, which GoI brings to cut elimination. The combination is important in understanding additive cut elimination. For this goal, we employ the indices to construct a GoI model for (non-indexed) MALL. We combine the Haghverdi–Scott categorical GoI situation (Haghverdi and Scott 2006) with the indices in such a way that that the original MLL GoI situation represents a collapse to the singleton index $\{*\}$. The dynamics of cut elimination is captured by a feedback mechanism determined by traces of morphisms interpreting proofs. We further augment the situation with two kinds of actions, identical and zero, over the symmetries interpreting the cut rule. These two actions provide representations of matches and of mismatches amongst locations. These come into play during a Gentzen-style cut-elimination procedure, in which one encounters noncommunication of individual proofs, due to the additive parallelism. Crucial instances of GoI situation such as Rel_+ and Hilb_2 (Haghverdi and Scott 2006) are directly lifted to our framework, the latter of which is the operator algebraic origin of the Girard project.

We study Girard's execution formula (Girard 1989) in the general categorical setting of a traced symmetric monoidal category. The execution formula accommodates indices, and faithfully simulates MALL cut elimination by a hybrid method relating the indexed syntax to the relational semantics. Each location in the relation interpreting a proof is first assigned an endomorphism on a reflexive object U. The cut rule before execution is interpreted as a tensor product of two premise morphisms, more loosely than their composition. This interpretation allows extraction of the dynamical meaning of the cut, which the usual categorical composition hides by virtue of its static approach. In the loose interpretation, there remain redundant indices when interpreting rules: however, they are shown to disappear, while running the Execution formula in terms of the categorical trace structure. The disappearance of indices is modelled by zero morphisms, which exist in the traced monoidal categories for GoI. Proof-theoretically, the zero morphisms allow us to interpret discarding subproofs specific to additive cut elimination, and in the way of theory of indices, they provide a way to to interpret mismatches amongst locations. In traced monoidal categories, the zero morphisms are supposed to act partially on symmetries for cut formulas, and also to act partially on retractions and co-retractions of the reflexive objects. The latter action arises via tracing along the zero morphisms which takes feedback into account along with the zero. We prove zero convergence, which means that execution formulas converge to zero when two proofs interact with mismatched locations. Thus the execution formulas terminate to the denotational interpretations of proofs, while diminishing sets of indices in order to recover the relational composition. This is realised by properly coupling indices to trace axioms, especially for 'generalized yanking' and 'dinaturality'. The former axiom directly designates that traces are primitive enough to retrieve the categorical composition in a monoidal category, and the latter axiom concerns the interaction of bidirectional flow of morphisms.

In contrast to the precursory work of Duchesne (2009) concerning both indices and GoI, the present paper accommodates the indices directly in GoI semantics in order to simulate (nonindexed) MALL cut elimination. The diminution of sets of indices is a typical dynamic aspect our GoI captures using the zero morphisms of our traced monoidal category. The precursor in Duchesne (2009), on the contrary, first accommodates the index into the static category of Rel, using the semantic method of localisation, of which the indexed syntax provides a precise description. Then (nonindexed) dynamic GoI action over the indexed denotational semantics is investigated to characterise the static fix points as the denotational interpretation.

We prove two main results: (i) (Invariance of the execution formula during MALL normalisation): The execution formula in our dynamic categorical modelling is shown to converge to the denotational interpretation of proofs in the static categorical model. This characterises the normalisation of proofs by categorical invariants. (ii) (Diminution of indices while running the execution formula): The execution formula may converge to 0, making the redundant indices disappear. Part (i) is seen as a pointwise collection of invariants, as previously established for the multiplicatives (Girard 1989; Haghverdi and Scott 2006). Part (ii) is specific to the additives: Proof-theoretically, it reflects erasure of subproofs as well as additive (co) contraction and superposition, in cut elimination. Category-theoretically, it ensures that our categorical ingredient (the execution formula) is fine grained enough to retrieve a static monoidal category as well as a relational category handling indices.

The rest of this paper is organised as follows: Section 2 introduces a syntax $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ for indexed MALL with a cut list as well as its relational counterpart $\mathsf{Rel}^{[\mathfrak{c}]}$. A fundamental lemma is proved, which connects a provable $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ sequent to an indexed subset of the interpretation of a MALL proof with cuts. In Section 3, MALL proof reduction for cut elimination is lifted to MALL(I) proof transformation with diminishing sets of indices. Section 4 concerns our MALL GoI interpretation by means of the indexed system in a traced symmetric monoidal category with zero morphism. Execution formulas are run indexwise, and the main theorem is proved.

2. MALL(*I*) with Cut List and Relational Semantics

2.1 MALL(I) with cut list

(**Inference rules of $\mathsf{MALL}^{[\mathfrak{c}]}$ with cut formulas**)

We accommodate a stack to record cut formulas into the syntax of the multiplicative-additive linear logic MALL. To stress this, the system is written as $\mathsf{MALL}^{[\mathfrak{c}]}$. To accommodate the stack into the additive fragment, one has to work with superpositions that arise inside the stack as well as in the conclusion (outside the stack).

A $\mathsf{MALL}^{[\mathfrak{c}]}$ sequent $\vdash [\Delta], \Gamma$ with a cut list is a set Γ of formula occurrences together with pairwise dual formulas occurrences Δ inside the brackets. Each dual pair in Δ is written $A, A^\perp$. Sequents are proved using the following rules:

$$\frac{}{\vdash A, A^\perp}\ \mathrm{ax} \qquad \frac{\vdash [\Delta_1],\ \Gamma_1, A \quad \vdash [\Delta_2],\ \Gamma_2, B}{\vdash [\Delta_1, \Delta_2],\ \Gamma_1, \Gamma_2, A \otimes B}\ \otimes \qquad \frac{\vdash [\Delta],\ \Gamma, A, B}{\vdash [\Delta],\ \Gamma, A \parr B}\ \parr$$

$$\frac{\vdash [\Delta_1],\ \Gamma_1, A \quad \vdash [\Delta_2],\ A^\perp, \Gamma_2}{\vdash [\Delta_1, \Delta_2, A, A^\perp],\ \Gamma_1, \Gamma_2}\ \mathrm{cut}$$

$$\frac{\vdash [\Delta_1, \Sigma],\ \Gamma, A_1 \quad \vdash [\Delta_2, \Sigma],\ \Gamma, A_2}{\vdash [\Delta_1, \Delta_2, \Sigma],\ \Gamma, A_1 \& A_2}\ \& \quad \frac{\vdash [\Delta],\ \Gamma, A_1}{\vdash [\Delta],\ \Gamma, A_1 \oplus A_2}\ \oplus_1 \quad \frac{\vdash [\Delta],\ \Gamma, A_2}{\vdash [\Delta],\ \Gamma, A_1 \oplus A_2}\ \oplus_2$$

Note: In the &-rule, not only is Γ superposed in the conclusion, but so is Σ in the stack. The superposition amongst cut formulas inside the stack causes the well-known additive (co-) contraction that arises in MALL cut elimination. The formulas occurrences Σ is not deterministically chosen in the premises, so that Σ in general is neither empty (i.e. never superimpose cuts) nor maximal (i.e. superimpose as many cuts as possible). Thus, the &-rule has several possible instances depending on the choice of Σ. The exchange rule is eliminated under the assumption that formula occurrences are implicitly tracked by the premises and the conclusion of a rule.

We extend the above accommodation of cut lists to Bucciarelli–Ehrhard indexed system $\mathsf{MALL}(I)$ (Bucciarelli and Ehrhard 2000). To stress this, the system is written as $\mathsf{MALL}^{[c]}(I)$. The extension stipulates that a set of indices is consistently associated with each formula (including cut formulas) and sequent (including cut lists).

We fix an index set I, once and for all. Each formula A of $\mathsf{MALL}(I)$ is associated with a set $d(A) \subseteq I$, called the *domain* of A.

(**MALL**(I) **formulas and domains**)
Formulas in the domain J are defined by the following grammar: $\mathbf{0}_\emptyset$ and $T_\emptyset$ are formulas of the domain $\emptyset$. For any $J, K, L \subseteq I$ such that $K \cap L = \emptyset$ and $K \cup L = J$,

$$X_J ::= \quad \mathbf{1}_J \quad | \quad \bot_J \quad | \quad X_J \otimes X_J \quad | \quad X_J \parr X_J \quad | \quad X_K \oplus X_L \quad | \quad X_K \& X_L.$$

For any $\mathsf{MALL}(I)$ formula A with $d(A) = J$, its negation $A^\perp$ with $d(A^\perp) = J$ is defined using the De Morgan duality for the MALL formula.

(**Restriction**)
For a $\mathsf{MALL}(I)$ formula A with $d(A) = J$ and $K \subseteq J$, the restriction $A\restriction_K$ of A by K is defined to be a $\mathsf{MALL}(I)$ formula in the domain $J \cap K$ as follows:

$$\mathbf{0}_\emptyset\restriction_K = \mathbf{0}_\emptyset \text{ and } T_\emptyset\restriction_K = T_\emptyset. \quad \bot_J\restriction_K = \bot_{J\cap K} \text{ and } \mathbf{1}_J\restriction_K = \mathbf{1}_{J\cap K} \quad (A \otimes B)\restriction_K = A\restriction_K \otimes B\restriction_K$$
$$(A \parr B)\restriction_K = A\restriction_K \parr B\restriction_K \quad (A \oplus B)\restriction_K = A\restriction_K \oplus B\restriction_K \quad (A\&B)\restriction_K = A\restriction_K \& B\restriction_K$$

It trivially follows that $(A^\perp)\restriction_K = (A\restriction_K)^\perp$. If Γ is a sequence of $\mathsf{MALL}(I)$ formulas $A_1, \ldots, A_n$ of domains J, we define $\Gamma\restriction_K = A_1\restriction_K, \ldots, A_n\restriction_K$.

(**Inference rules of MALL**$^{[c]}(I)$ **with cut lists**)
We augment $\mathsf{MALL}^{[c]}$ with indices. This makes it possible to accommodate a stack recording cut formulas to the original $\mathsf{MALL}(I)$ (Bucciarelli and Ehrhard 2000). Although this is straightforward for the multiplicative fragment, careful treatment is required for the listing of cut formulas in the additive fragment. Two kinds of sequences of formulas are considered in our $\mathsf{MALL}^{[c]}(I)$-syntax: One is a sequence Ξ whose all formulas occurrences have a same domain J uniformly, which is denoted by $d(\Xi) = J$. The other is a sequence Ξ whose any formula occurrence has a domain contained in J, which is denoted by $d(A) \subseteq J$. Each sequent is of the form $\vdash_J [\Delta], \Gamma$, in which $d(\Gamma) = J$ and $d(\Delta) \subseteq J$ with $d(A) = d(A^\perp) \subseteq J$ for any pairwise dual formulas A and $A^\perp$ in Δ within the stack.

Note: The uniformity requirement that all formulas in Γ have the same domain I does not apply to the stack Δ of the cut formulas. Formulas from different cuts have various domains contained in J.

Axioms and cut:

$$\vdash_J \mathbf{1}_J \qquad \vdash_\emptyset \Gamma, T_\emptyset \qquad \frac{\vdash_J [\Delta_1],\ \Gamma_1, A \qquad \vdash_J [\Delta_2],\ A^\perp, \Gamma_2}{\vdash_J [\Delta_1, \Delta_2, A, A^\perp],\ \Gamma_1, \Gamma_2}\ \text{cut}$$

Note that $d(A) = d(A^\perp) = J$ for cut formulas A and $A^\perp$.

Multiplicative rules:

$$\frac{\vdash_J [\Delta],\ \Gamma}{\vdash_J [\Delta],\ \Gamma, \bot_J}\ \bot_J \qquad \frac{\vdash_J [\Delta_1],\ \Gamma_1, A \quad \vdash_J [\Delta_2],\ \Gamma_2, B}{\vdash_J [\Delta_1, \Delta_2\,],\ \Gamma_1, \Gamma_2, A\otimes B}\ \otimes \qquad \frac{\vdash_J [\Delta],\ \Gamma, A, B}{\vdash_J [\Delta],\ \Gamma, A ⅋ B}\ ⅋$$

Additive rules:

$$\frac{\vdash_{J_1} [\Delta_1, \Sigma\lceil_{J_1}\ ],\ \Gamma\lceil_{J_1}, A_1 \quad \vdash_{J_2} [\Delta_2, \Sigma\lceil_{J_2}\ ],\ \Gamma\lceil_{J_2}, A_2}{\vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Sigma\ ],\ \Gamma, A_1 \& A_2}\ \&$$

Note that the superposed context Γ encompasses the *whole* domain $J = J_1 + J_2$, while the superposed context Σ in the stack has a domain *contained in* J.

$$\frac{\vdash_J [\Delta],\ \Gamma, A_1}{\vdash_J [\Delta],\ \Gamma, A_1 \oplus A_2}\ \oplus_1 \qquad \frac{\vdash_J [\Delta],\ \Gamma, A_2}{\vdash_J [\Delta],\ \Gamma, A_1 \oplus A_2}\ \oplus_2$$

Note $d(A_{3-i}) = \emptyset$ in each rule $\oplus_i$ $(i = 1, 2)$.

MALL(I) has no propositional variables; the only atomic formulas are the constants. Then, the usual identity axiom is readily derived:

Lemma 2.1 (Identity). *$\vdash_J A, A^\perp$ is provable for any MALL(I) formula A of domain J.*

Lemma 2.2 (Restricting provable sequents). *If $\vdash_J [\Delta],\ \Gamma$ is provable, then so is $\vdash_{J\cap K} [\Delta\lceil_K\ ]$, $\Gamma\lceil_K$ for any $K \subseteq I$.*

For each inference rule of MALL$^{[c]}(I)$, if the conclusion sequent has the domain $\emptyset$, then so does the premise sequent(s). Thus, the rules for sequents deriving the empty domain are identified with the rules of MALL$^{[c]}$. As a consequence, every MALL$^{[c]}(I)$-proof π for $\vdash_\emptyset [\Delta], \Gamma$ contains only sequents of the empty domain. Hence π is considered as a MALL-proof for $\vdash [\Delta], \Gamma$. To sum up,

Lemma 2.3. *MALL$^{[c]}(I)$ is a conservative extension of MALL$^{[c]}$.*

Accordingly, in the sequel MALL$^{[c]}$ is identified with MALL$^{[c]}(\emptyset)$.

2.2 MALL$^{[c]}(I)$ and Relational Semantics Rel$^{[c]}$

It is well known that the category Rel of sets and relations constitutes a denotational semantics of MALL, that is, the interpretation is invariant, $(\pi_{[\Delta],\Gamma})^* = (\pi'_{[\Delta'],\Gamma})^*$, for any reduction $\pi_{[\Delta],\Gamma} \rhd \pi'_{[\Delta'],\Gamma}$ of MALL cut elimination. In particular, the denotation of π is equal to that of a cut-free π' when Δ' is empty. The cut rule is interpreted by *relational composition* in Rel, and this interpretation makes the semantics denotational.

Definition 2.4 (Denotational interpretation $(\pi_{[\Delta],\Gamma})^*$ in Rel). *Every MALL proof $\pi_{[\Delta],\Gamma}$ of a sequent $\vdash [\Delta]\ \Gamma$ is interpreted as a subset of an associated set of the conclusion (without the cut list),*

$$(\pi_{[\Delta],\Gamma})^* \subseteq |\Gamma| \tag{1}$$

Note that in the interpretation, the cut formulas Δ become invisible by virtue of the relational composition: Two relations $R_1 \subseteq G_1 \times D$ and $R_2 \subseteq D \times G_2$ compose in Rel

$$R_2 \circ R_1 = \{(g_1, g_2) \mid \exists\, d \in D\, (g_1, d) \in R_1 \wedge (d, g_2) \in R_2\} \subseteq G_1 \times G_2$$

The interpretation π^* of (1) is known in Bucciarelli and Ehrhard (2000) as the relational interpretation of MALL proofs in a compact closed category $\mathsf{Rel}_\times$ with biproducts $+$. The interpretation is specified as follows accordingly to the MALL rules, for which we refer to the above $\mathsf{MALL}^{[\mathfrak{c}]}$-rules ridden of the cut lists. First, every formula A is interpreted as a set $|A|$, and every sequence $\Gamma = A_1, \ldots, A_n$ as $|\Gamma| = |A| \times \cdots \times |A_n|$: When A is $A_1 \otimes A_2$ or $A_1 \parr A_2$, $|A| = |A_1| \times |A_2|$, and when A is $A_1 \oplus A_2$ or $A_1 \& A_2$, $|A| = (\{1\} \times |A_1|) \cup (\{2\} \times |A_2|)$.

Then
(axiom) $\pi^* = \{(a, a) \mid a \in |A|\} \subseteq |A, A^\perp|$
(cut rule) $\pi^* = \{(\gamma_1, \gamma_2) \mid (\gamma_1, a) \in \pi_1^* \text{ and } (a, \gamma_2) \in \pi_2^*\} \subseteq |\Gamma_1, \Gamma_2|$
(&-rule) $\pi^* = \{(\gamma, (1, a)) \mid (\gamma, a) \in \pi_1^*\} \cup \{(\gamma, (2, a) \mid (\gamma, a) \in \pi_2^*\} \subseteq |\Gamma, A_1 \& A_2|$
($\oplus_i$-rule) $\pi^* = \{(\gamma, (i, a)) \mid (\gamma, a) \in \pi_i^*\} \subseteq |\Gamma, A_1 \oplus A_2|$
($\otimes$-rule) $\pi^* = \{(\gamma_1, \gamma_2, (a_1, a_2)) \mid (\gamma_1, a_1) \in \pi_1^* \text{ and } (\gamma_2, a_2) \in \pi_2^*\} \subseteq |\Gamma, A_1 \otimes A_2|$
($\parr$-rule) $\pi^* = \{(\gamma, (a_1, a_2)) \mid (\gamma, a_1, a_2) \in \pi'^*\} \subseteq |\Gamma, A_1 \parr A_2|$

Our aim in this paper is to investigate a dynamics of cuts hidden in such a static categorical composition. We begin by interpreting proofs in Rel but without performing cuts by means of relational composition. To stress this interpretation with the unexecuted cuts, the categorical framework is denoted by $\mathsf{Rel}^{[\mathfrak{c}]}$, in which the cut list $[\Delta]$ is interpreted explicitly.

To deal with the additives in $\mathsf{Rel}^{[\mathfrak{c}]}$, we have to work with a sublist and the set of all the sublists: Let Δ be $C_1, C_1^\perp, \ldots, C_m, C_m^\perp$. For a subset S of $\{1, \ldots, m\}$, let Δ_S denote the sublist $\ldots C_i, C_i^\perp \ldots$ where i ranges in S. Then the set $\mathfrak{sl}(\Delta)$ of all the sublists (including Δ and $\Delta_\emptyset$) is defined as

$$\mathfrak{sl}(\Delta) := \{\Delta_S \mid S \subseteq \{1, \ldots, m\}\} \tag{2}$$

Consequently, we interpret (2) as an object in Rel in terms of the disjoint union of each interpretations of the sublists:

$$[\,\mathfrak{sl}(\Delta)] = \sum_{S \subseteq \{1,\ldots,m\}} |\Delta_S|, \tag{3}$$

in which $|\Delta_S|$, for a nonempty sequence, is the usual interpretation of the sequence in Rel and $|\Delta_\emptyset| := \{*\} = |\bot|$. The disjoint sum Σ is taken over different S's.

In what follows in this paper, when S is clear from context, a sublist Δ_S is often abbreviated by $\widehat{\Delta}$, whose hat indicates a pairwise deletion of some cut formulas.

Lemma 2.5. *If $\Delta = \Delta_1, \ldots, \Delta_k$ such that Δ_i are lists of pairwise dual formulas, then*

$$[\mathfrak{sl}(\Delta)] \cong [\mathfrak{sl}(\Delta_1)] \times \cdots \times [\mathfrak{sl}(\Delta_k)]$$

For example, a particular choice of each Δ_i is $A_i, A_i^\perp$.

In what follows in Sections 2 and 3, $\cong$ denotes an iso modulo the symmetry of the set-theoretical cartesian product. In the sequel, the symmetry corresponds to the exchange of formula occurrences. As the exchange is always clear from the context and fixed, we use the terminologies $\subseteq_\cong$ and $\in_\cong$ consistently as follows: $A \subseteq_\cong B$ (resp. $a \in_\cong B$) means that A is a subset (resp. a member) of $\sigma(B)$ where σ is the exchange for $\cong$.

Definition 2.6 (Interpretation $|\pi_{[\Delta],\Gamma}|$ of proofs with unexecuted cuts in $\mathsf{Rel}^{[\mathfrak{c}]}$). *Every MALL proof $\pi_{[\Delta],\Gamma}$ of a sequent $\vdash [\Delta],\ \Gamma$ is interpreted by*

$$|\pi_{[\Delta],\Gamma}| \subseteq [\mathfrak{sl}(\Delta)] \times |\Gamma|,$$

which is defined inductively and in the same manner as in Definition 2.4, except for the cut rule to make the interpretation differ from the standard (1) in that Δ *is visible without performing the relational composition.*

(cut rule)

When π *is* $$\dfrac{\overset{\pi^1}{\vdash [\Delta_1],\ \Gamma_1, A} \quad \overset{\pi^2}{\vdash [\Delta_2],\ A^\perp, \Gamma_2}}{\vdash [\Delta_1, \Delta_2, A, A^\perp],\ \Gamma_1, \Gamma_2}\ \text{cut}$$

$$|\pi_{[\Delta_1,\Delta_2 A,A^\perp],\Gamma_1,\Gamma_2}| :\cong |\pi^1_{[\Delta_1],\Gamma_1,A}| \times |\pi^2_{[\Delta_2],A^\perp,\Gamma_2}|$$

$$\subseteq [\mathfrak{sl}(\Delta_1)] \times |\Gamma_1| \times |A| \times [\mathfrak{sl}(\Delta_2)] \times |A^\perp| \times |\Gamma_2| \subseteq_{\cong} [\mathfrak{sl}(\Delta_1, \Delta_2)] \times |A| \times |A^\perp| \times |\Gamma_1| \times |\Gamma_2|$$

The symmetry used for the definition is the exchange between the conclusion of the cut and merging those of π_i*'s. In obtaining the last inclusion, Lemma 2.5 is used because the two lists* Δ_1 *and* Δ_2 *are disjoint.*

(&-rule)

When π *is* $$\dfrac{\overset{\pi^1}{\vdash [\Delta_1, \Sigma\ ],\ \Gamma, A_1} \quad \overset{\pi^2}{\vdash [\Delta_2, \Sigma\ ],\ \Gamma, A_2}}{\vdash [\Delta_1, \Delta_2, \Sigma\ ],\ \Gamma, A_1 \& A_2}\ \&$$

$$|\pi_{[\Delta_1,\Delta_2,\Sigma],\Gamma,\ A_1\&A_2}| :=$$

$$\{\ (\lambda_1, (1, a_1)) \mid (\lambda_1, a_1) \in |\pi^1_{[\Delta_1,\Delta],\Gamma,A_1}|\ \} + \{\ (\lambda_2, (2, a_2)) \mid (\lambda_2, a_2) \in |\pi^2_{[\Delta_2,\Sigma],\Gamma,A_2}|\ \} \subseteq$$

$$(\ [\mathfrak{sl}(\Delta_1, \Sigma)] \times |\Gamma| \times \{1\} \times |A_1|\ ) + (\ [\mathfrak{sl}(\Delta_2, \Sigma)] \times |\Gamma| \times \{2\} \times |A_2|\ ) \subseteq [\mathfrak{sl}(\Delta_1, \Delta_2, \Sigma)] \times |\Gamma| \times |A_1 \& A_2|$$

For the last inclusion, the monotonicity, $[\mathfrak{sl}(\Delta_i, \Sigma)] \subseteq [\mathfrak{sl}(\Delta_1, \Delta_2, \Sigma)]$ *is used.*

We extend Bucciarelli–Ehrhard translation of Definitions 2.7–2.8 to accommodate cut formulas inside the stack.

Definition 2.7 (Translation of indexed relation γ to $\mathsf{MALL}(I)$ sequent $\Gamma\langle\gamma\rangle$ Bucciarelli and Ehrhard 2000). *To any MALL formula* A *and a family* $a \in |A|^J$*, a formula* $A\langle a\rangle$ *of* $\mathsf{MALL}(I)$ *is associated, with domain* J *so that* $A\langle a\rangle\upharpoonright_\emptyset$ *is* A*. For a sequence* $\Gamma = A_1, \ldots, A_n$ *of MALL formulas, every* $\gamma \in |\Gamma|^J$ *is written uniquely as* $\gamma = \gamma^1 \times \cdots \times \gamma^n$ *with* $\gamma^m \in |A_m|^J$*, and we set* $\Gamma\langle\gamma\rangle = A_1\langle\gamma_1\rangle, \ldots, A_n\langle\gamma_n\rangle$*.*

- For $A = \mathbf{0}$ or $A = T$, if $J \neq \emptyset$, we have $|A|^J = \emptyset$, and $A\langle a\rangle$ is undefined. If $J = \emptyset$, $|A|^J$ has exactly one element, namely, the empty family $\emptyset$, and we set $\mathbf{0}\langle\emptyset\rangle = \mathbf{0}$ and $T\langle\emptyset\rangle = T$.
- If $A = \mathbf{1}$ or $A = \perp$, a is the constant family, and we set $\mathbf{1}\langle(\ast)_{\mathbf{J}}\rangle = \mathbf{1}_{\mathbf{J}}$ and $\perp\langle(\ast)_J\rangle = \perp_J$.
- If $A = B \otimes C$, then $a = b \times c$ with $b \in |B|^J$ and $c \in |C|^J$, and we set $A\langle a\rangle = B\langle b\rangle \otimes C\langle c\rangle$ which is a well-formed formula of $\mathsf{MALL}(I)$ of domain J. Here $b \times c$ denotes the mediating morphism of the set-theoretical cartesian product.
  Similarly, for $A = B \parr C$, we set $A\langle a\rangle = B\langle b\rangle \parr C\langle c\rangle$.
- If $A = B \oplus C$, then $a = b + c$ with $b \in |B|^K$ and $c \in |C|^L$ and $K + L = J$. Then we set $A\langle a\rangle = B\langle b\rangle \oplus C\langle c\rangle$ which is a well-formed formula of $\mathsf{MALL}(I)$ of domain J.
  Similarly for $A = B \,\&\, C$, we set $A\langle a\rangle = B\langle b\rangle \,\&\, C\langle c\rangle$.

(**Notation**)
Let X be a set and $J = J_1 + J_2$. Every $x \in X^J$ yields the restrictions $x_i = x\upharpoonright_{J_i} \in X^{J_i}$ with $i = 1, 2$. Conversely, the two restrictions allow us to recover x. We write this as $x = x_1 \frown x_2$.

Definition 2.8 (Translation to $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ sequent $\vdash_J [\Delta\langle\!\langle\delta\rangle\!\rangle], \Gamma\langle\gamma\rangle$). *Let Δ be a sequence of pairwise dual MALL formulas and $\delta \in [\mathfrak{sl}(\Delta)]^J$ for some $J \subseteq I$. Then the MALL(I) sequence $\Delta\langle\!\langle\delta\rangle\!\rangle$ of pairwise dual formulas is associated such that $d(\Delta\langle\!\langle\delta\rangle\!\rangle) \subseteq J$ and $\Delta\langle\!\langle\delta\rangle\!\rangle\upharpoonright_\emptyset$ is Δ as follows.*

First, we write $\Delta = \Delta_1, \ldots, \Delta_n$, where each Δ_i is a list of two dual formulas A_i and $A_i^\perp$. By Lemma 2.5, $\delta = \delta^1 \times \cdots \times \delta^n$, so $\delta^i \in [\mathfrak{sl}(\Delta_i)]^J$. Because $\mathfrak{sl}(\Delta_i) = \{\Delta_i, \Delta_\emptyset\}$, we have $[\mathfrak{sl}(\Delta_i)] = |A_i, A_i^\perp| + \{*\}$. Recall that $\{*\}$ interprets the empty list in (3). Thus every δ^i makes J divide into $J = J_i + K_i$ to yield $\delta^i = \delta^i\upharpoonright_{J_i} \frown \delta^i\upharpoonright_{K_i}$ so that $\delta^i\upharpoonright_{J_i} \in |A_i, A_i^\perp|^{J_i}$ and $\delta^i\upharpoonright_{K_i} \in |*|^{K_i}$. (explicitly $J_i = \{x \mid \delta_i(x) \in |A_i, A_i^\perp|\}$ and $K_i = \{x \mid \delta_i(x) = *\}$.) Then, using the $\delta^i\upharpoonright_{J_i}$, we define

$$\Delta\langle\!\langle\delta\rangle\!\rangle = \Delta_1\langle\delta^1\upharpoonright_{J_1}\rangle, \ \ldots, \ \Delta_n\langle\delta^n\upharpoonright_{J_n}\rangle,$$

in which the two dual formulas in each $\Delta_i\langle\delta^n\upharpoonright_{J_i}\rangle$ have the same domain $J_i \subseteq J$.

Then, by employing Definition 2.7, for a given MALL sequent $\vdash [\Delta], \Gamma$, every $\nu \in (|\mathfrak{sl}(\Delta)| \times |\Gamma|)^J$ is associated with a MALL(I) sequent, for which we write $\nu = \delta \times \gamma$, so that $\delta \in [\mathfrak{sl}(\Delta)]^J$ and $\gamma \in |\Gamma|^J$:

$$\vdash_J ([\Delta], \ \Gamma)\langle\nu\rangle = \vdash_J [\Delta\langle\!\langle\delta\rangle\!\rangle], \ \Gamma\langle\gamma\rangle \tag{4}$$

Here $\vdash_J ([\Delta], \ \Gamma)\langle\nu\rangle$ restricted to $\emptyset$ is $\vdash [\Delta], \Gamma$. Note that all the formulas in $\Gamma\langle\gamma\rangle$ have domain J, while each formula in $\Delta\langle\!\langle\delta\rangle\!\rangle$ has a domain contained in J. The $\Delta\langle\!\langle\delta\rangle\!\rangle$'s inside the stack become a list of pairwise dual MALL(I) formulas in which each pair has the same domain.

The translations commute with restriction of indices, and Lemma 2.2 can be restated:

Lemma 2.9 (Restricting translation).

- *For any $\gamma \in |\Gamma|^J$, it holds that $\Gamma\langle\gamma\upharpoonright_{J\cap K}\rangle = \Gamma\langle\gamma\rangle\upharpoonright_K$.*
- *For any $\delta \in [\mathfrak{sl}(\Delta)]^J$, it holds that $\Delta\langle\!\langle\delta\upharpoonright_{J\cap K}\rangle\!\rangle = \Delta\langle\!\langle\delta\rangle\!\rangle\upharpoonright_K$.*
- *If $\vdash_J [\ \Delta\langle\!\langle\delta\rangle\!\rangle\ ], \ \Gamma\langle\gamma\rangle$ is provable, then so is $\vdash_{J\cap K} [\ \Delta\langle\!\langle\delta\rangle\!\rangle\upharpoonright_{J\cap K}\ ], \ \Gamma\langle\gamma\rangle\upharpoonright_{J\cap K}$.*

2.3 Fundamental lemma

Indexed linear logic arises essentially due to its tight connection to the relational semantics. The connection is realised by a fundamental lemma due to Bucciarelli & Ehrhard (proposition 20 of Bucciarelli and Ehrhard 2000) establishing a correspondence between indexed sets in Rel and indexed sequents in MALL(I). The former is semantic in MALL, while the latter is syntactic in MALL(I). The fundamental lemma is shown to be preserved under our extended syntax and semantics, designed to accommodate cut formulas in $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ and in $\mathsf{Rel}^{[\mathfrak{c}]}$, respectively.

Proposition 2.10 (Fundamental lemma à la Bucciarelli–Ehrhard). *For $\nu \in (|\mathfrak{sl}(\Delta)| \times |\Gamma|)^J$ with $J \subseteq I$, the following two statements are equivalent and induce a relationship $\rho\upharpoonright_\emptyset = \pi$ between π of (i) and ρ of (ii):*

(i) There exists a $\mathsf{MALL}^{[\mathfrak{c}]}$ proof π such that

$$\nu \in |\pi_{[\Delta],\Gamma}|^J.$$

(ii) There exists a $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ proof ρ of the sequent

$$\vdash_J ([\Delta], \ \Gamma)\langle\nu\rangle.$$

Proof. See Lemmas B.1 and B.2 in the Appendix B.1. □

3. Lifting MALL Reduction over Indices

This section describes how our indexed syntax $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ analyses Gentzen-style reduction of cut elimination for nonindexed MALL. Every MALL reduction with cut elimination is shown to be lifted to a directed transformation between two $\mathsf{MALL}(I)$ proofs. These transformations diminish sets of the indices of proofs overall.

Definition 3.1 ($\mathsf{MALL}^{[\mathfrak{c}]}(I)$ proof transformation $\blacktriangleright^{I}$ with diminishing sets of indices). *A $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ transformation $\blacktriangleright^{I}$ with diminishing sets of indices, written as $\rho_{\vdash_J [\Delta],\,\Gamma} \blacktriangleright^{I} \rho'_{\vdash_{J'} [\Delta'],\,\Gamma}$, is a transformation from one $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ proof ρ for $\vdash_J [\Delta],\ \Gamma$ to another, ρ' for $\vdash_{J'} [\Delta'],\ \Gamma$ with $J' \subseteq J$, satisfying the following condition:*

- *(Restriction to the empty domain)*
  Restricting the two $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ proofs to $\emptyset$ gives rise to a $\mathsf{MALL}^{[\mathfrak{c}]}$ reduction $\pi_{[\Delta],\Gamma} \ \triangleright\ \pi'_{[\Delta'],\Gamma}$ by cut elimination.
  Schematically, this can be written as

$$
\begin{array}{ccc}
\rho_{\vdash_J [\Delta],\,\Gamma} & \blacktriangleright^{I} & \rho'_{\vdash_{J'} [\Delta'],\,\Gamma} \\
\Big\downarrow {\restriction_{\emptyset}} & & \Big\downarrow {\restriction_{\emptyset}} \\
\pi_{[\Delta],\Gamma} & \triangleright & \pi'_{[\Delta'],\Gamma}
\end{array}
$$

The transformation $\rho \blacktriangleright^{I} \rho'$ is called a lifting of $\pi \triangleright \pi'$. The lifting simulates a MALL proof reduction for cut elimination in terms of $\mathsf{MALL}(I)$ proof transformation.

The lifting in Definition 3.1 is not unique for a given $\mathsf{MALL}^{[\mathfrak{c}]}$ reduction, as for any subset J'' of J', $\rho \ \blacktriangleright^{I}\ \rho'\restriction_{J''}$ obviously becomes a lifting for ρ and ρ' under this definition.

Example 3.2. The following is a $\mathsf{MALL}^{[\mathfrak{c}]}(I)$ reduction with diminishing sets of indices whose restriction to $\emptyset$ is a Gentzen reduction eliminating the pairwise dual additive connectives & and $\oplus$ in the cut formulas:

$$
\dfrac{
\dfrac{\left\{ \begin{array}{c} \vdots\ \pi^{i} \\ \vdash_{J_i} [\Delta_i, \Omega\restriction_{J_i}], \Gamma\restriction_{J_i}, A_i \end{array} \right\}^{i=1,2}}{\vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Omega], \Gamma, A_1 \& A_2}\ \&
\qquad
\dfrac{\begin{array}{c} \vdots\ \pi^{3} \\ \vdash_{J_1+J_2} [\Delta_3], \Xi, A_1^{\perp} \end{array}}{\vdash_{J_1+J_2} [\Delta_3], \Xi, A_1^{\perp} \oplus A_2^{\perp}}\ \oplus_1
}{\vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Omega, \Delta_3, (A_1 \& A_2), (A_2^{\perp} \oplus A_1^{\perp})], \Gamma, \Xi}\ \mathrm{cut}
$$

$$
\blacktriangleright^{I} \qquad
\dfrac{\begin{array}{c} \vdots\ \pi^{1} \\ \vdash_{J_1} [\Delta_1, \Omega], \Gamma, A_1 \end{array} \quad \begin{array}{c} \vdots\ \pi^{3} \\ \vdash_{J_1} [\Delta_3\restriction_{J_1}], \Xi\restriction_{J_1}, A_1^{\perp}\restriction_{J_1} \end{array}}{\vdash_{J_1} [\Delta_1, \Omega, \Delta_3\restriction_{J_1}, A_1, A_1^{\perp}\restriction_{J_1}], \Gamma, \Xi}\ \mathrm{cut}
$$

The sets of the indices are diminished from $J_1 + J_2$ to J_1 as a result of erasing the subproof π^2 within the proof transformation.

Proposition 3.3 (Lifting to indexed transformation). *Let $\nu \in |\pi_{[\Delta],\Gamma}|^{J}$ and consider a MALL reduction $\pi_{[\Delta],\Gamma} \ \triangleright\ \pi'_{[\Delta'],\Gamma}$. Then there exist $J' \subseteq J$ and $\nu' \in |\pi_{[\Delta'],\Gamma}|^{J'}$ lifting the given reduction:*

$$\begin{array}{ccc} \rho_{\vdash_J([\Delta],\Gamma)\langle\nu\rangle} & \blacktriangleright^I & \rho'_{\vdash_{J'}([\Delta'],\Gamma)\langle\nu'\rangle} \\ \downarrow \upharpoonright_\emptyset & & \downarrow \upharpoonright_\emptyset \\ \pi_{[\Delta],\Gamma} & \rhd & \pi'_{[\Delta'],\Gamma} \end{array}$$

ρ and ρ' are MALL proofs ensured by the fundamental lemma (Proposition 2.10) for the sequents $\vdash_J ([\Delta],\ \Gamma)\langle\nu\rangle$ and $\vdash_{J'} ([\Delta'],\ \Gamma)\langle\nu'\rangle$, respectively. Hence, we can also denote the lifting by

$$\nu \in |\pi_{[\Delta],\Gamma}|^J \quad \blacktriangleright^I \quad \nu' \in |\pi'_{[\Delta'],\Gamma}|^{J'}.$$

Note: There is no straight connection between ν and ν' such as the former is the restriction to the latter.

Proof. For every kind of reduction $\rhd$, we can directly construct ν' together with J'. There are three crucial cases:
(Crucial case 1)

$$\dfrac{\overline{\vdash B, B^\perp} \qquad \begin{array}{c}\vdots\ \pi' \\ \vdash [\Delta], A, \Gamma\end{array}}{\vdash [\Delta, B^\perp, A], B, \Gamma}\ \text{cut}$$

(with A and B being different occurrences of the same formula) reduces to

$$\begin{array}{c}\vdots\ \pi' \\ \vdash [\Delta], B, \Gamma\end{array}$$

(identifying the occurrence of A with B).

Let $\epsilon \in |\mathrm{ax}_{B,B^\perp}|^J$ and $\tau \in |\pi'_{[\Delta],A,\Gamma}|^J$. Then, for each $j \in J$, we have $\epsilon_j = (b_j, b_j)$ with $b_j \in |B| = |B^\perp|$, and $\tau_j = (\delta_j, a_j, \lambda_j)$ with $\delta_j \in [\mathfrak{sl}(\Delta)]$, $a_j \in |A|$ and $\lambda_j \in |\Gamma|$. Note that $\nu_j = (\delta_j, b_j, a_j, b_j, \lambda_j)$. We define

$$J' = \{j \in J \mid b_j = a_j\} \quad \text{and} \quad \nu' = \tau \upharpoonright_{J'}.$$

(Crucial case 2)
This case is the MALL reduction arisen by Example 3.2 above, when restricting to the empty domain $\emptyset$ and identifying MALL($\emptyset$) with MALL.

By π's last rule, $\nu \cong \tau \times \lambda$ with $\tau \in |\&(\pi^1, \pi^2)|^J$ and $\lambda \in |\oplus_1(\pi^3)|^J$, where the conclusion of π^i with $i = 1, 2$ is $\vdash [\Delta_i, \Omega], \Gamma, A_i$ and that of π^3 is $\vdash [\Delta_3],\ \Xi, A_1^\perp$. Then the &-rule of the left premise divides J into $J = J_1 + J_2$. We define

$$J' = J_1 \quad \text{and} \quad \nu' = \tau \upharpoonright_{J_1} \times \lambda'$$
where $\lambda' \in |\pi^3|^{J'}$ is defined by $\lambda'_j := (x, a)$ if $\lambda_j = (x, (1, a))$.

(Crucial case 3) Here π

$$\dfrac{\begin{array}{c}\vdots\ \rho \\ \vdash [\Xi], \Delta, A\end{array} \qquad \dfrac{\left\{\begin{array}{c}\vdots\ \pi^i \\ \vdash [\Sigma_i, \Omega], A^\perp, \Gamma, B_i\end{array}\right\}^{i=1,2}}{\vdash [\Sigma_1, \Sigma_2, \Omega], A^\perp, \Gamma, B_1 \& B_2}\ \&}{\vdash [\Xi, \Sigma_1, \Sigma_2, \Omega, A, A^\perp], \Delta, \Gamma, B_1 \& B_2}\ \text{cut}$$

reduces to π'

$$\dfrac{\left\{ \dfrac{\begin{array}{c}\vdots\ \rho \\ \vdash [\Xi], \Delta, A\end{array} \quad \begin{array}{c}\vdots\ \pi^i \\ \vdash [\Sigma_i, \Omega], A^\perp, \Gamma, B_i\end{array}}{\vdash [\Xi, \Sigma_i, \Omega, A, A^\perp], \Delta, \Gamma, B_i} \text{ cut} \right\}^{i=1,2}}{\vdash [\Sigma_1, \Sigma_2, A, A^\perp,\ A, A^\perp, \Xi, \Omega], \Delta, \Gamma, B_1 \& B_2} \ \&$$

Note that in the last &-rule of π', Ξ and Ω inside the stack are chosen to be superposed.
By π's last rule, $\nu \cong \lambda \times \tau$, so $\lambda \in |\rho_{[\Xi],\Delta,A}|^J$ and $\tau \in |\&(\pi^1, \pi^2)|^J$. The last &-rule of the right premise divides J into $J = J_1 + J_2$ so that $\tau \cong \tau_1 \frown \tau_2$ and $\tau_i \in |\pi^i_{[\Sigma_i,\Omega],A^\perp,\Gamma,B_i}|^{J_i}$. Then $\lambda\lceil_{J_i} \times \tau_i \in_\cong |\operatorname{cut}(\rho, \pi^i)|^{J_i}$. We define

$$J' = J \quad \text{and} \quad \nu' \cong (\lambda\lceil_{J_1} \times \tau_1) \frown (\lambda\lceil_{J_2} \times \tau_2).$$

□

4. MALL GoI Interpretation

4.1 Execution formula with zero action on symmetries of cuts

4.1.1 Interpretation of indexed point in MALL proof

Our categorical framework is a minimal part of the Haghverdi–Scott GoI situation (Haghverdi and Scott 2006) with a reflexive object U in a traced symmetric monoidal category $\mathcal{C}$ with tensor unit I. Ours in addition requires that $\mathcal{C}$ has zero morphisms, in particular, a zero endomorphism 0_U on U:

$$(\mathcal{C},\ \otimes,\ I,\ s_{U,U},\ j: U \otimes U \lhd U: k,\ 0_U) \tag{5}$$

$s_{U,U}$ is a symmetry $U \otimes U \longrightarrow U \otimes U$ of tensor product. $j: U^2 \lhd U: k$ denotes a pair of morphisms j and k, respectively, from U^2 to U and the other way around. j and k are called, respectively, *co-retraction* and *retraction* for the reflexive U when $k \circ j = U \otimes U = U^2$. The m-ary tensor folding $\underbrace{\star \otimes \cdots \otimes \star}_{m}$ is denoted by $\star^m$ both for object $\star$ or morphism $\star$. The trace structure will be introduced later in (14).

We require the commutativity of the pair (j, k) and the zero 0_U;

$$k \circ 0_U \circ j = 0_U \otimes 0_U \tag{6}$$

Indeed, (6) is equivalent to the two commutativity $j \circ (0_U \otimes 0_U) = 0_U \circ j$ and $(0_U \otimes 0_U) \circ k = k \circ 0_U$.

Note: The zero morphism 0_U is absorbing with respect to composition, but not with respect to tensor. That is $f \otimes 0_U$ and $0_U \otimes f$ are not in general 0_{U^2} for any endomorphism f on U.

Lemma 4.1 (tensoring zero). $0_U \otimes 0_U = 0_{U^2}$. *More generally,* $0_U^m = 0_{U^m}$ *for any natural number* m.

Proof. The first assertion is derived by the condition (6). The general assertion is by iterating the condition $(k^n \otimes U) \circ (k \circ 0_U \circ j) \circ (j^n \otimes U) = 0_U^{n+2}$. □

The zero endomorphism 0_U acts on the symmetry s as follows.

Definition 4.2 (zero action $(s_{U,U})^0$). 0 *action on the symmetry* $s_{U,U}$ *on* U^2 *is defined to* annihilate *the symmetry* $s_{U,U}$ *to the zero endomorphism on* U^2.

$$(s_{U,U})^0 := 0_{U \otimes U}$$

Alternatively, the action is defined to be the following decomposition in terms of conjugation (both precomposing and composing):

$$(s_{U,U})^0 := (0_U \otimes 0_U) \circ s_{U,U} \circ (0_U \otimes 0_U) = 0_U \otimes 0_U = 0_{U \otimes U}$$

We abbreviate $s_{U,U}$ and $(s_{U,U})^0$ as s and s^0, respectively.

To avoid collapsing the categorical framework whose GoI interpretation becomes the degenerate zero, we assume

$$s \text{ is nonzero; that is, } s \text{ and } s^0 \text{ are distinct endomorphisms on } U^2 \text{ in } \mathcal{C}. \tag{7}$$

This is a technical assumption for the main theorem (Theorem 4.16) of this paper to characterise the diminution of the index set in terms of the convergence to the zero, distinguishable from the other morphisms.

The zero morphism, which is required in our framework, exists in crucial examples of GoI situations: (i) Rel_+ is Rel with the disjoint union $+$ of sets as $\otimes$ and a reflexive object $\mathbb{N}$. The *empty relation* on $\mathbb{N}$ is the zero morphism. Furthermore $0_{\mathbb{N}} + 0_{\mathbb{N}} = 0_{\mathbb{N}+\mathbb{N}}$ sufficient to the condition (6). (ii) The monoidal subcategories Pfn and PInj of Rel_+, both in which resides the zero morphism. PInj is known to be equivalent to the original category Hilb_2 of Hilbert spaces and partial isometries for Girard's GoI 1 (Girard 1989).

Note: The above examples of GoI situations happen to be sum-style monoidal structures (Haghverdi and Scott 2011), whose $\otimes$ is given by the disjoint union. The style is known to capture the notion of feedback as data flow in terms of streams of tokens around graphical networks. However our categorical framework (5) in the present paper is the general one, hence does not assume that the monoidal product is sum-style.

The ith constituent x_i of $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in |\pi_{[\Delta],\Gamma}|$ of Definition 2.6 corresponds, in terms of the membership relation, to the ith occurrence of formulas $\widehat{\Delta}, \Gamma$ with a unique sublist $\widehat{\Delta}$.

Lemma 4.3 (Tag of x_i with $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in |\widehat{\Delta}| \times |\Gamma|$ for $\boldsymbol{x} \in |\pi_{[\Delta],\Gamma}|$). *Every element $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in |\pi_{[\Delta],\Gamma}|$ interpreting π in Definition 2.6 belongs to $|\widehat{\Delta}| \times |\Gamma|$ with a unique sublist $\widehat{\Delta}$ of Δ. That is, for $\boldsymbol{x} = (x_1, \ldots, x_\ell)$, there exists a unique sublist $\widehat{\Delta}$ such that the ith constituent $x_i \in |A_i|$ for the ith formula A_i in $\widehat{\Delta}$ (resp. in Γ) when $i \leq 2m$ (resp. $i > 2m$), where $2m$ is the number of formulas in $\widehat{\Delta}$. The formula A_i is called the* tag *of the ith constituent x_i of $\boldsymbol{x}$ in $|\pi|$. Note the sublist $\widehat{\Delta}$ is determined not only by π but also by $\boldsymbol{x}$, as shown in the construction (8) in the following proof.*

Proof. As $|\pi_{[\Delta],\Gamma}| \cong |\pi_{[\Delta],\Gamma}|^{\{*\}}$, by (4) in Definition 2.8 when J is the singleton set $\{*\}$, every $\boldsymbol{x} = (x_1, \ldots, x_\ell)$ factors $\boldsymbol{x} = \boldsymbol{x}' \times \boldsymbol{x}''$ so that

$$\vdash_{\{*\}} ([\Delta], \Gamma)\langle \boldsymbol{x} \rangle = \vdash_{\{*\}} [\Delta\langle\!\langle \boldsymbol{x}' \rangle\!\rangle],\ \Gamma\langle \boldsymbol{x}'' \rangle$$

While all the formulas in the sequence $\Gamma\langle \boldsymbol{x}'' \rangle$ of $\mathsf{MALL}(I)$ formulas has domain $\{*\}$, each formula in the sequence $\Delta\langle\!\langle \boldsymbol{x}' \rangle\!\rangle$ has the domain either $\{*\}$ or $\emptyset$. Thus the unique sublist $\widehat{\Delta}$ is determined by the following two steps: (i) Ridding $\Delta\langle\!\langle \boldsymbol{x}' \rangle\!\rangle$ of all the formulas D such that $d(D) = \emptyset$, which yields the subsequence of $\Delta\langle\!\langle \boldsymbol{x}' \rangle\!\rangle$. (ii) $\widehat{\Delta}$ is defined to be the subsequence of (i) restricted to $\emptyset$ (i.e. forgetting the domain) in order to obtain non-indexed formulas.

To be short for (i) and (ii), the sublist $\widehat{\Delta}$ is the unique one making $\widehat{\Delta}\langle\!\langle \boldsymbol{x}' \rangle\!\rangle$ agree with $\Delta\langle\!\langle \boldsymbol{x}' \rangle\!\rangle$ ridden of all the formulas whose domain is $\emptyset$. (8)

Note whenever a cut formula occurs in $\widehat{\Delta}$, so does its dual formula, hence $\boldsymbol{x}' = (x_1, \ldots, x_{2m})$ for a natural number m, then $\boldsymbol{x}'' = (x_{2m+1}, \ldots, x_\ell)$. By Definitions 2.7 and 2.8, the membership relation for the assertion of x_i and the ith formula in $\widehat{\Delta}, \Gamma$ follows. □

In what follows, we make permutations $(x_{\tau(1)}, \ldots, x_{\tau(\ell)})$ amongst the constituents implicit so that $\boldsymbol{x}$ is up to the permutation. This is because the permutation corresponds to the exchange rule eliminated from our syntax. The permutations will be reflected by the symmetry of monoidal product of $\mathcal{C}$, in the following interpretation $[\![\boldsymbol{x}]\!]$, which we denote by $\cong$.

Definition 4.4 (Endomorphism $[\![\boldsymbol{x}]\!]$ on tensor folding U's and tensor folding $\sigma_{\boldsymbol{x}}$ of symmetry s and of zero s^0).

- *Every $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in |\pi_{[\Delta],\Gamma}|$ is interpreted as an endomorphism $[\![\boldsymbol{x}]\!]_\pi$ on the tensor product U^ℓ together with an endomorphism $\sigma^\pi_{\boldsymbol{x}}$ on a subfactor U^{2m} of U^ℓ. The endomorphism $\sigma^\pi_{\boldsymbol{x}}$ interprets cut rules in $\pi_{[\Delta],\Gamma}$ and is an m-ary tensor folding of morphisms which are either $s^1 = s$ or $s^0 = 0$ (cf. Definition 4.2) both on U^2:*

$$[\![\boldsymbol{x}]\!]_\pi : U^\ell \longrightarrow U^\ell \quad \text{and} \quad \sigma^\pi_{\boldsymbol{x}} = \otimes_{i=1}^m s^{\eta(i)} \quad \text{where } \eta_{\boldsymbol{x}} \text{ is a } \{0,1\}\text{-valued function.} \tag{9}$$

  To distinguish each ith component U of U^ℓ, we label each component U_{x_i} with x_i by abuse of notation, since the label x_i is always clearly specified to designate the ith constituent of $\boldsymbol{x}$. Then $x_i \in |A_i|$ so that A_i is the tag of x_i. Under this labelling, $\eta(i)$ of $s^{\eta(i)}$ on $U_a \otimes U_{a'}$ is defined to be the Kronecker delta $\delta_{a,a'}$, where the tags of a and a' are pairwise dual formulas in $\widehat{\Delta}$. That is,

$$\sigma^\pi_{\boldsymbol{x}} = \otimes\, s^{\delta_{a,a'}} \qquad \text{where } a \text{ and } a' \text{ range so that their tags are pairwise dual cut formulas in } \widehat{\Delta}. \tag{10}$$

  We define $([\![\boldsymbol{x}]\!]_\pi, \sigma^\pi_{\boldsymbol{x}})$ by induction on the construction of the proof π.
- *We simultaneously define that a component U_{x_i} such that the tag of x_i is a formula in Γ is* contracted *by the induction on π* [1]*. Since U_{x_i} appears both in the co-domain and in the domain of $[\![\boldsymbol{x}]\!]$, every contracted component in the domain (resp. co-domain) of $[\![\boldsymbol{x}]\!]$ is a domain (resp. co-domain) of a unique retraction (resp. co-retraction), called* associated retraction *(resp.* associated co-retraction*)* [2]*. We simply say i is* contracted *when so is U_{x_i}.*

In the definition, π_1 and π_2 denote the two premise proofs of the binary rules, and π' of the unary rules.

(Axiom)
$\boldsymbol{x} = (\bar{*}, *) \in |A^\perp, A|$ with $A = 1$ and $A^\perp = \perp$. We define $[\![\boldsymbol{x}]\!]_\pi$ to be a symmetry $s_{U_{\bar{*}}, U_*}$ on $U_{\bar{*}} \otimes U_*$ of $\mathcal{C}$. Because π is cut-free, $\sigma^\pi_{\boldsymbol{x}}$ is empty by definition.
$\boldsymbol{x}$ has no contracted component so that neither U_* nor $U_{\bar{*}}$ are contracted.

(Cut rule)
$\boldsymbol{x} = (\boldsymbol{v}^1, \boldsymbol{v}^2, a, a', \boldsymbol{w}^1, \boldsymbol{w}^2) \in |\widehat{\Delta_1}| \times |\widehat{\Delta_2}| \times |A| \times |A^\perp| \times |\Gamma_1| \times |\Gamma_2|$, so $\boldsymbol{x}_1 = (\boldsymbol{v}^1, \boldsymbol{w}^1, a)$ and $\boldsymbol{x}_2 = (\boldsymbol{v}^2, a', \boldsymbol{w}^2)$ belong respectively to $|\pi_1|$ and to $|\pi_2|$.
We define

$$[\![\boldsymbol{x}]\!]_\pi \cong [\![\boldsymbol{x}_1]\!]_{\pi_1} \otimes [\![\boldsymbol{x}_2]\!]_{\pi_2} \quad \text{and} \quad \sigma^\pi_{\boldsymbol{x}} \text{ is } \sigma^{\pi_1}_{\boldsymbol{x}_1} \otimes \sigma^{\pi_2}_{\boldsymbol{x}_2} \otimes (s_{U_a, U_{a'}})^{\delta_{a,a'}}$$

That is, if $a = a'$ (resp. $a \neq a'$), then $\sigma^\pi_{\boldsymbol{x}}$ on $U_a \otimes U_{a'}$ is s (resp. s^0), and $\sigma^\pi_{\boldsymbol{x}}$ on the remaining components is $\sigma_{\boldsymbol{x}_1} \otimes \sigma_{\boldsymbol{x}_2}$. Note the definition makes sense because $\sigma^{\pi_1}_{\boldsymbol{x}_1} \otimes \sigma^{\pi_2}_{\boldsymbol{x}_2}$ acts on the domain distinct both from U_a and $U_{a'}$.

We say the cut (of the last rule of π) *matches* (resp. *mismatches*) in $\boldsymbol{x}$ if $a = a'$ (resp. otherwise).

($\parr$-rule)
$\boldsymbol{x} = (\boldsymbol{v}, (a, b))$, so that $\boldsymbol{x}' = (\boldsymbol{v}, a, b) \in |\pi'|$. Note $A \parr B$ is the tag of (a, b) in π, while A (resp. B) is the tag of a (resp. b) in the premise. $[\![\boldsymbol{x}]\!]_\pi$ is obtained directly from $[\![\boldsymbol{x}']\!]_{\pi'}$ on $U^{\ell+1} = U^\ell \times U_{(a,b)}$

by the retraction $U_a \otimes U_b \lhd U_{(a,b)}$. That is, $[\![\, \boldsymbol{x} \,]\!]_\pi = [\![\, \boldsymbol{x'} \,]\!]_{\pi'}^{(j,k)} = (U^{\ell-1} \otimes j) \circ [\![\, \boldsymbol{x'} \,]\!]_{\pi'} \circ (U^{\ell-1} \otimes k)$. We also define $\sigma_{\boldsymbol{x}}^{\pi}$ by $\sigma_{\boldsymbol{x'}}^{\pi'}$.

($\otimes$-rule)
$\boldsymbol{x} = (\boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{w}^1, \boldsymbol{w}^2, (a,b)) \in |\widehat{\Delta_1}| \times |\widehat{\Delta_2}| \times |\Gamma_1| \times |\Gamma_2| \times |A| \times |B|$, so that $\boldsymbol{x}_1 = (\boldsymbol{v}^1, \boldsymbol{w}^1, a)$ and $\boldsymbol{x}_2 = (\boldsymbol{v}^2, \boldsymbol{w}^2, b)$ are respectively from $|\pi_1|$ and $|\pi_2|$. Note $A \otimes B$ is the tag of (a,b) in π, while A (resp. B) is the tag of a (resp. b) in π_1 (resp. in π_2). The endomorphism $[\![\, \boldsymbol{x} \,]\!]$ is obtained directly from $[\![\, \boldsymbol{x}_1 \,]\!]_{\pi_1} \otimes [\![\, \boldsymbol{x}_2 \,]\!]_{\pi_2}$ on $U^{\ell+1} = U^{\ell} \times U_{(a,b)}$ by the retraction $U_a \otimes U_b \lhd U_{(a,b)}$. That is, $[\![\, \boldsymbol{x} \,]\!]_\pi \cong ([\![\, \boldsymbol{x}_1 \,]\!]_{\pi_1} \otimes [\![\, \boldsymbol{x}_2 \,]\!]_{\pi_2})^{(j,k)} = (U^{\ell-1} \otimes j) \circ ([\![\, \boldsymbol{x}_1 \,]\!]_{\pi_1} \otimes [\![\, \boldsymbol{x}_2 \,]\!]_{\pi_2}) \circ (U^{\ell-1} \otimes k)$. We also define $\sigma_{\boldsymbol{x}}^{\pi}$ by $\sigma_{\boldsymbol{x}_1}^{\pi_1} \otimes \sigma_{\boldsymbol{x}_2}^{\pi_2}$.

In the above both multiplicatives rules ($\otimes$ and $\invamp$), the introduced $U_{(a,b)}$ in the domain (resp. co-domain) is a contracted component, and the assoc-ret (resp. assoc-coret) is $k : U_{(a,b)} \rhd U_a \otimes U_b$ (resp. $j : U_a \otimes U_b \lhd U_{(a,b)}$). Other contracted components are those of $\left[\!\left[\, (\boldsymbol{v}, a, b) \,\right]\!\right]$ for $\invamp$ and $\left[\!\left[\, (\boldsymbol{v}_1, \boldsymbol{w}_1, a) \,\right]\!\right]$ and $\left[\!\left[\, (\boldsymbol{v}_2, \boldsymbol{w}_2, b) \,\right]\!\right]$ for $\otimes$ distinct from the components U_a and U_b. Note that $(\boldsymbol{v}, a, b) \in |\text{premise of } \invamp|$ and $(\boldsymbol{v}_1, \boldsymbol{w}_1, a) \in |\text{left premise of} \otimes|$ and $(\boldsymbol{v}_2, \boldsymbol{w}_2, b) \in |\text{right premise of} \otimes|$.

(&-rule)
$\boldsymbol{x}$ is either $(\boldsymbol{v}, (1,a))$ or $(\boldsymbol{v}, (2,a))$, so that $(\boldsymbol{v}, a)$ are either from $|\pi_1|$ or $|\pi_2|$, respectively. Note $A_1 \& A_2$ is the tag of (i,a) in π, while A_i is the tag of a in π_1 or in π_2 when $i=1$ or $i=2$, respectively. We define $[\![\, \boldsymbol{x} \,]\!]_\pi = \left[\!\left[\, (\boldsymbol{v}, a) \,\right]\!\right]_{\pi_i}$ by relabelling the component U_a either by $U_{(1,a)}$ or $U_{(2,a)}$ for the domain (equally for the codomain) of $[\![\, \boldsymbol{x} \,]\!]_\pi$. We also define $\sigma_{\boldsymbol{x}}^{\pi}$ by $\sigma_{(\boldsymbol{v},a)}^{\pi_i}$.

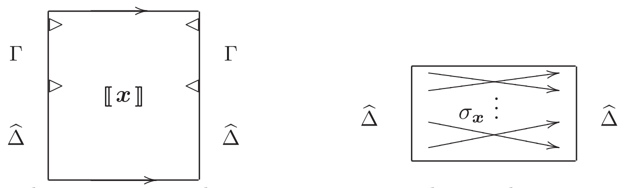
($\oplus_i$-rule) Same as &-rule but using the unique premise π' deterministically.

In the above both additive rules (& and $\oplus_i$), contracted components are those of $\left[\!\left[\, (\boldsymbol{v}, a) \,\right]\!\right]$ under the relabelling U_a by $U_{(i,a)}$ for the component of the domain (equally of the codomain). Note that $(\boldsymbol{v}, a)$ belongs to one of |left premise| and |right premise| in &-rule depending on $i=1$ or $i=2$, and obviously to |the unique premise| in $\oplus_i$-rule.

In the sequel, the pair of Definition 4.4 is simply written $([\![\, \boldsymbol{x} \,]\!], \sigma_{\boldsymbol{x}})$ by omitting π, since the proof π will be always specified clearly from the context.

(**Remark on Definition 4.4**) [**The endomorphism $[\![\, \boldsymbol{x} \,]\!]$ as an I/O box**]
The endomorphism $[\![\, \boldsymbol{x} \,]\!]$ is seen as an input/output (I/O) box on the $(n+2m)$-ary tensor folding of U, whose inputs/outputs are the formulas occurring in $\Gamma, \widehat{\Delta}$, in which Γ contains n occurrences of formulas, and a sublist $\widehat{\Delta}$ contains $2m$ occurrences of (pairwise dual) formulas. The formulas are the tags of x_is where $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in |\pi_{[\Delta],\Gamma}|$.
The endomorphism $\sigma_{\boldsymbol{x}}$ is seen as a more special box consisting of m-ary tensor folding of $\{s, s^0\}$ for the I/O formulas in the sublist $\widehat{\Delta}$. See Figure 1 below for $([\![\, \boldsymbol{x} \,]\!], \sigma_{\boldsymbol{x}})$.

A characterisation of contracted component is derived:



$[\![\, \boldsymbol{x} \,]\!]$ with assoc-rets $\rhd$ and assoc-corets $\lhd$ $\quad$ $\sigma_{\boldsymbol{x}}$ whose each cross is either s or s^0

Figure 1. $([\![\, \boldsymbol{x} \,]\!], \sigma_{\boldsymbol{x}})$ of Definition 4.4 and assoc-rets and assoc-corets of Definition 4.4.

Lemma 4.5. *U_{x_i} is a contracted component of $[\![\boldsymbol{x}]\!]$ for $\boldsymbol{x} \in [\![\pi]\!]$ if and only if x_i's tag contains a multiplicative connective (i.e. $\otimes$ or $⅋$).*

Proof. Straightforward accordingly to the inductive step of Definition 4.4, in which the retraction and the co-retraction are used only for multiplicatives rules ($⅋$ or $\otimes$) so that $[\![\boldsymbol{x}]\!]$ is constructed $[\![\boldsymbol{x}']\!]^{(j,k)}$ or $([\![\boldsymbol{x}_1]\!] \otimes [\![\boldsymbol{x}_2]\!])^{(j,k)}$, respectively. $\square$

(**Labelling associated retractions $\rhd_i$ and co-retractions $\rhd_i$**)
Every associated retraction $\rhd$ (resp. associated co-retraction $\lhd$) is by Definition 4.4 uniquely labelled with a contracted i such that the tag of x_i is a formula in Γ. The labelling is written $\rhd_i$ (resp. $\lhd_i$). In what follows, the labelling is made implicit except when an explicit labelling makes an explanation easier to understand.

(**Assoc-rets and assoc-corets in I/O box $[\![\boldsymbol{x}]\!]$**)
When the endomorphism $[\![\boldsymbol{x}]\!]$ is seen as the I/O box, the assoc-rets and the assoc-corets are those $\rhd$'s and $\lhd$'s whose domains and co-domains lie respectively amongst the inputs and the outputs of $[\![\boldsymbol{x}]\!]$. By the construction of $[\![\boldsymbol{x}]\!]$, they lie pairwise in the inputs and the outputs. See Figure 1 for $[\![\boldsymbol{x}]\!]$ depicting the occurrence of the assoc-rets $\rhd$'s and the assoc-corets $\lhd$'s.

(**Convention omitting** Id_U**s**) When an indicated occurrence of a contracted component U is clear from the context, $\rhd \circ [\![\boldsymbol{x}]\!]$ (resp. $[\![\boldsymbol{x}]\!] \circ \lhd$) is an abbreviation for the composition $(\mathrm{Id}_U \otimes \cdots \otimes \mathrm{Id}_U \otimes \rhd \otimes \mathrm{Id}_U \otimes \cdots \otimes \mathrm{Id}_U) \circ [\![\boldsymbol{x}]\!]$ (resp. $[\![\boldsymbol{x}]\!] \circ (\mathrm{Id}_U \otimes \cdots \otimes \mathrm{Id}_U \otimes \lhd \otimes \mathrm{Id}_U \otimes \cdots \otimes \mathrm{Id}_U)$, where the domain of $\rhd$ (resp. the codomain of $\lhd$) is the contracted U. This abbreviation is generalised for plural indicated occurrences of contracted components $U_1, \ldots U_r$ in U^ℓ as follows: $(\otimes^r \rhd) \circ [\![\boldsymbol{x}]\!]$ (resp. $[\![\boldsymbol{x}]\!] \circ (\otimes^r \lhd)$) stands for the composition (resp. precomposition) to $[\![\boldsymbol{x}]\!]$ by the morphism tensoring $\rhd$ (resp. $\lhd$) on the contracted components indicated and Id_U on the remaining components. Note because $r \leq \ell$, the abbreviation is for omitting identities on Us.

In the sequel, the two abbreviations are pictured as above respectively. Using a notation of the r-ary tensor folding of $\rhd$ (resp. $\lhd$), it is also written by $\rhd^r \circ [\![\boldsymbol{x}]\!]$ (resp. $[\![\boldsymbol{x}]\!] \circ \lhd^r$). This convention is equally employed when indicated occurrences of assoc-(co)rets are clear from the context.

Under this convention, for any contracted component in the co-domain (resp. domain) of $[\![\boldsymbol{x}]\!]$, it holds;

$$\lhd \circ \rhd \circ [\![\boldsymbol{x}]\!] = [\![\boldsymbol{x}]\!] \qquad (\textit{resp.} \quad [\![\boldsymbol{x}]\!] \circ \lhd \circ \rhd = [\![\boldsymbol{x}]\!]) \tag{11}$$

Figure 2. Equation (11).

That is, $\lhd \circ \rhd$ composes (resp. precomposes) with any retraction (resp. co-retraction) as the identity. In other word, $\lhd \circ \rhd$ is a projector on a contracted component in the codomain (resp. domain) by composition (resp. precomposition). Pictorially,

In Equation (11), the assoc-coret (resp. assoc-ret) occurs explicitly as the last composed $\lhd$ (resp. the first precomposed $\rhd$). Thus, the leftmost $\lhd$ (resp. rightmost $\rhd$) in (11) is seen labelled $\lhd_i$ (resp. $\rhd_i$) such that the tag of x_i is a formula occurrence in Γ, where $\boldsymbol{x} = (x_1, \ldots, x_\ell) \in |\pi_{[\Delta],\Gamma}|$. Since every contracted component occurs pairwise in the co-domain and the domain of $[\![\boldsymbol{x}]\!]$, the two equations can be written successively all at once;

$$\lhd_i \circ \rhd \circ [\![\boldsymbol{x}]\!] \circ \lhd \circ \rhd_i = [\![\boldsymbol{x}]\!]$$

The co-domain and the domain U^ℓ of $[\![\boldsymbol{x}]\!]$ have in general several contracted components Us labelled with x_is, where i ranges in the set $\mathfrak{r}$ of the contraced is. Thus, for all the several pairs of contracted components in the domain and the co-domain of $[\![\boldsymbol{x}]\!]$, the parallel compositions and precompositions with $\lhd \circ \rhd$s to each contracted components act as the identity on $[\![\boldsymbol{x}]\!]$:

$$\begin{aligned} [\![\boldsymbol{x}]\!] &= (\otimes_{i \in \mathfrak{r}} (\lhd_i \circ \rhd)) \circ [\![\boldsymbol{x}]\!] \circ (\otimes_{i \in \mathfrak{r}} (\lhd \circ \rhd_i)) \\ &= (\lhd \circ \rhd)^r \circ [\![\boldsymbol{x}]\!] \circ (\lhd \circ \rhd)^r = (\lhd^r \circ \rhd^r) \circ [\![\boldsymbol{x}]\!] \circ (\lhd^r \circ \rhd^r) \end{aligned} \tag{12}$$

where r is the cardinality of $\mathfrak{r}$, hence is a number of the contracted components, and the third equality is by $(\lhd \circ \rhd)^r = \lhd^r \circ \rhd^r$, as $(-)^r$ is the r-ary tensor folding. Since the last composed $\lhd^r$ (resp. the first precomposed $\rhd^r$) in the rightmost expression of (12) are the explicit occurrences of the assoc-rets (resp. assoc-corets), the endomorphism $[\![\boldsymbol{x}]\!]$ is written so that all the assoc-rets $\rhd^r$ and the assoc-corets $\lhd^r$ can be made explicit as follows:

$$[\![\boldsymbol{x}]\!] = \lhd^r \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}} \circ \rhd^r \qquad \text{where} \qquad [\![\boldsymbol{x}]\!]^{\mathrm{o}} = \rhd^r \circ [\![\boldsymbol{x}]\!] \circ \lhd^r \tag{13}$$

Roughly speaking, $[\![\boldsymbol{x}]\!]^{\mathrm{o}}$ is $[\![\boldsymbol{x}]\!]$ stripped of all the assoc-rets and assoc-corets. which is depicted in the following Figure 3:

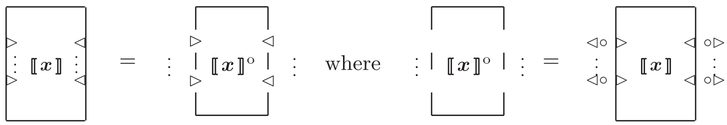


Figure 3. Equation (13).

4.1.2 The action $\epsilon_{\boldsymbol{x}}$ annihilating associated (co)retractions

This subsection is concerned with defining the action $\epsilon_{\boldsymbol{x}}$ (Definition 4.8) over the associated retractions (resp. co-retractions) in Definition 4.4 above. The action $\epsilon_{\boldsymbol{x}}$ arises from $\sigma_{\boldsymbol{x}}$ of (9) when the feedback on the trace of $\mathcal{C}$ is taken into account, and annihilates, using the zero morphism 0_U, a certain class of retractions and co-retractions. This class is defined in Definition 4.8 in terms of zero input and output.

In what follows, we shall see how feedback stemming from Gentzen cut elimination for a MALL proof π acts on the assoc-rets and the assoc-corets of $[\![\boldsymbol{x}]\!]$ for $\boldsymbol{x} \in |\pi_{[\Delta],\Gamma}|$. The action is stipulated in terms of the zero morphism added in our framework. First, in a categorical framework of Girard's GoI project, the feedback is modelled by the trace structure (cf. Haghverdi and Scott 2006) defined by the seven axioms below:

$$\mathsf{Tr}^Z_{X,Y} : \mathcal{C}(X \otimes Z, Y \otimes Z) \longrightarrow \mathcal{C}(X, Y) \tag{14}$$

There are three kinds of naturality axioms: *naturality* in X and *naturality* in Y, and *dinaturality* in Z. The other axioms are *vanishing I,II*, *superposing* and *yanking*. See Appendix A.1 for the seven axioms: the three naturalities and the four axioms.

In our setting of Definition 4.4, the endomorphism $[\![\boldsymbol{x}]\!]$ is on U^{n+2m} so that n and $2m$ are the numbers of formulas respectively in Γ and in a sublist $\widehat{\Delta}$, and $\sigma_{\boldsymbol{x}}$ is on the subfactor U^{2m}. Then the feedback is calculated by

$$\mathsf{ex}\,(\sigma, \boldsymbol{x}) := \mathsf{Tr}^{U^{2m}}_{U^n, U^n} \left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!] \right) \tag{15}$$

See Figure 4.

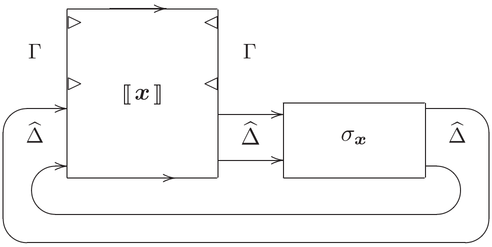


Figure 4. Equation(15): ex $(\sigma, \boldsymbol{x})$ with feedback.

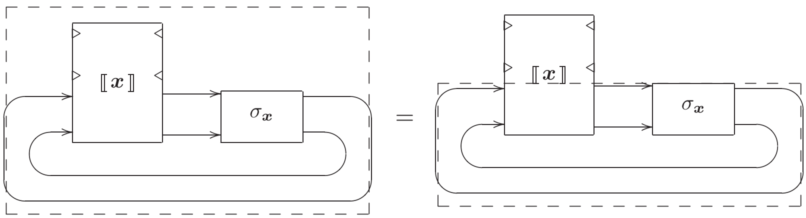


Figure 5. Equation (16): naturality of assoc-rets and assoc-corets.

Note that when $\boldsymbol{x} \in |\pi_{[\Delta],\Gamma}|$ comes from a proof π of the multiplicative fragment, the equation is exactly the GoI interpretation of the proof $\pi_{[\Delta],\Gamma}$ (cf. Haghverdi and Scott 2006). This is because in the multiplicative fragment, the index set I becomes redundantly the singleton $\{*\}$, thus $|\pi_{[\Delta],\Gamma}| = \{\boldsymbol{x}\}$, whereby $\sigma_{\boldsymbol{x}}$ is a simple tensor folding of the symmetry s (free of 0 morphism).

By the naturalities of traces, the assoc-corets (resp. the assoc-rets) of $[\![\boldsymbol{x}]\!]$ commute with $\mathsf{Tr}^{U^{2m}}_{U^n,U^n}$, hence taking a trace of (11) composed with $\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}$ yields for any i such that ith component of $\boldsymbol{x} = (x_1, \ldots, x_\ell)$ is contracted.

$$\begin{aligned}\mathsf{Tr}^{U^{2m}}_{U^n,U^n}((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ (\lhd_i \circ \rhd \circ [\![\boldsymbol{x}]\!])) &= \lhd_i \circ \mathsf{Tr}^{U^{2m}}_{U^n,U^n}((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ (\rhd \circ [\![\boldsymbol{x}]\!]))\\ &= \lhd_i \circ \rhd \circ \mathsf{ex}(\sigma, \boldsymbol{x})\\ (\text{resp. } \mathsf{Tr}^{U^{2m}}_{U^n,U^n}((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ ([\![\boldsymbol{x}]\!] \circ \lhd \circ \rhd_i)) &= \mathsf{ex}(\sigma, \boldsymbol{x}) \circ \lhd \circ \rhd_i).\end{aligned}$$

Thus all the assoc-rets and assoc-corets of $[\![\boldsymbol{x}]\!]$ are written explicitly;

$$\begin{aligned}\mathsf{Tr}^{U^{2m}}_{U^n,U^n}((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]) &= (\otimes_{i\in\mathfrak{r}} \lhd_i) \circ \mathsf{Tr}^{U^{2m}}_{U^n,U^n}\left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}}\right) \circ (\otimes_{i\in\mathfrak{r}} \rhd_i)\\ &= \lhd^r \circ \mathsf{Tr}^{U^{2m}}_{U^n,U^n}\left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}}\right) \circ \rhd^r \qquad (16)\end{aligned}$$

where $\mathfrak{r}$ is the set of all contracted is and r is the cardinality of $\mathfrak{r}$.
Equation(16) is depicted in Figure 5, in which the dotted squares are the scopes of the traces and the shifting of the scopes are naturalities of the $\rhd$s and the $\lhd$s.

While inside the sole $[\![\boldsymbol{x}]\!]$, the assoc-rets and the assoc-corets (written explicitly in (13)) do not interact with zero morphisms because the construction of $[\![\boldsymbol{x}]\!]$ of Definition 4.4 is free from the zero morphisms. Remember that the zero morphisms reside only in $\sigma_{\boldsymbol{x}}$ as subfactors (cf. (9)). However, when they are put inside the context $\mathsf{Tr}^{U^{2m}}_{U^n,U^n}((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ -)$ (written explicitly in (16)), they may interact with zero morphisms arising from $\sigma_{\boldsymbol{x}}$ via the feedback of the trace. That is, the trace in a monoidal category takes feedback into account, hence makes the zeros stemming from $\sigma_{\boldsymbol{x}}$ interact with the assoc-rets and the assoc-corets of $[\![\boldsymbol{x}]\!]$. This yields a certain action $\epsilon_{\boldsymbol{x}}$ on the assoc-(co)rets of $[\![\boldsymbol{x}]\!]$, as defined in Definition 4.8 below.

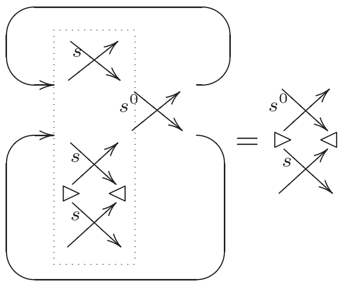
Definition 4.6 (zero input (resp. output) of assoc-coret (resp. assoc-ret) with respect to the interpretation $\sigma_{\boldsymbol{x}}$ of cuts).
*(**zero input of assoc-coret** $\lhd$) An assoc-coret $\lhd$ of $[\![\boldsymbol{x}]\!]$ is said to have* zero input *with respect to $\sigma_{\boldsymbol{x}}$ when $\lhd$ decomposes in* ex $(\sigma, \boldsymbol{x})$ *either as $\lhd \circ (0_U \otimes U)$ or as $\lhd \circ (U \otimes 0_U)$.*

*(**zero output of assoc-ret** $\rhd$) An assoc-ret $\rhd$ of $[\![\boldsymbol{x}]\!]$ is said to have* zero output *with respect to $\sigma_{\boldsymbol{x}}$ when $\rhd$ decomposes in* ex $(\sigma, \boldsymbol{x})$ *either as $(0_U \otimes U) \circ \rhd$ or as $(U \otimes 0_U) \circ \rhd$.*

Why do we use the terminology zero input (resp. output)? The object $U \otimes U$ of $\lhd$'s domain (resp. $\rhd$'s co-domain) can be regarded as having two inputs (resp. outputs), one left component U and the other right one. Then the decomposition in each case says that one of two inputs (resp. outputs) is zero.

Pictorially, $\overset{0_U}{\underset{U}{\rightrightarrows}}\lhd$ or $\overset{U}{\underset{0_U}{\rightrightarrows}}\lhd$ for the zero input and $\rhd\overset{0_U}{\underset{U}{\rightrightarrows}}$ or $\rhd\overset{U}{\underset{0_U}{\rightrightarrows}}$ for the zero output.

Note that Definition 4.6 is alternatively stated as follows: When the assoc-coret (resp. assoc-ret) is written explicitly as ex $(\sigma, \boldsymbol{x}) = \lhd \circ \mathsf{g}$ (resp. ex $(\sigma, \boldsymbol{x}) = \mathsf{g} \circ \rhd$) (cf. (11)), the assoc-coret $\lhd$ (resp. assoc-ret $\rhd$) has zero input (resp. output) iff either $0_U \otimes U$ or $U \otimes 0_U$ acts trivially on g by composing (resp. precomposing) to the indicated component $U \otimes U$.



Example 4.7. Let $\pi_{[A \& A, A^\perp \oplus A^\perp]\ A^\perp, A \otimes B, B^\perp}$ be a proof obtained by a $\otimes$-rule between π_1 of Appendix C (the first paragraph) and $ax_{B^\perp, B}$. Let $\boldsymbol{x} := \nu_2 \times (\bar{\star}, \star) \in |\pi|$, where ν_2 is in Appendix C and $(\bar{\star}, \star) \in |ax_{B^\perp, B}|$. Then $[\![\boldsymbol{x}]\!]$ has the unique pair of assoc-ret and assoc-coret both interpreting the $\otimes$-rule. See the left-hand dotted rectangle representing $[\![\boldsymbol{x}]\!]$ with the assoc-ret and the assoc-coret. The pair of assoc-ret and assoc-coret appears explicitly in the first and the second of the following equations (in which $\sigma_{\boldsymbol{x}} = s^0_{U,U}$):

$$\mathsf{Tr}^{U^2}_{U^3, U^3}((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]) = \lhd \circ \mathsf{Tr}^{U^2}_{U^4, U^4}\left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}}\right) \circ \rhd$$
$$= \lhd \circ (s^0_{U,U} \otimes s_{U,U}) \circ \rhd,$$

where $[\![\boldsymbol{x}]\!]^{\mathrm{o}}$ is $[\![\boldsymbol{x}]\!]$ without the assoc-ret and the assoc-coret.

See the above figure whose LHS and RHS are the first and the last equations, respectively, The assoc-coret (resp. assoc-ret) has zero input (resp. output) because the right picture depicts $\lhd$ (resp. $\rhd$) having a zero input (resp. output) from the northwest (resp. to the northeast). Hence $0_U \otimes U$ composes (resp. precomposes) to $s^0_{U,U} \otimes s_{U,U}$ trivially.

Definition 4.8 (action $\epsilon_{\boldsymbol{x}}$ on assoc-rets and assoc-corets of $[\![\boldsymbol{x}]\!]$). *The endomorphism $\sigma_{\boldsymbol{x}}$ of Definition 4.4 for $\boldsymbol{x} \in |\pi_{[\Delta], \Gamma}|$ yields the following action $\epsilon_{\boldsymbol{x}}$ on the assoc-rets and the assoc-corets of $[\![\boldsymbol{x}]\!]$. The action $\epsilon_{\boldsymbol{x}}$ acts on each assoc-ret and assoc-coret as* either zero *or the* identity *by (pre)composition on them, as follows:*

$$\rhd^{\epsilon_{\boldsymbol{x}}} = \begin{cases} \rhd^0 & \text{if } \rhd \text{ has a zero output w.r.t } \sigma_{\boldsymbol{x}} \\ \rhd & \text{otherwise} \end{cases} \qquad \lhd^{\epsilon_{\boldsymbol{x}}} = \begin{cases} \lhd^0 & \text{if } \lhd \text{ has a zero input w.r.t } \sigma_{\boldsymbol{x}} \\ \lhd & \text{otherwise} \end{cases}$$

where zero actions $\rhd^0$ and $\lhd^0$ *are defined respectively as follows:*

$$\rhd^0 := 0_{U, U \otimes U} = k \circ 0_U = (0_U \otimes 0_U) \circ k \qquad \lhd^0 := 0_{U \otimes U, U} = 0_U \circ j = j \circ (0_U \otimes 0_U)$$

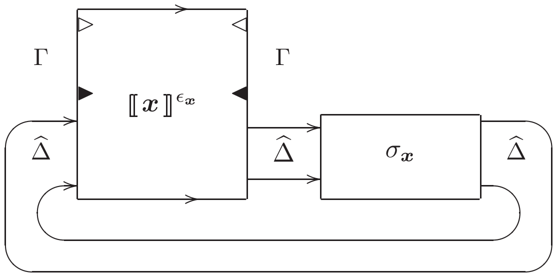


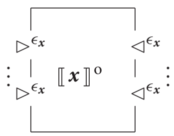
Figure 6. Execution formula Ex $(\sigma, \boldsymbol{x})$, where ▶ (resp. ◀) denotes $\rhd^0$ (resp. $\lhd^0$).

That is, the zero annihilates *the pair of assoc-ret and assoc-coret* $j : U \otimes U \lhd U : k$ *to the pair of the zero morphisms* $j^0 : U \otimes U \lhd^0 U : k^0$, *where* $j^0 = \lhd^0$ *and* $k^0 = \rhd^0$.

The action $\epsilon_{\boldsymbol{x}}$ of Definition 4.8 is by definition conjugate on the pairwise tensor foldings $(\otimes_{i\in\mathfrak{r}} \rhd_i, \otimes_{i\in\mathfrak{r}} \lhd_i) = (\rhd^r, \lhd^r)$ of the assoc-rets and the assoc-corets represented in (13), where $\mathfrak{r} = \{i \mid$ i is contracted with $\boldsymbol{x} = (x_1, \ldots, x_\ell)\}$. Hence we may formulate the action on $[\![\boldsymbol{x}]\!]$ by conjugation:

$$[\![\boldsymbol{x}]\!]^{\epsilon_x} := (\lhd^{\epsilon_x})^r \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}} \circ (\rhd^{\epsilon_x})^r \tag{17}$$

Pictorially,

$[\![\boldsymbol{x}]\!]^{\epsilon_x} :=$ 

This action of $\epsilon_{\boldsymbol{x}}$, by naturalities, extends to the action $\epsilon_{\boldsymbol{x}}$ on the corresponding retractions and co-retractions in (16):

$$\begin{aligned}
(\mathsf{Tr}^{U^{2m}}_{U^n,U^n} ((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]))^{\epsilon_x} &:= (\lhd^{\epsilon_x})^r \circ \mathsf{Tr}^{U^{2m}}_{U^n,U^n} \left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}}\right) (\rhd^{\epsilon_x})^r \\
&= \mathsf{Tr}^{U^{2m}}_{U^n,U^n} \left((\lhd^{\epsilon_x})^r \circ (\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}} \circ (\rhd^{\epsilon_x})^r\right) && \text{by nats} \\
&= \mathsf{Tr}^{U^{2m}}_{U^n,U^n} \left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ (\lhd^{\epsilon_x})^r \circ [\![\boldsymbol{x}]\!]^{\mathrm{o}} \circ (\rhd^{\epsilon_x})^r\right) && \text{by (16)} \\
&= \mathsf{Tr}^{U^{2m}}_{U^n,U^n} \left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\epsilon_x}\right) && \text{by (17)}
\end{aligned}$$

Note by (15) that the LHS of the first equation is $\mathsf{ex}\,(\sigma, \boldsymbol{x})^{\epsilon_x}$. Recall that r is the number of the assoc-rets $\rhd_i$ (equally the assoc-corets $\lhd_i$) of $[\![\boldsymbol{x}]\!]$ such that i is contracted.

4.1.3 The Execution formula

Definition 4.9 (Execution formula $\mathsf{Ex}\,(\sigma, \boldsymbol{x})$ for $\boldsymbol{x} \in |\pi_{[\Delta],\Gamma}|$). *For every* $\boldsymbol{x} \in |\pi_{[\Delta],\Gamma}|$, *the endomorphism Ex* $(\sigma, \boldsymbol{x})$ *is defined by*

$$\begin{aligned}
Ex\,(\sigma, \boldsymbol{x}) &:= \mathsf{ex}\,(\sigma, \boldsymbol{x})^{\epsilon_x} \\
&= Tr^{U^{2m}}_{U^n,U^n} \left((\mathrm{Id} \otimes \sigma_{\boldsymbol{x}}) \circ [\![\boldsymbol{x}]\!]^{\epsilon_x}\right),
\end{aligned}$$

where $([\![\boldsymbol{x}]\!], \sigma_{\boldsymbol{x}})$ *is the pair of the endomorphism on* U^{n+2m} *and on the subfactor* U^{2m} *in Definition 4.4 and* $\epsilon_{\boldsymbol{x}}$ *is the action in Definition 4.8 on the assoc-rets and the assoc-corets of* $[\![\boldsymbol{x}]\!]$. *The domains (resp. the co-domains) of the assoc-rets (resp. the assoc-corets) lie amongst the subfactor* U^n *in the domain (resp. the co-domain) of* $[\![\boldsymbol{x}]\!]$. *See Figure 6.*

Example 4.10. Let $\boldsymbol{x}$ be of Example 4.7. Since $\epsilon_{\boldsymbol{x}}$ acts as zero both on the unique assoc-ret $\rhd$ and on the unique assoc-coret $\lhd$, $\mathsf{Ex}\,(\sigma, \boldsymbol{x}) = \lhd^0 \circ (s^0_{U,U} \otimes s_{U,U}) \circ \rhd^0 = 0_{U^3,U^3}$.

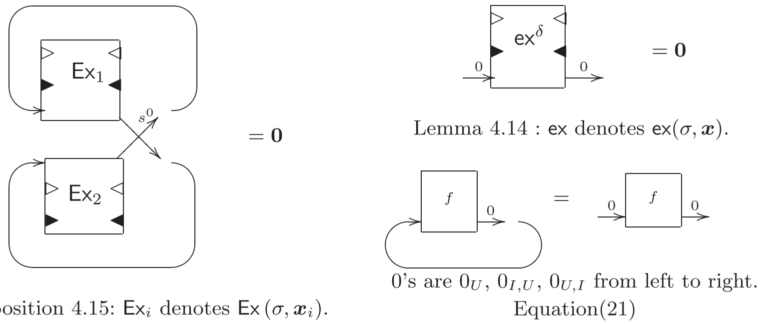


Figure 7. Proposition 4.15 and Lemma 4.14 and Equation (21) pictorially, where ▶ and ◀ denote respectively $\triangleright^0$ and $\triangleleft^0$.

Finally, the execution formula is run pointwise for every enumerated set ν in interpretation of a proof in $\mathsf{Rel}^{[\mathfrak{c}]}$.

Definition 4.11 (Execution formula $\mathsf{Ex}_J(\sigma, \nu)$ for $\nu \in |\pi_{[\Delta],\Gamma}|^J$). *Let $\pi_{[\Delta],\Gamma}$ be a MALL$^{[\mathfrak{c}]}$ proof. For every $\nu \in |\pi_{[\Delta],\Gamma}|^J$, $Ex(\sigma, \nu) \in |\Gamma|^J$ is defined indexwise by:*

$$(\, Ex_J(\sigma, \nu)\,)_j = Ex\left(\sigma, \nu_j\right) \quad \textit{for every index } j \in J$$

4.2 Zero convergence of execution formula

This subsection concerns the main proposition (Proposition 4.15), which says that communicating two proofs via mismatched pair yields zero convergence of Ex. We start with the tracing zero lemma derivable from some trace axioms.

Lemma 4.12 (Tracing zero). *For any natural number $n \geq 1$,*

$$Tr^U_{U^n,U^n}\left(0_{U^{n+1}}\right) = 0_{U^n} \tag{18}$$

Proof. First by Lemma 4.1 $0_{U^m} = (0_U)^m$ for any natural number m. Then by superposing $\mathsf{Tr}^U_{U^{n+1},U^{n+1}}\left(0_{U^{n+2}}\right) = 0_{U^n} \otimes \mathsf{Tr}^U_{U,U}\left(0_{U^2}\right)$, it suffices to prove the assertion for $n = 1$. Second, observe the equation[3]

$$(0_U \otimes U) \circ s_{U,U} \circ (0_U \otimes U) = 0_U \otimes 0_U = (U \otimes 0_U) \circ s_{U,U} \circ (U \otimes 0_U) \tag{19}$$

Thus $\mathsf{Tr}^U_{U,U}\left(0_{U^2}\right) = 0_U \circ \mathsf{Tr}^U_{U,U}\left(s_{U,U}\right) \circ 0_U = 0_U \circ U \circ 0_U$, where the first equation is by naturalities and the second equation is by yanking. □

We prepare the following Lemma 4.14, which will directly entail the main Proposition 4.15.

Definition 4.13 (action δ_x). *For $\boldsymbol{x} \in |\pi_{[\Delta],\Gamma}|$, let us put $[\![\boldsymbol{x}]\!]$ into the context $(\,\mathrm{Id} \otimes 0_U) \circ (\,-\,) \circ (\,\mathrm{Id} \otimes 0_U)$, allowing interaction of the assoc-rets and the assoc-corets of $[\![\boldsymbol{x}]\!]$ with the two zeros 0_U in the context. Zero input (resp. zero output) of assoc-ret (resp. assoc-coret) in this context is defined in the same manner, yielding the action, say $\delta_{\boldsymbol{x}}$, on the assoc-rets and the assoc-corets of $[\![\boldsymbol{x}]\!]$ same as in Definition 4.8 (but simpler without the feed back): That is, the morphism $\triangleright^{\delta_{\boldsymbol{x}}}$ is defined to be $\triangleright^0$ (resp. $\triangleright$) if $\triangleright$ decomposes in $(\,\mathrm{Id} \otimes 0_U) \circ [\![\boldsymbol{x}]\!] \circ (\,\mathrm{Id} \otimes 0_U)$ either as $(0_U \otimes U) \circ \triangleright$ or $(U \otimes 0_U) \circ \triangleright$ (resp. otherwise). Symmetrically, $\triangleleft^{\delta_{\boldsymbol{x}}}$ is defined to be $\triangleleft^0$ (resp. $\triangleleft$) if $\triangleleft$ decomposes in $(\,\mathrm{Id} \otimes 0_U) \circ [\![\boldsymbol{x}]\!] \circ (\,\mathrm{Id} \otimes 0_U)$ either as $\triangleleft \circ (0_U \otimes U)$ or $\triangleleft \circ (U \otimes 0_U)$ (resp. otherwise).*

Lemma 4.14 (lemma for Proposition 4.15).

$$(\,\mathrm{Id} \otimes 0_U) \circ \ \mathsf{ex}\,(\sigma, \boldsymbol{x})^{\delta_{\boldsymbol{x}}} \circ (\,\mathrm{Id} \otimes 0_U) = 0_{U^n},$$

where $\mathsf{ex}\,(\sigma, \boldsymbol{x})^{\delta_x}$ *is* $\mathsf{ex}\,(\sigma, \boldsymbol{x})$ *of (15) whose* $[\![\, \boldsymbol{x} \,]\!]$ *is replaced by* $[\![\, \boldsymbol{x} \,]\!]^{\delta_x}$ *using the action* $\delta_{\boldsymbol{x}}$ *of Definition 4.13.*

See Figure 7 (upper right) depicting the equation. The lemma holds up to the the permutations τ *on* U^n *so that the left* $\mathsf{ex}\,(\sigma, \boldsymbol{x})$ *is read by* $\tau^{-1} \circ \mathsf{ex}\,(\sigma, \boldsymbol{x}) \circ \tau$*. Hence the assertion is independent of the choice of* U *for the* 0_U*. The choice is of one formula occurrence from* Γ*, as each occurrence is interpreted by the distinct* U*.*

Proof. Induction on the construction of π for $\boldsymbol{x}$ in Definition 4.4. In the proof, Equation(19) in the proof of Lemma 4.12 is used. In the following, for $i = 1, 2$, $\boldsymbol{x}_i$ are the premises of $\boldsymbol{x}$ (i.e. $\boldsymbol{x}_1 = \boldsymbol{y}$ and $\boldsymbol{x}_2 = \boldsymbol{z}$ in Definition 4.4), and ex_i denotes $\mathsf{ex}\,(\sigma, \boldsymbol{x}_i)$.

(axiom)
$(\,\mathrm{Id} \otimes 0_U) \circ [\![\, ax \,]\!] \circ (\,\mathrm{Id} \otimes 0_U) = (\,\mathrm{Id} \otimes 0_U) \circ s_{U,U} \circ (\,\mathrm{Id} \otimes 0_U) = 0_U \otimes 0_U.$

($\otimes$-rule)(case 1) U is introduced by the $\otimes$-rule.

$$(\,\mathrm{Id}_1 \otimes 0_U \otimes \mathrm{Id}_2\,) \circ \lhd \circ (\mathsf{ex}_1 \otimes \mathsf{ex}_2) \circ \rhd \circ (\,\mathrm{Id}_1 \otimes 0_U \otimes \mathrm{Id}_2\,)$$
$$= \lhd \circ (((\,\mathrm{Id}_1 \otimes 0_U) \circ\ \mathsf{ex}_1 \circ (\,\mathrm{Id}_1 \otimes 0_U)) \otimes ((\,\mathrm{Id}_2 \otimes 0_U) \circ\ \mathsf{ex}_2 \circ (\,\mathrm{Id}_2 \otimes 0_U)) \circ \rhd = 0_{U^{n_1}} \otimes 0_{U^{n_2}}$$

The last equation is by I.H.'s on $[\![\, \boldsymbol{x}_1 \,]\!]$ and $[\![\, \boldsymbol{x}_2 \,]\!]$.

($\otimes$-rule) (case 2) other than case 1:
In this case, the U for the 0_U of $\mathrm{Id} \otimes 0_U$ is a factor from the (co)domain of $[\![\, \boldsymbol{x}_i \,]\!]$. We assume without loss of generality that $i = 1$. Then, $(\,\mathrm{Id} \otimes 0_U) \circ\ \mathsf{ex}_1 \circ (\,\mathrm{Id} \otimes 0_U) = 0_{U^{n_1}}$ by I.H on $\boldsymbol{x}_1$. This directly implies that the co-retraction and the retraction (j, k) interpreting the $\otimes$-rule are acted by δ as zero, denoted by (j^0, k^0), since j's output and k's input both on $[\![\, \boldsymbol{x}_1 \,]\!]$ are zeros by the I.H. Hence, when (j, k) is written by $(\lhd, \rhd)$,

$$\lhd^0 \circ (0_{U^{n_1}} \otimes \mathsf{ex}_2) \circ \rhd^0 = \lhd \circ (0_{U^{n_1}} \otimes (0_U \otimes \mathrm{Id}\,) \circ\ \mathsf{ex}_2 \circ (0_U \otimes \mathrm{Id}\,)) \circ \rhd$$
$$= \lhd \circ (0_{U^{n_1}} \otimes 0_{U^{n_2}}\,) \circ \rhd = 0_{U^n}$$

The first equation is by the assumption and the second equation is by I.H. on $[\![\, \mathrm{x}_2 \,]\!]$.

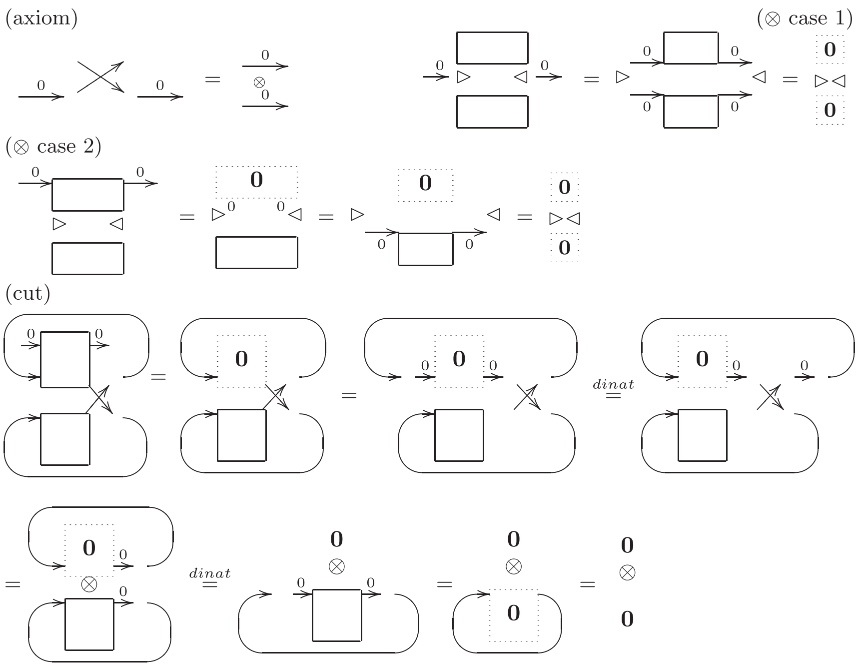
(cut rule)
By the rule,

$$\mathsf{ex}(\sigma, \boldsymbol{x}) = \mathsf{Tr}^{U^2}_{U^n, U^n}\left((\,\mathrm{Id}_1 \otimes s_{U,U} \otimes \mathrm{Id}_2\,) \circ (\mathsf{ex}_1 \otimes \mathsf{ex}_2)\right)$$

We assume without loss of generality that the U for the 0_U of the $\mathrm{Id} \otimes 0_U$ is a factor from the (co)domain of $[\![\, \boldsymbol{x}_1 \,]\!]$. Then, LHS of the assertion is equal to

$$
\begin{aligned}
&(\,\mathrm{Id} \otimes 0_U) \circ \mathsf{Tr}^{U^2}_{U^n, U^n}\left((\,\mathrm{Id}_1 \otimes s_{U,U} \otimes \mathrm{Id}_2\,) \circ (\mathsf{ex}^\delta_1 \otimes \mathsf{ex}^\delta_2)\right) \circ (\,\mathrm{Id} \otimes 0_U) \\
&= \mathsf{Tr}^{U^2}_{U^n, U^n}\left((\,\mathrm{Id} \otimes 0_U) \circ ((\,\mathrm{Id}_1 \otimes s_{U,U} \otimes \mathrm{Id}_2\,) \circ (\mathsf{ex}^\delta_1 \otimes \mathsf{ex}^\delta_2)) \circ (\,\mathrm{Id} \otimes 0_U)\right) && \text{naturalities} \\
&= \mathsf{Tr}^{U^2}_{U^n, U^n}\left((\,\mathrm{Id}_1 \otimes s_{U,U} \otimes \mathrm{Id}_2\,) \circ ((\,\mathrm{Id} \otimes 0_U) \circ\ \mathsf{ex}^\delta_1\ \circ (\,\mathrm{Id} \otimes 0_U)) \otimes \mathsf{ex}^\delta_2)\right) && \text{by the asm.} \\
&= \mathsf{Tr}^{U^2}_{U^n, U^n}\left((\,\mathrm{Id}_1 \otimes s_{U,U} \otimes \mathrm{Id}_2\,) \circ (0_{U^{n_1+1}} \otimes \mathsf{ex}^\delta_2)\right) && \text{I.H. on } [\![\, \mathrm{x}_1 \,]\!] \\
&= \mathsf{Tr}^{U^2}_{U^n, U^n}\left((\,\mathrm{Id}_1 \otimes (0_U \otimes U) \circ s_{U,U} \circ (0_U \otimes U) \otimes \mathrm{Id}_2\,) \circ (0_{U^{n_1+1}} \otimes \mathsf{ex}^\delta_2)\right) && \text{dinaturality} \\
&= \mathsf{Tr}^{U}_{U^{n_1}, U^{n_1}}\left(0_{U^{n_1+1}}\right) \otimes \mathsf{Tr}^{U}_{U^{n_2}, U^{n_2}}\left((0_U \otimes \mathrm{Id}\,) \circ\ \mathsf{ex}^\delta_2\right) && \text{(19) and superposing} \\
&= \mathsf{Tr}^{U}_{U^{n_1}, U^{n_1}}\left(0_{U^{n_1+1}}\right) \otimes \mathsf{Tr}^{U}_{U^{n_2}, U^{n_2}}\left((0_U \otimes \mathrm{Id}\,) \circ\ \mathsf{ex}^\delta_2 \circ (0_U \otimes \mathrm{Id}\,)\right) && \text{dinaturality} \\
&= 0_{U^{n_1}} \otimes 0_{U^{n_2}} && \text{I.H. on } [\![\, \boldsymbol{x}_2 \,]\!]
\end{aligned}
$$

The first dinaturality is via the decomposition $0_{U^{n_1+1}} = (U^{n_1} \otimes 0_U) \circ\ 0_{U^{n_1}} \circ\ (U^{n_1} \otimes 0_U)$ and the second dinaturality is via the decomposition $0_U = 0_U \circ\ 0_U$.



In all the cases above, the upper (resp. lower) square represents ex_1 (resp. ex_2), and each $\mathbf{0}$ inside the dotted square arises by an induction hypothesis.

Figure 8. Pictorial Proof of Lemma 4.14.

(⅋-rule and additives)
Direct from the construction.
See Figure 8 for a pictorial proof depicting the above rewriting in each case. □

Proposition 4.15 (Mismatch gives rise to zero convergence of Ex). *For two MALL proofs $\pi^i_{[\Delta_i],\Gamma_i,A_i}$ with $i = 1, 2$, let $\boldsymbol{x}_i = \lambda_i \times (\gamma_i, a_i) \in |\pi^i_{[\Delta_i],\Gamma_i,A_i}|$ so that $a_i \in |A_i|$ with $a_1 \neq a_2$ and A_1 and A_2 are pairwise dual formulas. Then*

$$Ex\left(\sigma_{a_1,a_2}, Ex(\sigma, \boldsymbol{x}_1) \otimes Ex(\sigma, \boldsymbol{x}_2)\right) = 0_{U^{n_1+n_2}} \qquad \text{where } \sigma_{a_1,a_2} = s^{\delta_{a_1,a_2}}$$

Note that the LHS of the assertion is, by Definition 4.9, the following, in which ϵ is the action arising from $\sigma_{a_1,a_2} = s^0_{U_{a_1},U_{a_2}}$ of Definition 4.8:

$$\mathsf{Tr}^{U^{2(1+m_1+m_2)}}_{U^{n_1+n_2},U^{n_1+n_2}}\left((\mathrm{Id} \otimes \sigma_{a_1,a_2} \otimes \sigma_{\boldsymbol{x}_1} \otimes \sigma_{\boldsymbol{x}_2}) \circ (\llbracket \boldsymbol{x}_1 \rrbracket^{\epsilon_{x_1}} \otimes \llbracket \boldsymbol{x}_2 \rrbracket^{\epsilon_{x_1}})^{\epsilon}\right) \tag{20}$$
$$= \mathsf{Tr}^{U^{2(1+m_1+m_2)}}_{U^{n_1+n_2},U^{n_1+n_2}}\left((\mathrm{Id} \otimes \sigma_{a_1,a_2}) \circ ((\mathsf{Ex}(\sigma, \boldsymbol{x}_1) \otimes \mathsf{Ex}(\sigma, \boldsymbol{x}_2))^{\epsilon}\right) \quad \text{by nats and vanish IIs}$$

In the above $\llbracket \boldsymbol{x}_j \rrbracket$ is an endomorphism on $U^{2(m_j+1)+n_j}$ with the subfactor U^{2m_j} for $\sigma_{\boldsymbol{x}_j}$.

Before the proof of Proposition 4.15, let us observe a general equation derivable from certain trace axioms (dinaturality and yanking), where $f : X \otimes U \longrightarrow Y \otimes U$ and $0_{U,I}$ (resp. $0_{I,U}$) is the zero morphism from U to the tensor unit I (resp. the other way around).

$$\mathsf{Tr}^U_{X,Y}\left((\mathrm{Id} \otimes 0_U) \circ f\right) = (\mathrm{Id} \otimes 0_{U,I}) \circ f \circ (\mathrm{Id} \otimes 0_{I,U}) \tag{21}$$

Note first that the zero morphisms $0_{U,I}$ and $0_{I,U}$ above are derivable from 0_U using the trace:

$$0_{U,I} = \mathsf{Tr}^U_{U,I}\left(U \otimes U \stackrel{0_U \otimes 0_U}{\longrightarrow} U \otimes U \stackrel{j}{\longrightarrow} U \cong I \otimes U\right)$$

$$0_{I,U} = \mathsf{Tr}^U_{U,I}\left(I \otimes U \cong U \stackrel{k}{\longrightarrow} U \otimes U \stackrel{0_U \otimes 0_U}{\longrightarrow} U \otimes U\right)$$

See Figure 7 (lower right) depicting Equation (21).

(proof of (21))
By the decomposition $0_U = 0_{I,U} \circ 0_{U,I}$, the LHS is $\mathsf{Tr}^I_{X\otimes I, Y \otimes I}\left((\mathrm{Id} \otimes 0_{U,I}) \circ f \circ (\mathrm{Id} \otimes 0_{I,U})\right)$, by dinaturality, which is equal to the RHS by vanishing. (end of proof of (21))

Finally we go to:

Proof of Proposition 4.15. We prove the following instance of the proposition using Equation (20), where $n = n_1 + n_2$, since $\sigma_{a_1,a_2} = s^0 = 0^2_U$:

$$\mathsf{Tr}^{U^2}_{U^n,U^n}\left((\mathrm{Id} \otimes 0^2_U) \circ (\mathsf{Ex}_1 \otimes \mathsf{Ex}_2)^\epsilon\right) = 0_{U^{n_1+n_2}} \qquad \text{where } \mathsf{Ex}_i = \mathsf{Ex}\,(\sigma, \boldsymbol{x}_i)$$

For this, it suffices to show the following stronger equation, as ex_i is Ex_i ridden of the zero action on the assoc-rets and the assoc-corets (cf. Definition 4.8):

$$\mathsf{Tr}^{U^2}_{U^n,U^n}\left((\mathrm{Id} \otimes 0^2_U) \circ (\mathsf{ex}_1 \otimes \mathsf{ex}_2)^\epsilon\right) = 0_{U^{n_1+n_2}} \qquad \text{where } \mathsf{ex}_i = \mathsf{ex}\,(\sigma, \boldsymbol{x}_i)$$

By superposing (after the distribution of ϵ over $\otimes$), the LHS is equal to

$$\mathsf{Tr}^U_{U^{n_1},U^{n_1}}\left((\mathrm{Id} \otimes 0_U) \circ \mathsf{ex}^\epsilon_1\right) \otimes \mathsf{Tr}^U_{U^{n_2},U^{n_2}}\left((\mathrm{Id} \otimes 0_U) \circ \mathsf{ex}^\epsilon_2\right) \tag{22}$$

On the other hand, Lemma 4.14 and (21) say for all $i = 1, 2$

$$\mathsf{Tr}^U_{U^{n_i},U^{n_i}}\left((0_U \otimes \mathrm{Id}) \circ \mathsf{ex}^\delta_i\right) = 0_{U^{n_i}}$$

Since the two actions ϵ and δ coincide again by (21), the formula (22) becomes equal to $0_{U^{n_1}} \otimes 0_{U^{n_2}} = 0_{U^n}$. □

4.3 Main theorem

This section concerns the main theorem of this paper.

Theorem 4.16 (Ex is invariant and diminishes sets of indices).
Let $\nu \in |\pi_{[\Delta],\Gamma}|^J \blacktriangleright^I \nu' \in |\pi'_{[\Delta'],\Gamma}|^{J'}$ be any MALL$^{[c]}(I)$ proof transformation. Then

(i)

$$Ex_J\,(\sigma, \nu)\restriction_{J'} = Ex_{J'}\left(\sigma, \nu'\right) \qquad \text{and} \qquad \forall j \in J \setminus J' \quad Ex\left(\sigma, \nu_j\right) = 0$$

(ii) In particular, when π' is cut-free so that Δ' is empty, then

$$Ex_J\,(\sigma, \nu)\restriction_{J'} = \left[\!\left[\, \nu' \,\right]\!\right] \qquad \text{and} \qquad J' = \{j \in J \mid Ex\left(\sigma, \nu_j\right) \neq 0\ \}$$

Proof. We prove (i) according to the cases of Proposition 3.3, since (ii) follows directly from (i) as follows: For a cut-free π', $\sigma_{\nu'}$ is empty, hence $\mathsf{Ex}\left(\sigma, \nu_j\right) = \left[\!\left[\, \nu'_j \,\right]\!\right]$, where $\left[\!\left[\, \nu'_j \,\right]\!\right] \neq 0$ holds directly both from the construction of Definition 4.4 and from the non-collapsing assumption (7).

The invariance of (i) is direct by the yanking axiom in case 1, and by induction on the proof π in cases 2 and 3: This proof method directly comes as an instance of the known method in the symmetric traced monoidal category modelling multiplicative GoI (Haghverdi and Scott 2006).

Thus we prove the zero convergence for the diminution of J. The following crucial cases are those of the proof of Proposition 3.3.

(Crucial case 1)
Each instance of ν at $j \in J \setminus J'$ is $\nu_j = (\delta_j, b_j, a_j, b_j, \lambda_j)$, so that $a_j \neq b_j$ since $j \notin J'$, then $\mathsf{Ex}\left(\sigma, \nu_j\right) = 0$ by Proposition 4.15.

(Crucial case 2)
$J = J_1 + J_2$ diminishes into J_1. Each instance of τ at $j_2 \in J_2$ is $\tau_{j_2} = (\omega, \delta_2, \gamma, (2, a_2)) \in |\&(\pi^1, \pi^2)|$. Thus each instance of ν at $j_2 \in J_2$ is $\nu_{j_2} = (\omega, \delta_2, \delta_3, (2, a_2), (1, a_1), \gamma, \xi) \in |\pi|$. Since $(2, a_2) \neq (1, a_1)$, we have $\mathsf{Ex}\left(\sigma, \nu_{j_2}\right) = 0$ by Proposition 4.15.

(Crucial case 3)
J does not diminish in this case. □

Appendix C is read as an elucidating example of Theorem 4.16.

5. Conclusion and Future Work

This paper offers two main contributions:

(i) Presenting an indexed MALL system for stacking cut formulas and its relational counterpart to simulate MALL proof reduction of cut elimination.
(ii) Constructing an execution formula for the interpretation of MALL proofs equipped with indices. The MALL proof reduction is characterised by the convergence of the execution formula to the denotational interpretation. Furthermore, the zero convergence of the execution formula characterises the diminution of indices, which is specific to additive cut elimination.

Our explicit use of indexed-syntactical manipulations directly overcomes known difficulties in additive GoI. We hope that this paper, from the perspective of indexed linear logic, will shed light on an approachable understanding of the preceding literature on additive GoI, from precursory ones (Duchesne 2009; Girard 1995) to more recent developments (Girard 2011; Seiller 2016).

We discuss some future directions.
For a genuine MALL GoI without bypassing via indexed logic, a syntax-free counterpart is required to replace the indices. We construct such a genuine GoI (Hamano 2018) using an algebraic ingredient: a scalar extension of Girard's $*$-algebra of partial isometries over a boolean polynomial semi-ring. The genuine GoI may help us connect our syntactic manipulation of indices to Girard's semantic use of clauses for predicates in the precursory MALL GoI (Girard 1995).

In a syntactic direction, the status of Gentzen cut elimination for MALL(I) remains open since the present paper only concerns lifting the image to the indices of MALL cut reduction. The status will complement the reduction-free cut elimination, known to be derivable from the Fundamental lemma 2.10 (cf. Bucciarelli and Ehrhard 2000, 2001; Hamano and Takemura 2008).

Extending the present paper to the exponentials is challenging to use Bucciarelli–Ehrhard LL(I) (Bucciarelli and Ehrhard 2001) for modelling GoI. This will involve extending our methodology of a traced monoidal category with a zero morphism to the whole GoI situation (Haghverdi and Scott 2006, 2011), compatibly with the multisets interpretation of the exponential connective in Rel. The explicit accommodation of the indices to GoI will give a novel approach to the (non indexed) GoI modelling for the exponentials.

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Notes

1 For the choice of x_i (i.e. a choice of a formula (not in the cut list Δ but) in Γ), the construction is free from cut.
2 assoc-ret (resp. assoc-coret) for short.
3 In a more general setting, the natural iso $(b \otimes U) \circ s_{U,U} \circ (a \otimes U) \cong a \otimes b \cong (U \otimes a) \circ s_{U,U} \circ (U \otimes b)$, for any endomorphisms a and b on U.

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Appendix A. Axioms of Traced Monoidal Category

Definition A.1 (Trace axioms of the family $\mathsf{Tr}^Z_{X,Y}$ of (14) Haghverdi and Scott 2006; Joyal et al. 1996).

(1) ***(Natural** in X)*

$$Tr^Z_{X,Y}\left(f\right) \circ g = Tr^Z_{X',Y}\left(f \circ (g \otimes Z)\right) \quad where \quad f : X \otimes Z \longrightarrow Y \otimes Z \quad and \quad g : X' \to X$$

(2) ***(Natural** in Y)*

$$g \circ Tr^Z_{X,Y}\left(f\right) = Tr^Z_{X',Y}\left((g \otimes Z) \circ f\right) \quad where \quad f : X \otimes Z \longrightarrow Y \otimes Z \quad and \quad g : Y \to Y'$$

(3) ***(Dinatural** in Z)*

$$Tr^Z_{X,Y}\left((Y \otimes g) \circ f\right) = Tr^{Z'}_{X,Y}\left(f \circ (X \otimes g)\right) \quad where \quad f : X \otimes Z \longrightarrow Y \otimes Z \quad and \quad g : Z' \to Z$$

(4) ***(Vanishing I**)*

$$Tr^I_{X,Y}\left(f \otimes I\right) = f \quad where \quad f : X \longrightarrow Y$$

(5) ***(Vanishing II**)*

$$Tr^{Z \otimes W}_{X,Y}\left(f\right) = Tr^Z_{X,Y}\left(Tr^W_{X \otimes Z, Y \otimes Z}\left(f\right)\right) \quad where \quad f : X \otimes Z \otimes W \longrightarrow Y \otimes Z \otimes W$$

(6) (**Superposing**)

$$g \otimes Tr^Z_{X,Y}\left(f\right) = Tr^Z_{W\otimes X, V\otimes Y}\left(g \otimes f\right) \quad \textit{where} \quad f : X \otimes Z \longrightarrow Y \otimes Z \quad \textit{and} \quad g : W \longrightarrow V$$

(7) (**Yanking**)

$$Tr^X_{X,X}\left(s_{X,X}\right) = X \quad \textit{for the symmetry} \quad s_{X,X} : X \otimes X \longrightarrow X \otimes X$$

Lemma A.2 **(Generalised Yanking Haghverdi and Scott 2011).** *Let $s_{Z,Y}$ denote the symmetry from $Z \otimes Y$ to $Y \otimes Z$.*

$$Tr^Z_{X,Y}\left(s_{Z,Y} \circ (f \otimes g)\right) = g \circ f \qquad \textit{where} \quad f : X \longrightarrow Z \quad \textit{and} \quad g : Z \longrightarrow Y$$

Proof. $LHS \overset{nat}{=} \mathrm{Tr}^Z_{Z,Y}\left(s_{Z,Y} \circ (Z \otimes g)\right) \circ f \overset{dinat}{=} \mathrm{Tr}^Y_{Z,Y}\left((Y \otimes g) \circ s_{Z,Y}\right) \circ f$. Inside the trace $(Y \otimes g) \circ s_{Z,Y} = s_{Y,Y} \circ (g \otimes Y)$, thus $\mathrm{Tr}^Y_{Z,Y}\left(s_{Y,Y} \circ (g \otimes Y)\right) \overset{nat}{=} \mathrm{Tr}^Y_{Y,Y}\left(s_{Y,Y}\right) \circ g \overset{yank}{=} g$ □

Appendix B. Omitted Proofs

B.1 Proof for Proposition 2.10 (Fundamental Lemma)

Lemma B.1 ((i) implies (ii)). *Let $\pi_{[\Delta],\Gamma}$ be a proof of a sequent $\vdash [\Delta],\ \Gamma$ in MALL$^{[\mathfrak{c}]}$. Let $\delta \times \gamma \in |\pi_{[\Delta],\Gamma}|^J$ (for some $J \subseteq I$) with $\delta \in [\mathfrak{sl}(\Delta)]^J$ and $\gamma \in |\Gamma|^J$. The sequent $\vdash_J [\Delta\langle\!\langle\delta\rangle\!\rangle],\ \Gamma\langle\gamma\rangle$ has a proof ρ in MALL$^{[\mathfrak{c}]}(I)$ such that $\rho\lceil_\emptyset = \pi$.*

Proof. By construction on the MALL proof π. The proof figures are referred in Definition 2.6.

(cut rule)
$\delta \times \gamma \cong \delta_1 \times \delta_2 \times \gamma_1 \times \gamma_2$ with $\delta_1 \times \gamma_1 \in \pi^1_{[\Delta_1],\Gamma_1,A}$ and $\delta_2 \times \gamma_2 \in \pi^2_{[\Delta_2],A^\perp,\Gamma_2}$. By I.H's on $(\delta_i \times \gamma_i)$ s, there are MALL(I)-proofs of the sequents $\vdash_J [\Delta_1\langle\!\langle\delta_1\rangle\!\rangle],\ \Gamma_1\langle\gamma_1'\rangle, A\langle\gamma_1''\rangle$ and $\vdash_J [\Delta_2\langle\!\langle\delta_2\rangle\!\rangle],\ A^\perp\langle\gamma_2'\rangle, \Gamma_2\langle\gamma_2''\rangle$ with $\gamma_i = \gamma_i' \times \gamma_i''$. Note that $A\langle\gamma_1''\rangle$ and $A^\perp\langle\gamma_2'\rangle$ are dual formulas since they have the same domain J. Hence, the cut between the dual formulas is applied to prove $\vdash_J [\ \Delta_1\langle\!\langle\delta_1\rangle\!\rangle, \Delta_2\langle\!\langle\delta_2\rangle\!\rangle, A\langle\gamma_1''\rangle, A^\perp\langle\gamma_2'\rangle\ ],\ \Gamma_1\langle\gamma_1'\rangle, \Gamma_2\langle\gamma_2''\rangle$. The assertion follows since $\Delta_1\langle\!\langle\delta_1\rangle\!\rangle \times \Delta_2\langle\!\langle\delta_2\rangle\!\rangle = (\Delta_1, \Delta_2)\langle\!\langle\delta_1 \times \delta_2\rangle\!\rangle$.

(&-rule)
Let $\nu = |\pi_{[\Delta],\Gamma}|$. Then $\nu = \{(x_1, z, y, (1, a_1)) \mid (x_1, z, y, a_1) \in \nu_1\} + \{(x_2, z, y, (2, a_2)) \mid (x_2, z, y, a_2) \in \nu_2\} \cong \nu_1 \frown \nu_2$ with $\gamma_i \in |\pi^i_{[\Delta_i,\Delta],\Gamma,A_i}|^{J_1}$ and $J = J_1 + J_2$. By I.H's on π^i s, there are MALL(I)-proofs of $\vdash_{J_i} [\ \Delta_i\langle\!\langle\delta_i'\rangle\!\rangle, \Delta\langle\!\langle\delta_i''\rangle\!\rangle\ ],\ \Gamma\langle\gamma_i'\rangle, A_i\langle\gamma_i''\rangle$ with $\delta_i = \delta_i' \times \delta_i''$ and $\gamma_i = \gamma_i' \times \gamma_i''$. Because $\Gamma\langle\gamma_1' \frown \gamma_2'\rangle\lceil_{J_i} = \Gamma\langle\gamma_i'\rangle$ and $\Delta\langle\!\langle\delta_1'' \frown \delta_2''\rangle\!\rangle\lceil_{J_i} = \Delta\langle\!\langle\delta_i''\rangle\!\rangle$ by Lemma 2.2, the &-rule is applied to prove $\vdash_{J_1+J_2} [\ \Delta_1\langle\!\langle\delta_1'\rangle\!\rangle, \Delta_2\langle\!\langle\delta_2'\rangle\!\rangle, \Delta\langle\!\langle\delta_1'' \frown \delta_2''\rangle\!\rangle\ ],\ \Gamma\langle\gamma_1' \frown \gamma_2'\rangle,\ A_1\langle\gamma_1''\rangle \,\&\, A_2\langle\gamma_2''\rangle$. □

Lemma B.2 ((ii) implies (i)). *Let $\vdash [\Delta],\ \Gamma$ be a sequent of MALL$^{[\mathfrak{c}]}$. Let $\nu \in (\mathfrak{sl}(\Delta) \times \Gamma)^J$ (for some $J \subseteq I$) and let ρ be a proof of $\vdash_J ([\Delta],\ \Gamma)\langle\nu\rangle$ in MALL$^{[\mathfrak{c}]}(I)$. Then $\nu \in |(\rho\lceil_\emptyset)_{[\Delta],\Gamma}|^J$*

Proof. By the construction on the MALL(I) proof ρ.

(cut rule)

$$\rho \text{ is } \quad \frac{\overset{\rho^1}{\vdash_J [\Delta_1],\ \Gamma_1, A} \qquad \overset{\rho^2}{\vdash_J [\Delta_2],\ A^\perp, \Gamma_2}}{\vdash_J [\Delta_1, \Delta_2, A, A^\perp],\ \Gamma_1, \Gamma_2}\ \text{cut}\ .$$

The conclusion is written as $\vdash_J ([\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2)\langle\nu\rangle$. From the construction, $\nu \cong \nu_1 \times \nu_2$ so that the conclusions of ρ^1 and ρ_2 are, respectively, $\vdash_J ([\Delta_1], \Gamma_1, A)\langle\nu_1\rangle$ and $\vdash_J ([\Delta_2], A^\perp, \Gamma_2)\langle\nu_2\rangle$. By I.H's on ρ^is, $\nu_1 \in |(\rho^1\restriction_\emptyset)_{[\Delta_1],\Gamma_1,A}|^J$ and $\nu_2 \in |(\rho^2\restriction_\emptyset)_{[\Delta_2],A^\perp,\Gamma_2}|^J$. The assertion follows since
$|(\rho^1\restriction_\emptyset)_{[\Delta_1],\Gamma_1,A}|^J \times |(\rho^2\restriction_\emptyset)_{[\Delta_2],A^\perp,\Gamma_2}|^J \cong |(\rho\restriction_\emptyset)_{[\Delta_1,\Delta_2,A,A^\perp],\Gamma_1,\Gamma_2}|^J$.
(&-rule)

$$\rho \text{ is } \dfrac{\overset{\rho^1}{\vdash_{J_1} [\Delta_1, \Sigma\,],\;\; \Gamma, A_1} \qquad \overset{\rho^2}{\vdash_{J_2} [\Delta_2, \Sigma\,],\;\; \Gamma, A_2}}{\vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Sigma\,],\;\; \Gamma, A_1 \& A_2}\ \&$$

ν is of the form $\nu_1 \frown \nu_2$ so that the conclusions of ρ_i s are $\vdash_{J_i} ([\Delta_i, \Sigma],\ \Gamma, A_i)\langle\nu_i\rangle$. By I.H's on ρ^is, $\nu_i \in |(\rho^i\restriction_\emptyset)_{[\Delta_i,\Sigma],\Gamma_1,A_i}|^{J_i}$. The assertion follows since

$$|(\rho^1\restriction_\emptyset)_{[\Delta_1,\Sigma],\Gamma,A_1}|^{J_1} \frown |(\rho^1\restriction_\emptyset)_{[\Delta_2,\Sigma],\Gamma,A_2}|^{J_2} \cong |(\rho\restriction_\emptyset)_{[\Delta_1,\Delta_2,\Sigma],\Gamma,A_1\&A_2}|^{J_1+J_2} \qquad \square$$

Appendix C. Indices and Additive Cut Elimination

This appendix elucidates the fundamental idea of the paper. The appendix may read as a prologue of the paper by readers yet familiar with MLL GoI interpretation on a reflexive object in a traced monoidal category.

Consider a sequence $\pi_1 \rhd \pi_2 \rhd \pi_3$ of cut eliminations for proofs in the additive fragment of MALL. In our sequent notation, pairwise cut formulas, if present, are stored inside a stack $[\,\cdot\,]$ in a sequent. The first reduction, intrinsic to the additives, eliminates a & in a cut, whereby the subproof ax_2 is pruned. The second reduction eliminates a redundant cut against an axiom:

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,\text{ax}_1 \quad \dfrac{}{\vdash A^\perp, A}\,\text{ax}_2}{\vdash A^\perp, A\&A}\,\& \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,\text{ax}_3}{\vdash A^\perp \oplus A^\perp, A}\,\oplus_1}{\vdash [A\&A, A^\perp \oplus A^\perp]\ A^\perp, A}\,\text{cut} \quad \rhd \quad \dfrac{\dfrac{}{\vdash A^\perp, A}\,\text{ax}_1 \quad \dfrac{}{\vdash A^\perp, A}\,\text{ax}_3}{\vdash [A, A^\perp]\ A^\perp, A}\,\text{cut}$$

$$\rhd \quad \dfrac{}{\vdash A^\perp A}\,\text{ax}_1$$

Step 1 (Interpretation $|\pi|$ in Rel with unperformed cuts and indices for additives)
We begin by interpreting proofs in Rel but without relational composition. For this, the cut rule is interpreted same as the tensor rule. This interpretation is consistent with the syntactic convention, starting from Girard's GoI 1, which puts the cut formulas into a stack.

For simplicity and in accordance with the fact that the multiplicative dual elements 1 and $\perp$ are interpreted in Rel as the singleton set, we take $A = 1$ and, dually, $A^\perp = \perp$, so that $|1| = |\perp|$ is the singleton, whose unique element is denoted $*$ or $\bar{*}$ (obviously, $* = \bar{*}$), depending on whether it comes from $|A|$ or $|A^\perp|$, respectively.

An axiom is interpreted in Rel by the diagonal, so that $|\text{ax}_i| = \{(\bar{*}, *)\} \subseteq |A^\perp| \times |A|$. The proof π_2 is interpreted as $|\text{cut}(\text{ax}_1, \text{ax}_2)| = \{(\bar{*}, *, \bar{*}, *)\} \subseteq |A^\perp| \times [\,|A| \times |A^\perp|\,] \times |A|$, in which the pair $(*, \bar{*})$ in the cut slot from $[\,|A| \times |A^\perp|\,]$ remains explicit, rather than being hidden by relational composition through $* = \bar{*}$. Note that both interpretations $|\pi_2|$ and $|\pi_3|$ are singletons. More generally, it is straightforward to see that any proof in the multiplicatives can be interpreted by a singleton whenever literals are interpreted by singletons. However, this is not the case for the additives. When interpreting π_1 with additive rules, singletons prove insufficient, and this is where the indices become necessary: The left and right premises of π_1 are interpreted, respectively, by $|\&(\text{ax}_1, \text{ax}_2)| = \{(\bar{*}, (1, *)), (\bar{*}, (2, *))\} \subseteq |A^\perp| \times (|A| + |A|)$ and $|\oplus_1(\text{ax}_3)| = \{((1, \bar{*}), *)\} \subseteq (|A^\perp| + |A^\perp|) \times |A|$. A set of indices $J = \{1, 2\}$ is employed to describe these two interpretations: the first yields $\delta \in |\&(\text{ax}_1, \text{ax}_2)|^J$ so that $\delta_1 = (\bar{*}, (1, *))$ and $\delta_2 = (\bar{*}, (2, *))$, and the second yields $\tau \in |\oplus_1(\text{ax}_3)|^J$, so $\tau_1 = \tau_2 = ((1, \bar{*}), *)$.

Then π_1 is interpreted by ν:

$$\nu := \delta \times \tau \in |\mathrm{cut}(\&(\mathrm{ax}_1, \mathrm{ax}_2), \oplus_1(\mathrm{ax}_3))|^J \subseteq (|A^\perp| \times [\ (|A|+|A|) \times (|A^\perp|+|A^\perp|)\ ] \times |A|)^J,$$

and therefore $(\delta \times \tau)_1 = (\delta_1, \tau_1) = (\bar{*}, (1, *), (1, \bar{*}), *)$ and $(\delta \times \tau)_2 = (\delta_2, \tau_2) = (\bar{*}, (2, *), (1, \bar{*}), *)$, where $\delta \times \tau$ denotes the mediating morphism of the set-theoretical cartesian product. Summing up, $|\pi_1| = \{\nu_j \mid j \in J = \{1, 2\}\}$, where $\nu_1 = (\bar{*}, (1, *), (1, \bar{*}), *)$ and $\nu_2 = (\bar{*}, (2, *), (1, \bar{*}), *)$.

Step 2 ($\mathsf{Ex}_J\ (\sigma, \nu)$ **for** $\nu \in |\pi|^J$**: Executing cuts using trace structures)**
In addition to Step 1, our GoI interpretation runs an execution formula for $|\pi_i|$ to perform cut elimination against the unperformed cut formulas, syntactically in the stack and semantically in the noncompositional interpretation.

Each point in $|\pi|$ is interpreted as an endomorphism on a certain tensor folding of a *reflexive object* U in a traced monoidal category $\mathcal{C}$ with a zero morphism on U. The object U uniformly interprets each element in the interpretation of the conclusion of π; e.g. in $|\pi_1|$, U has the elements $\bar{*}$, $*$, $(1, \bar{*})$, $(1, *)$ and $(2, *)$. In the following, these points are identified with their interpretation U.

For the most primitive case, e.g. for $|\pi_3|$, each diagonal point $(\bar{*}, *)$ interpreting the axiom is interpreted as a *symmetry* of $\mathcal{C}$:

$$s_{\bar{*},*} : U_{\bar{*}} \otimes U_* \longrightarrow U_{\bar{*}} \otimes U_* \tag{C1}$$

The unique point of $|\pi_2|$ is interpreted by the endomorphism $\mathsf{Ex}\ (\sigma_{\mathrm{cut}}, |\pi_2|)$ on $U_{\bar{*}} \otimes U_*$, in which σ_{cut}, as the interpretation of the cut, is the symmetry $s_{*,\bar{*}}$ acting on the cut formulas:

$$\mathsf{Ex}\ (\sigma_{\mathrm{cut}}, |\pi_2|) = \mathsf{Tr}^{*\otimes\bar{*}}_{\bar{*}\otimes*,\bar{*}\otimes*} \left((\bar{*} \otimes \sigma_{\mathrm{cut}} \otimes *) \circ (s_{\bar{*},*} \otimes s_{\bar{*},*}) \right) \tag{C2}$$

Note that the symmetries s's occurring in (C2) interpret respective axioms.
This is equal to (C1) in $\mathcal{C}$ by the trace axioms. The adjacent diagrams illustrate (C1) and (C2), where the equality is found in the diagram for (C2) by chasing arrows with respect to both composition and feedback.

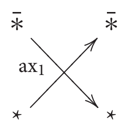

Diagram of (C1)

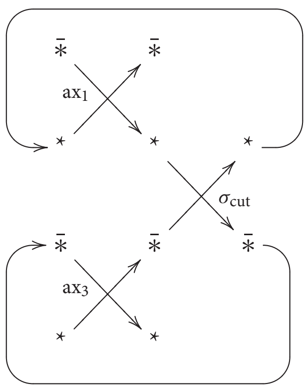

Diagram of (C2)

The GoI interpretation of π_1 is that for the indexed $\nu \in |\pi_1|^J$ in Step 1, which is defined pointwise (for $j \in J = \{1, 2\}$) at ν_1 and ν_2, in which σ_{ν_1} and σ_{ν_2} are stipulated respectively by σ_{cut} and 0, where σ_{cut} is a symmetry $s_{(1,*),(1,\bar{*})}$ for the cut formulas while 0 is the zero morphism 0_{U^2} resulting by zero action on a symmetry $s_{(2,*),(1,\bar{*})}$:

$$\mathsf{Ex}\left(\sigma_{\nu_1}, \nu_1\right) = \mathsf{Tr}^{(1,*)\,\otimes\,(1,\bar{*})}_{\bar{*}\otimes *\,,\,\bar{*}\otimes *}\left((\bar{*} \otimes \sigma_{\text{cut}} \otimes *) \circ (s_{\bar{*},(1,*)} \otimes s_{(1,\bar{*}),*})\right), \tag{C3}$$

in which σ_{cut} arises because of the matching $(1, *) = (1, \bar{*})$.

$$\mathsf{Ex}\left(\sigma_{\nu_2}, \nu_2\right) = \mathsf{Tr}^{(2,*)\,\otimes\,(1,\bar{*})}_{\bar{*}\otimes *\,,\,\bar{*}\otimes *}\left((\bar{*} \otimes\ 0\ \otimes *) \circ (s_{\bar{*},(2,*)} \otimes s_{(1,\bar{*}),*})\right) \tag{C4}$$

in which 0 arises because of the mismatch $(2, *) \neq (1, \bar{*})$.

Here (C3) is equivalent to (C2), while (C4) reduces to 0 in $\mathcal{C}$ by virtue of the trace axioms with zero morphisms. The next two diagrams, for (C3) and (C4), illustrate that (C4) yields a zero morphism because chasing any arrow results in passing through 0.

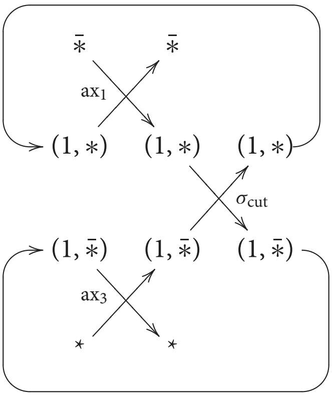

Diagram of (C3), $j = 1$

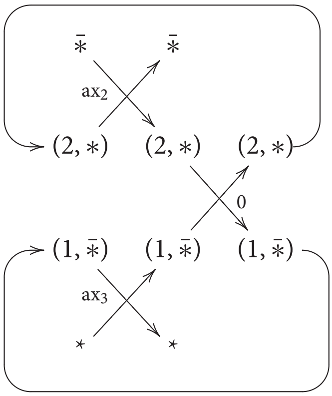

Diagram of (C4), $j = 2$

At $j = 2$, $\mathsf{Ex}\left(\sigma_{\nu_2}, \nu_2\right) = 0$, so we delete the index 2, reducing J into the singleton $\{1\}$. For index 1, $\mathsf{Ex}\left(\sigma_{\nu_1}, \nu_1\right)$ is identical to the symmetry (C1) of $|\pi_3|$ and, hence, to the denotational interpretation.

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