A NOTE ON REDUCTIONS OF IDEALS RELATIVE TO AN ARTINIAN MODULE

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Introduction. The concept of reduction and integral closure of ideals relative to Artinian modules were introduced in [7]; and we summarize some of the main aspects now.

Let A be a commutative ring (with non-zero identity) and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of A. Suppose that M is an Artinian module over A. We say that $\mathfrak{a}$ is a *reduction* of $\mathfrak{b}$ *relative* to M if $\mathfrak{a} \subseteq \mathfrak{b}$ and there is a positive integer s such that

$$(0:_M \mathfrak{a}\mathfrak{b}^s) = (0:_M \mathfrak{b}^{s+1}).$$

An element x of A is said to be *integrally dependent* on $\mathfrak{a}$ relative to M if there exists $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers) such that

$$\left(0:_M \sum_{i=1}^{n} x^{n-i}\mathfrak{a}^i\right) \subseteq (0:_M x^n).$$

It is shown that this is the case if and only if $\mathfrak{a}$ is a reduction of $\mathfrak{a} + Ax$ relative to M; moreover

$$\bar{\mathfrak{a}} = \{x \in A : x \text{ is integrally dependent on } \mathfrak{a} \text{ relative to } M\}$$

is an ideal of A called the *integral closure* of $\mathfrak{a}$ *relative* to M and is the unique maximal member of

$$\mathscr{C} = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of } A \text{ which has } \mathfrak{a} \text{ as a reduction relative to } M\}.$$

In [3] the concept of the relevant component of an ideal I (denoted by I^*) of a Noetherian ring R was introduced; moreover the arguments in [3], [5] prove that I^* is an interesting and useful ideal.

Now, an interesting question arises: whether there are, in the Artinian situation, some companion results to those discussed for instance in [3].

The purpose of this paper is to show that the Artinian property of the A-module M enables us to define and develop a satisfactory concept of the relevant component of an ideal relative to M; and the author hopes that this note presents topics for further research.

1. Notation and preliminary results. Throughout the paper M is an Artinian module over the commutative ring A (with non-zero identity) and $\mathfrak{a}$ is an ideal of A. We use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{Z}$ to denote the set of integers.

Definition and Remark 1.1. The *relevant component* of the ideal $\mathfrak{a}$ of A *relative* to M is denoted by $\mathfrak{a}^*$ and defined as follows:

$$\mathfrak{a}^* = \operatorname{ann}_A\left(\bigcap_{i \geq 0} \mathfrak{a}^i(0:_M \mathfrak{a}^{i+1})\right).$$

It follows from the minimal condition that, for large enough k, $\mathfrak{a}^* = \operatorname{ann}_A(\mathfrak{a}^k(0:_M \mathfrak{a}^{k+1}))$, also we note that $\mathfrak{a} \subseteq \mathfrak{a}^*$ and

$$\mathfrak{a}^* = (0 :_A \mathfrak{a}^k(0 :_M \mathfrak{a}^{k+1})) = ((0 :_M \mathfrak{a}^k) :_A (0 :_M \mathfrak{a}^{k+1})). \tag{1}$$

In what follows we shall show that, in fact, when $\mathfrak{a}$ is a co-regular ideal (for definition of a co-regular ideal, see 1.6 below)

$$(0 :_M \mathfrak{a}^{*k}) = (0 :_M \mathfrak{a}^k) \quad \text{for all large enough } k.$$

Thus, $\mathfrak{a}$ is a very special kind of reduction of $\mathfrak{a}^*$ relative to M, since we have, for all large enough k,

$$(0 :_M \mathfrak{a}^k) = (0 :_M \mathfrak{a}^{*k}) \subseteq (0 :_M \mathfrak{a}\mathfrak{a}^{*k-1}) \subseteq (0 :_M \mathfrak{a}^2\mathfrak{a}^{*k-2}) \subseteq \ldots \subseteq (0 :_M \mathfrak{a}^k).$$

Hence, for all large enough k,

$$(0 :_M \mathfrak{a}^{*k}) = (0 :_M \mathfrak{a}\mathfrak{a}^{*k-1}) = \ldots = (0 :_M \mathfrak{a}^{k-1}\mathfrak{a}^*) = (0 :_M \mathfrak{a}^k).$$

For the main result, we need a few preliminary lemmas which will be given below.

LEMMA 1.2. *Suppose $x \in A$ is such that $xM = M$. Then there exists $r \in \mathbb{N}$ such that, for all $n \geqslant r$,*

$$x(0 :_M \mathfrak{a}^n) \supseteq (0 :_M \mathfrak{a}^{n-r}).$$

Proof. By the Artin–Rees lemma for Artinian modules [**1**, Proposition 3], there is $r \in \mathbb{N}$ such that, for all $n \geqslant r$,

$$(0 :_M x) + (0 :_M \mathfrak{a}^n) = (((0 :_M x) + (0 :_M \mathfrak{a}^r)) : \mathfrak{a}^{n-r}) \supseteq ((0 :_M x) : \mathfrak{a}^{n-r}). \tag{2}$$

On the other hand, we have, for all $n \in \mathbb{N}$,

$$(0 :_M x) + (0 :_M \mathfrak{a}^n) = (x(0 :_M \mathfrak{a}^n) : Ax). \tag{3}$$

So it follows from (2) and (3) that, for all $n \geqslant r$,

$$(x(0 :_M \mathfrak{a}^n) : Ax) \supseteq ((0 :_M x) : \mathfrak{a}^{n-r}). \tag{4}$$

Now, let $y \in (0 :_M \mathfrak{a}^{n-r})$, where $n \geqslant r$. Then, since $xM = M$, there is $m \in M$ such that $y = xm$. Hence $m \in ((0 :_M x) : \mathfrak{a}^{n-r})$. Thus $m \in (x(0 :_M \mathfrak{a}^n) : Ax)$ by (4). Therefore $y = xm \in x(0 :_M \mathfrak{a}^n)$. The proof is now complete.

For the main result (Theorem 2.2), we need the concept of a superficial element relative to a module, which we introduced and developed in [**8**, Chapter III, §7]. However, for readers' convenience, we recall the main points of the theory.

Let N be an A-module and $\mathfrak{b}$ a proper ideal of A. Then $x \in \mathfrak{b}^s$ ($s \in \mathbb{N}$) is called a *superficial element* of order s for $\mathfrak{b}$ *relative* to N if there is $c \in \mathbb{N}$ such that for all $n > c$

$$(0 :_N \mathfrak{b}^n) = x(0 :_N \mathfrak{b}^{n+s}) + (0 :_N \mathfrak{b}^c).$$

For the next two propositions, suppose that $\mathfrak{a} = (a_1, \ldots, a_k)$ is a finitely generated ideal of A and M an Artinian A-module and T an indeterminate. Let $R = A[a_1T, \ldots, a_kT]$ be the (small) Rees ring of A with respect to $\mathfrak{a}$ and graded in the usual way by $\mathbb{Z}$. Set $G = \bigoplus_{n \in \mathbb{Z}} G_n$, where, for $n \in \mathbb{Z}$,

$$G_n = \begin{cases} 0 & \text{if } n > 0, \\ (0 :_M \mathfrak{a}^{-n+1})/(0 :_M \mathfrak{a}^{-n}) & \text{if } n \leqslant 0. \end{cases}$$

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Let $x_1, \ldots, x_k$ be indeterminates over A and let $R' = \bigoplus_{n \in \mathbb{Z}} R'_n$ denote the ring $A[x_1, \ldots, x_k]$ graded in the usual way, so that $R'_n = 0$ for $n < 0$. Now turn G into a graded R'-module using the ideas of Kirby in [**1**, p. 54]: if n is an integer with $n < 0$, $m \in (0:_M \mathfrak{a}^{-n+1})$ and $1 \leq i \leq k$, put

$$x_i(m + (0:_M \mathfrak{a}^{-n})) = a_i m + (0:_M \mathfrak{a}^{-n-1}) \in G_{n+1}.$$

Next there exists a surjective ring homomorphism $\varphi: R' \to R$ such that x_i $(1 \leq i \leq k)$ is mapped into $a_i T$ and $\operatorname{Ker} \varphi \subseteq \operatorname{Ann}_{R'} G$. Thus G has a structure as an R-module. Now we easily deduce the following proposition. (For a proof, see [**8**, p. 82, 7.3].)

Proposition 1.3. *Let the notation and assumptions be the same as above. Then $x \in \mathfrak{a}^s$ $(s \in \mathbb{N})$ is a superficial element of order s for $\mathfrak{a}$ relative to M if and only if there is $c' \in \mathbb{N}$ such that, for all $n > c'$*

$$(xT^s)G_{-n-s} = G_{-n}.$$

The next proposition shows that whenever M is an Artinian A-module, superficial elements do exist.

Proposition 1.4 [**8**, p. 83, 7.4]. *Let M be an Artinian A-module and $\mathfrak{a}$ a proper ideal of A. Then there exists an element x of A such that x is a superficial element of order s for $\mathfrak{a}$ relative to M.*

Proof. By [**1**, Lemma 3], we can (and do) assume that $\mathfrak{a}$ is finitely generated. Suppose the notation is as in Proposition 1.3 and, further, $\mathfrak{b}$ denotes the ideal $\sum_{i=1}^{k} R(a_i T)$ of R. Then, by [**9**, Lemma 2.2], G is an Artinian R-module and hence graded Artinian (that is, satisfies the minimal condition for homogeneous submodules). By [**6**, Proposition 2.4], G has a reduced graded-secondary representation,

$$G = N_1 + \ldots + N_{r'} + N_{r'+1} + \ldots + N_t,$$

where each N_i is a graded secondary homogeneous submodule of G and $\sqrt{(0:N_i)}$ is a homogeneous prime ideal $\mathfrak{q}_i$, say, of R $(1 \leq i \leq t)$. Further suppose that the N_i are numbered so that

$$\mathfrak{b} \nsubseteq \mathfrak{q}_i \quad \text{for} \quad i = 1, \ldots, r',$$

$$\mathfrak{b} \subseteq \mathfrak{q}_i \quad \text{for} \quad i = r'+1, \ldots, t.$$

Then, by the same argument as in [**6**, Theorem 3.1], we find $c \in \mathbb{N}$ and $f_s \in R_s$ homogeneous of degree s $(s \in \mathbb{N})$, such that $f_s \notin \bigcup_{i=1}^{r'} \mathfrak{q}_i$ and $f_s G_{-m-s} = G_{-m}$ for all $m > c$.

Suppose that $f_s = xT^s$ for some $x \in \mathfrak{a}^s$. Then

$$(xT^s)G_{-m-s} = G_{-m} \quad \text{for all } m > c,$$

and the result follows from Proposition 1.3.

Lemma 1.5. *Let M be an Artinian A-module and $\mathfrak{a}$ an ideal of A. Then the following are equivalent:*

(i) $M = \mathfrak{a}M$,

(ii) $M = xM$ *for some* $x \in \mathfrak{a}$.

Proof. We adapt the proof of [**2**, 2.8]. Let $M = N_1 + \ldots + N_r$ be a reduced secondary representation with $\sqrt{(0:N_i)} = \mathfrak{p}_i$, $1 \leqslant i \leqslant r$.

(i)$\Rightarrow$(ii). Suppose $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some $1 \leqslant i \leqslant r$. Then, by [**1**, Lemma 3], there is a finitely generated ideal $\mathfrak{b}$ such that $\mathfrak{b} \subseteq \mathfrak{a}$ and $(0:_M \mathfrak{b}^n) = (0:_M \mathfrak{a}^n)$ for all $n \in \mathbb{N}$. Now for each $x \in \mathfrak{b}$ there is $t_x \in \mathbb{N}$ such that $x^{t_x} N_i = 0$. Thus, since $\mathfrak{b}$ is finitely generated, there is $t \in \mathbb{N}$ such that $\mathfrak{b}^t N_i = 0$, and so $N_i \subseteq (0:_M \mathfrak{b}^t) = (0:_M \mathfrak{a}^t)$. Hence $\mathfrak{a}^t N_i = 0$. So

$$M = \mathfrak{a}^t M = \sum_{j \neq i} \mathfrak{a}^t N_j \subseteq \sum_{j \neq i} N_j \neq M,$$

a contradiction. Thus $\mathfrak{a} \nsubseteq \mathfrak{p}_i$ for $1 \leqslant i \leqslant r$, and so $\mathfrak{a} \nsubseteq \bigcup_{i=1}^{r} \mathfrak{p}_i$.

Let $x \in \mathfrak{a} \setminus \bigcup_{i=1}^{r} \mathfrak{p}_i$. Then, by [**2**, 2.6], the endomorphism of M given by multiplication by x is surjective, i.e. $M = xM$. Since (ii)$\Rightarrow$(i) is obvious, the proof of the lemma is now complete.

Remark. The referee kindly pointed out to me that this lemma (with essentially the same proof) appears as Proposition 3.4 in a paper of Ooishi [**5**]; and he attributes the result to Matlis.

Definition 1.6. The ideal $\mathfrak{a}$ of A is called *co-regular* if $\mathfrak{a}M = M$. By Lemma 1.5, this is the case if and only if $xM = M$ for some $x \in \mathfrak{a}$. The element x then is called a *co-regular element.*

2. Main result. Throughout this section $\mathfrak{a}$ is a co-regular ideal of A.

For the main result we need another lemma which is given below.

Lemma 2.1. *Let M be an Artinian A-module and $\mathfrak{a}$ a proper ideal of A. Then there exists a superficial element of order s for $\mathfrak{a}$ relative to M which is co-regular as well.*

Proof. With the same notation as in Propositions 1.3 and 1.4, we note that $\mathfrak{a} \subseteq \mathfrak{p}_l \Leftrightarrow \mathfrak{b} = \sum_{j=1}^{k} R(a_j T) \subseteq \mathfrak{P}_l$ $(1 \leqslant l \leqslant r)$, where $\mathfrak{P}_l = \bigoplus_{n \in \mathbb{Z}} (\mathfrak{p}'_l)_n$ are primes and are such that

$$(\mathfrak{p}'_l)_n = \begin{cases} 0 & \text{if} \quad n < 0, \\ (\mathfrak{p}_l \cap \mathfrak{a}^n) T^n & \text{if} \quad n \geqslant 0 \end{cases}$$

(with the convention that $\mathfrak{a}^n = A$ for $n \leqslant 0$). Now, by the arguments used in Propositions 1.3 and 1.4 and the identity above,

$$\mathfrak{b} \nsubseteq \mathfrak{q}_i \qquad 1 \leqslant i \leqslant r',$$
$$\mathfrak{b} \nsubseteq \mathfrak{P}_i \qquad 1 \leqslant i \leqslant r.$$

So, by [**9**, Lemma 2.1], there is $h_0 \in \mathbb{Z}$ such that, for all $h \geqslant h_0$, $R_h \nsubseteq \left(\bigcup_{i=1}^{r} \mathfrak{P}_i\right) \cup \left(\bigcup_{i=1}^{r'} \mathfrak{q}_i\right)$.

Let $s \in \mathbb{N}$ be such that $s \geqslant h_0$ and $xT^s = f_s \in R_s \setminus \left(\bigcup_{i=1}^{r} \mathfrak{P}_i\right) \cup \left(\bigcup_{i=1}^{r'} \mathfrak{q}_i\right)$ for some $x \in \mathfrak{a}^s$. Then, by Propositions 1.3 and 1.4 and (5), x is a superficial element of order s relative to M, which is a co-regular element as well.

THEOREM 2.2. *Let M be an Artinian A-module and $\mathfrak{a}$ a co-regular ideal of A. Then $(0:_M \mathfrak{a}^{*k}) = (0:_M \mathfrak{a}^k)$, for all large k. Further, $\mathfrak{a}^*$ is the largest ideal with this property.*

Proof. By Lemma 2.1, there exists a positive integer s and an element $x \in \mathfrak{a}^s$ such that x is a superficial element of A of order s relative to M which is a co-regular element. Thus there is $c \in \mathbb{N}$ such that, for all large enough n,

$$(0:_M \mathfrak{a}^n) = x(0:_M \mathfrak{a}^{n+s}) + (0:_M \mathfrak{a}^c). \tag{6}$$

Now it follows from Lemma 1.2 and (6) that for all large enough n, say $n \geq t$,

$$x(0:_M \mathfrak{a}^{n+s}) = (0:_M \mathfrak{a}^n). \tag{7}$$

Let $m = t + s$. Then, by (7), we have, for all $j \geq 0$,

$$\begin{aligned}(0:_M \mathfrak{a}^{m+j}) = x(0:_M \mathfrak{a}^{m+j+s}) &\subseteq \mathfrak{a}^s(0:_M \mathfrak{a}^{m+j+s}) \\ &\subseteq \mathfrak{a}^{s-1}(0:_M \mathfrak{a}^{m+j+s-1}) \\ &\subseteq \cdots \subseteq \mathfrak{a}(0:_M \mathfrak{a}^{m+j+1}) \\ &\subseteq (0:_M \mathfrak{a}^{m+j}).\end{aligned}$$

Thus $\mathfrak{a}(0:_M \mathfrak{a}^{l+1}) = (0:_M \mathfrak{a}^l)$ for all $l \geq m$. Therefore

$$\mathfrak{a}^i(0:_M \mathfrak{a}^{l+i}) = (0:_M \mathfrak{a}^l) \quad \text{for all} \quad l \geq m, \qquad i \geq 1. \tag{8}$$

Now it follows from the definition of $\mathfrak{a}^*$ that, for large n,

$$\begin{aligned}(0:_M \mathfrak{a}^2) \supseteq (0:_M \mathfrak{a}^{*2}) = ((0:_M \mathfrak{a}^*):\mathfrak{a}^*) &\supseteq (\mathfrak{a}^n(0:_M \mathfrak{a}^{n+1}):_A \mathfrak{a}^*) \\ &\supseteq \mathfrak{a}^n((0:_M \mathfrak{a}^{n+1}):_A \mathfrak{a}^*) \\ &= \mathfrak{a}^n((0:_M \mathfrak{a}^*):_A \mathfrak{a}^{n+1}) \\ &\supseteq \mathfrak{a}^n(\mathfrak{a}^n(0:_M \mathfrak{a}^{n+1}):_A \mathfrak{a}^{n+1}) \\ &\supseteq \mathfrak{a}^{2n}((0:_M \mathfrak{a}^{n+1}):_A \mathfrak{a}^{n+1}) \\ &\supseteq \mathfrak{a}^{2n}(0:_M \mathfrak{a}^{2n+2}).\end{aligned}$$

Then, we deduce from this (by induction) and (8) that, for large k, $(0:_M \mathfrak{a}^k) \supseteq (0:_M \mathfrak{a}^{*k}) \supseteq \mathfrak{a}^{kn}(0:_M \mathfrak{a}^{kn+k}) = (0:_M \mathfrak{a}^k)$. Therefore $(0:_M \mathfrak{a}^k) = (0:_M \mathfrak{a}^{*k})$ for all large k.

For the last part, suppose that $\mathfrak{b}$ is a proper ideal of A such that $(0:_M \mathfrak{b}^k) = (0:_M \mathfrak{a}^k)$ for all large k, say, $k \geq t$. Let $l \geq 2t - 1$. Then

$$\begin{aligned}(0:_M (\mathfrak{a}+\mathfrak{b})^l) = \left(0:_M \sum_{i=0}^{l} \mathfrak{a}^i\mathfrak{b}^{l-i}\right) &= \bigcap_{i=0}^{l} (0:_M \mathfrak{a}^i\mathfrak{b}^{l-i}) \\ &= \left(\bigcap_{i=0}^{t-1} ((0:_M \mathfrak{b}^{l-i}):\mathfrak{a}^i)\right) \cap \left(\bigcap_{i=t}^{l} ((0:_M \mathfrak{a}^i):\mathfrak{b}^{l-i})\right).\end{aligned}$$

Now by our assumption if $0 \leq i \leq t-1$, then $(0:_M \mathfrak{b}^{l-i}) = (0:_M \mathfrak{a}^{l-i})$ and for $t \leq i \leq l$, $(0:_M \mathfrak{a}^i) = (0:_M \mathfrak{b}^i)$. Thus

$$\begin{aligned}(0:_M (\mathfrak{a}+\mathfrak{b})^l) &= \left(\bigcap_{i=0}^{t-1} ((0:_M \mathfrak{a}^{l-i}):\mathfrak{a}^i)\right) \cap \left(\bigcap_{i=t}^{l} ((0:_M \mathfrak{b}^i):\mathfrak{b}^{l-i})\right) \\ &= (0:_M \mathfrak{a}^l) \cap (0:_M \mathfrak{b}^l) = (0:_M \mathfrak{a}^l) = (0:_M \mathfrak{b}^l).\end{aligned}$$

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Therefore, if $x \in b$ and k is large enough, then

$$(0:_M(\mathfrak{a}+\mathfrak{b})^l) \subseteq \left(0:_M \sum_{i=0}^{l} x^{l-i}\mathfrak{a}^i\right) \subseteq (0:_M \mathfrak{a}^l).$$

So $(0:_M \sum_{i=0}^{l} x^{l-i}\mathfrak{a}^i) = (0:_M \mathfrak{a}^l)$ and hence $x(0:_M \mathfrak{a}^l) \subseteq (0:_M \mathfrak{a}^{l-1})$. Now the result follows from identity (1).

Corollary 2.3. *Let* $\mathfrak{a}$, $\mathfrak{a}^*$ *be the same as in* 2.2 *and* $\mathfrak{b}$ *an ideal of* A *such that* $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}^*$. *Then* $\mathfrak{b}^* = \mathfrak{a}^*$ *and in particular*, $\mathfrak{a}^{**} = \mathfrak{a}^*$.

Proof. Since $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}^*$, we have,

$$(0:_M \mathfrak{a}^{*k}) \supseteq (0:_M \mathfrak{b}^k) \supseteq (0:_M \mathfrak{a}^k) \quad \text{for all} \quad k \geqslant 0.$$

Hence, it follows from this and Theorem 2.2 that

$$(0:_M \mathfrak{b}^k) = (0:_M \mathfrak{a}^k) \quad \text{for all large } k. \tag{9}$$

Now, using (1) and (9), we deduce that $\mathfrak{b}^* = \mathfrak{a}^*$. Also

$$(0:_M \mathfrak{b}^k) = (0:_M \mathfrak{a}^{*k}) \quad \text{for all large } k.$$

Using identity (1) again, we see at once that $\mathfrak{a}^{**} = \mathfrak{b}^*(=\mathfrak{a}^*)$.

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