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ON EXPLICIT ESTIMATES FOR LINEAR FORMS IN THE VALUES OF A CLASS OF E-FUNCTIONS

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We apply methods of Mahler to obtain explicit lower bounds for certain combinations of E-functions satisfying systems of linear differential equations as studied by Makarov. Our results sharpen and generalise earlier results of Mahler, Shidlovskii, and Väänänen.

1. Introduction

Makarov [3] has found lower bounds for linear forms in the values of a certain class of E-functions, but the constants involved in his estimates are not given explicitly. In this note we apply the method of Mahler [1]. Firstly, we give an explicit expression for the constant appearing in the lower bound of [3], thereby obtaining an explicit result (Theorem 1). Secondly, the effective transference theorem for Corollary 1.2 is provided by Theorem 2 of the present paper. Corollaries 1.3 and 2.1 of the present paper give explicit results which sharpen those of [10] and Väänänen [9]. We also give some results which sharpen those of [3], [1], Shidlovskii [7] and Väänänen [8] by applying the results of this note to some special E-functions. We detail these applications in the last part of the section "Main results".

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2. Main results

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{Z}$ the domain of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{K}$ an algebraic number field (thus of finite degree over $\mathbb{Q}$), $O_{\mathbb{K}}$ the domain of integers of $\mathbb{K}$. An entire function $f(z)$ satisfying the following conditions is called an E-function:

(i) $f(z) = \sum_{l=0}^{\infty} \frac{a_l}{l!} z^l$, $a_l \in \mathbb{K}$, $\lceil a_l \rceil \le C^l$, $l = 0, 1, 2, \ldots$,

where $\lceil a_l \rceil$ denotes the maximum of the absolute values of a_l and its field conjugates, and $C \geq 1$ is a positive constant;

(ii) there is a sequence of national numbers $q_0, q_1, \ldots, q_l, \ldots$ such that

$$q_l a_j \in O_{\mathbb{K}} , \quad j = 0, 1, \ldots, l , \quad l = 0, 1, \ldots ,$$

and

$$q_l \le C^l , \quad l = 0, 1, \ldots .$$

The E-functions we consider below are the class of E-functions defined over the field $\mathbb{K} = \mathbb{Q}$. Let

$$f_{ij}(z) = \sum_{l=0}^{\infty} \frac{a_{ijl}}{l!} z^l , \quad i = 1, \ldots, k , \quad j = 1, \ldots, n_i ,$$

be a set of E-functions satisfying conditions (i) and (ii) and the following system of the differential equations

$$(1) \quad y'_{ij} = Q_{ij0}(z) + \sum_{l=1}^{n_i} Q_{ijl}(z) y_{il} , \quad i = 1, \ldots, k , \quad j = 1, \ldots, n_i ,$$

where $Q_{ijl}(z) \in \mathbb{C}(z)$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$. We may, without loss of generality, assume that $Q_{ijl}(z) \in \mathbb{Q}(z)$ as is pointed out by Shidlovskii [7].

First of all we introduce some notation. Let $T(z) \in \mathbb{Z}[z]$ be the

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least common denominator of the rational functions $Q_{ijl}(z)$; then $T(z)Q_{ijl}(z) \in \mathbb{Z}[z]$. Let

$$g = \max_{\substack{1\le i\le k\\ 1\le j\le n_i\\ 0\le l\le n_i}} \left(\deg T(z),\ \deg\left(T(z)Q_{ijl}(z)\right)\right) .$$

T denotes the maximum of the absolute values of the coefficients of $T(z)$ and the $T(z)Q_{ijl}(z)$. Set $\alpha = a/b \in \mathbb{Q}$ such that $\alpha T(\alpha) \ne 0$, where $b > 0$. Let

$$H = \max(|a|, b)\ ,\ B = 4C^2HT\ ,\quad L = n_1 + \ldots + n_k\ .$$

Denote the minimum value of the orders of the zero at $z = 0$ of all the functions $f_{ij}(z)$ by p , and their maximum by q . We define the constants σ and σ_1 as follows:

$\sigma = q$, $\sigma_1 = p$ if the set $\{f_{ij}(z)\}$ and 1 constitute an irreducible set of functions (see [7], p. 389 for the definition),

$\sigma = q + \delta$, $\sigma_1 = \delta$ otherwise,

where δ is a constant depending only on the functions $\{f_{ij}(z)\}$ and the system of differential equations (1). Finally, we define two functions

$$c(r) = (L+1)^2(g+\sigma+1)(\log B)^{\frac{1}{2}}r(\log r)^{\frac{1}{2}}\ ,$$

$$g(r) = e^{-2c(r)}r!\ .$$

We obtain the following results.

THEOREM 1. *Let* $\{f_{ij}(z)\}$ *be a set of* E*-functions defined as above which with* 1 *are linearly independent over* $\mathbb{C}(z)$ *and satisfy the system of differential equations* (1)*, and let* $\{x_{ij}\}$ $(i = 1, \ldots, k$, $j = 1, \ldots, n_i)$ *be an arbitrary given set of integers not all zero. Put*

$$x_i' = \max_{1\le j\le n_i} \left(|x_{ij}|\right)\ ,\quad \bar{x}_i = \max\left(1, x_i'\right)\ ,\quad x = \max_{1\le i\le k} \left(\bar{x}_i\right)\ .$$

If r *is the positive integer satisfying the inequality*

$$g(r-1) \le x < g(r)\ , \tag{2}$$

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then we have

$$r \geq B^{4(L+1)^4(g+\sigma+1)^2} + 1 \tag{3}$$

and

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} \left\| \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} f_{ij}(\alpha) \right\| > e^{-2(L+1)c(r)} ,$$

where $\|y\|$ *denotes the distance of the real number* y *from the nearest integer.*

COROLLARY 1.1. *Under the assumptions of Theorem* 1 *we have*

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} \left\| \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} f_{ij}(\alpha) \right\| > x^{-12(L+1)^3(g+\sigma+1)(\log B/\log\log x)^{\frac{1}{2}}}$$

if

$$x > B^{16(L+1)^4(g+\sigma+1)^2 B^{16(L+1)^4(g+\sigma+1)^2}} .$$

COROLLARY 1.2. *Under the hypotheses of Theorem* 1 *and the conditions* $n_i = 1$ $(i = 1, \ldots, n)$ *we have*

$$r \geq B^{4(k+1)^4(g+\sigma+1)^2} + 1$$

and

$$\bar{x}_1 \ldots \bar{x}_k \left\| \sum_{i=1}^{k} x_i f_i(\alpha) \right\| > e^{-2(k+1)c(r)} .$$

COROLLARY 1.3. *Under the conditions of Corollary* 1.2 *we have*

$$\bar{x}_1 \ldots \bar{x}_k \left\| \sum_{i=1}^{k} x_i f_i(\alpha) \right\| > x^{-12(k+1)^3(g+\sigma+1)(\log B/\log\log x)^{\frac{1}{2}}}$$

if

$$x > B^{16(k+1)^4(g+\sigma+1)^2 B^{16(k+1)^4(g+\sigma+1)^2}} .$$

THEOREM 2. *Suppose that the functions* $f_i(z)$ $(i = 1, \ldots, k)$, *with* 1 , *belong to an irreducible set of functions and let* $y \geq 2$ *be any* *integer. Let* r *be the positive integer such that* $g(r-1) \le y < g(r)$ *. Under the hypotheses of Corollary* 1.2 *we have*

(4) $$r \ge B^{4(k+1)^4(g+q+1)^2} + 1$$

and

(5) $$y\|yf_1(\alpha)\| \ldots \|yf_k(\alpha)\| > e^{-2(k+2)c(r)} .$$

COROLLARY 2.1. *Under the hypotheses of Theorem* 2 *we have*

$$y\|yf_1(\alpha)\| \ldots \|yf_k(\alpha)\| > y^{-12(k+2)(k+1)^2(g+q+1)(\log B/\log\log y)^{\frac{1}{2}}}$$

if

$$y > B^{16(k+1)^4(g+q+1)^2 B^{16(k+1)^4(g+q+1)^2}} .$$

Corollaries 1.3 and 2.1 are similar to Theorems 2 and 2' in [10] and Theorems 1 and 2 of Väänänen in [9], respectively. The constants here, however, are given in explicit form.

We now mention some examples:

(i) Consider a function

$$K_\lambda(z) = \sum_{h=1}^{\infty} \frac{(-1)^h}{h!(\lambda+1)\ldots(\lambda+h)} (z/2)^{2h} .$$

Suppose that $\lambda_1, \ldots, \lambda_m$ are rational numbers with $\lambda_i \ne -1, \pm 1/2, -2, \ldots$ $(i = 1, \ldots, m)$ and that the $\lambda_i \pm \lambda_l$ $(1 \le i < l \le m)$ are not integers. Let $\alpha_1, \ldots, \alpha_n$ be nonzero rational integers whose squares are distinct. Then the $2mn$ functions $K_{\lambda_i}(\alpha_j z)$, $K'_{\lambda_i}(\alpha_j z)$ $(1 \le i \le m$, $1 \le j \le n)$ are a set of E-functions which are linearly independent together with the identity over $\mathbb{Q}$ (see [3], p. 8) and satisfy the following system of differential equations:

$$\frac{d}{dz} y_{1ij} = y_{2ij} ,$$

$$\frac{d}{dz} y_{2ij} = -(2\lambda_i - 1)z^{-1} y_{2ij} - \alpha_j^2 y_{1ij} , \quad i = 1, \ldots, m , \quad j = 1, \ldots, n .$$

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Put

$$\lambda_i = \lambda_i'/d \quad (d > 0) \ , \quad \lambda' = \max_{1 \le i \le m} \left(|\lambda_i'|\right) \ ,$$

$$\alpha_j = \alpha_j'/\beta \quad (\beta > 0) \ , \quad \alpha = \max_{1 \le j \le n} \left(|\alpha_j'|\right) \ .$$

It is easy to compute that

$$C = 2\alpha\beta d^m e^{6(\lambda'+d)m} \ ,$$

$$T = 2\alpha^2\beta^2 d\lambda' \ ,$$

$$H = 1 \ , \quad g = 1 \ , \quad L + 1 = 2mn + 1 \ ,$$

$$B = 4C^2TH \ .$$

From Corollary 1.1 we obtain

$$\prod_{i=1}^{m} \prod_{j=1}^{n} X_{ij}^2 \left\| \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} K_{\lambda_i}(\alpha_j) + y_{ij} K'_{\lambda_i}(\alpha_j) \right\| > X^{-12(2mn+1)^3(2+\sigma)(\log B/\log\log x)^{\frac{1}{2}}} \ ,$$

if

$$X > B^{16(2mn+1)^4(2+\sigma)^2 B^{16(2mn+1)^4(2+\sigma)^2}} \ ,$$

where x_{ij}, y_{ij} $(1 \le i \le m \ , \ 1 \le j \le n)$ are any set of integers not all zero, $X_{ij} = \max\left(1, |x_{ij}|, |y_{ij}|\right)$, $X = \max_{1 \le i \le m, 1 \le j \le n} \left(X_{ij}\right)$. This result is an application of the theorem of [3], our result however explicitly provides the unspecified constants appearing in [3].

(ii) Suppose that $a_1, \ldots, a_k$ are distinct rational integers, and that $b > 0$ is a rational integer such that $\left(b, a_1, \ldots, a_k\right) = 1$. Consider a set of E-functions

$$1, e^{(a_1/b)z}, \ldots, e^{(a_k/b)z} \ .$$

Obviously, the hypotheses of Corollary 1.2 and Theorem 2 are satisfied, and these E-functions belong to an irreducible set of functions, so that $g = 0$ and $q = 0$. We obtain from Corollary 1.2 and Theorem 2 that

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$$\bar{x}_1 \ldots \bar{x}_k \left\| \sum_{i=1}^{k} x_i e^{a_i/b} \right\| > e^{-2(k+1)c(r)} ,$$

$$y\|ye^{a_1/b}\| \ldots \|ye^{a_k/b}\| > e^{-2(k+2)c(r)} ,$$

respectively, where

$$c(r) = (k+1)^2(\log B)^{\frac{1}{2}} r(\log r)^{\frac{1}{2}} ,$$

$$B = 4b \max_{1\le i\le k} \left(b, |a_i|\right) .$$

The two inequalities above are similar to those of Theorems 1 and 2 of [1], respectively. The exponent $-2(k+2)c(r)$ above constitutes a slight sharpening of the result obtained by Mahler in [1], Theorem 2, which has the exponent $-2k(k+1)c(r)$.

It follows from Theorem 1' of [7] that

$$\min_{\substack{|x_i|\le x \\ x_1^2+\ldots+x_k^2>0 \\ x_i\in\mathbb{Z}}} \left| x_1 e^{a_1/b} + \ldots + x_k e^{a_k/b} \right| > x^{1-k-\gamma k^{\frac{7}{2}}(\log\log x)^{\frac{1}{2}}}$$

if

$$x > \exp\left(\exp \gamma^2 k^5\right) ,$$

where γ is a constant independent of k . Our Corollary 1.3 implies: if

$$x \ge B^{16k^4 B^{16k^4}} ,$$

then

$$\min_{\substack{|x_i|\le x \\ x_1^2+\ldots+x_k^2>0 \\ x_i\in\mathbb{Z}}} \left| x_1 e^{a_1/b} + \ldots + x_k e^{a_k/b} \right| > x^{1-k-12k^3(\log B/\log\log x)^{\frac{1}{2}}} ,$$

again sharpening the earlier result.

(iii) Let λ be a rational number (but not a rational integer).

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Consider a set of E-functions

$$\phi_\lambda\big((a_1/b)z\big), \ \ldots, \ \phi_\lambda\big((a_k/b)z\big) \ ,$$

where

$$\phi_\lambda(z) = \sum_{l=0}^{\infty} \frac{z^l}{(\lambda+1)\ldots(\lambda+l)} \ .$$

By [5], $\phi_\lambda\big((a_1/b)z\big), \ldots, \phi_\lambda\big((a_k/b)z\big)$, with 1, belong to an irreducible set of functions which are linearly independent over $\mathbb{C}(z)$ and satisfy the following system of differential equations:

$$\frac{d}{dz}\phi_\lambda\big((a_i/b)z\big) = \lambda/z + \big((a_i/b)-(\lambda/z)\big)\phi_\lambda\big((a_i/b)z\big) \ , \quad i = 1, \ldots, k \ .$$

Put $\lambda = \lambda_1/d$, $(\lambda_1, d) = 1$, $d > 0$, $\alpha = 1$. It is easy to compute that

$$C = bd^2e^{12(|\lambda_1|+d)} \max_{1\le i\le k} \big(|a_i|\big) \quad \text{(by Mahler [2], p. 146)},$$

$$T = bd\lambda_1 \max_{1\le i\le k} \big(|a_i|\big) \ ,$$

$$H = 1 \ , \quad g = 1 \ , \quad q = 0 \ .$$

From Corollaries 1.3 and 2.1 we obtain

$$\bar{x}_1 \ldots \bar{x}_k \Big\| \sum_{i=1}^{k} x_i\phi_\lambda\big(a_i/b\big)\Big\| > x^{-24(k+1)^3(\log B/\log\log x)^{\frac{1}{2}}}$$

if

$$x > B^{64(k+1)^4}B^{64(k+1)^4} \ ,$$

and

$$y\|y\phi_\lambda\big(a_1/b\big)\| \ \ldots \ \|y\phi_\lambda\big(a_k/b\big)\| > y^{-24(k+1)^2(k+2)(\log B/\log\log y)^{\frac{1}{2}}}$$

if

$$y > B^{64(k+1)^4}B^{64(k+1)^4} \ .$$

These results are Theorems 1 and 2 of [8], respectively; however we explicitly compute the unspecified constants that appear in [8].

3. Lemmas

LEMMA 1. *Let* (g_{ij}) $(1 \le i \le m,\ 1 \le j \le n)$ *be a* $m \times n$ $(m < n)$ *matrix of integers. Put*

$$G_i = \sum_{j=1}^{n} |g_{ij}|, \quad i = 1, \ldots, m .$$

Then there are integers $x_1, \ldots, x_n$ *not all zero such that*

$$\sum_{j=1}^{n} g_{ij}x_j = 0, \quad i = 1, \ldots, m,$$

and

$$\max_{1 \le i \le n} \left(|x_i|\right) \le \left(G_1 \ldots G_m\right)^{1/(n-m)} .$$

This is Lemma 1 of [1].

LEMMA 2. *Let* $r_1, \ldots, r_k$ *and* R *be positive integers satisfying*

$$r = r_0 = \max\left(r_1, \ldots, r_k\right) \ge 2,$$

$$L < R \le \sum_{i=0}^{k} n_i r_i + L \quad \left(\text{where } n_0 = 1\right);$$

and let

$$m = \sum_{i=0}^{k} n_i r_i + L + 1 - R, \quad n = \sum_{i=0}^{k} n_i r_i + L + 1,$$

$$M = \left[(L+1)^m \left(2C^2\right)^{m(m-1)/2}\right]^{1/R} .$$

Then there are polynomials $P_{ij}(z) \in \mathbb{Z}[z]$ $(i = 0, 1, \ldots, k,$ $j = 1, \ldots, n_i)$ *which do not all vanish identically and have the following properties:*

(i) $\deg P_{ij}(z) \le r$, $\operatorname{ord} P_{ij}(z) \ge r - r_i$, $\overline{\left|P_{ij}\right|} \le r_i! 2^r M$,

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$i = 0, 1, \ldots, k$, $j = 1, \ldots, n_i$, where $\operatorname{ord} P_{ij}(z)$ *denotes the order of the zero at $z = 0$ of the polynomial $P_{ij}(z)$, and $\lceil P_{ij} \rceil$ denotes the height of $P_{ij}(z)$, that is, the maximum of the absolute values of the coefficients of $P_{ij}(z)$;*

(ii) let

$$F(z) = \sum_{i=0}^{k} \sum_{j=1}^{n_i} P_{ij}(z) f_{ij}(z) \quad \left(\text{where } f_{01}(z) \equiv 1\right)$$

$$= \sum_{h=m}^{\infty} r!\sigma_h (h!)^{-1} z^h = \sum_{h=m}^{\infty} \rho_h z^h ,$$

then

$$|\rho_h| \le (L+1) r! (h!)^{-1} (1+C)^h M , \quad h \ge m .$$

Proof. Let S be the set $S = \{(i, l) \mid 0 \le i \le k, r-r_i \le l \le r\}$, and write

$$P_{ij}(z) = r! \sum_{l=0}^{r} p_{ijl} (l!)^{-1} z^l , \quad i = 0, 1, \ldots, k , \quad j = 1, \ldots, n_i .$$

Property *(i)* implies $p_{ijl} = 0$ if $(i, l) \overline{\in} S$ and $j = 1, \ldots, n_i$. Property *(ii)* implies $\operatorname{ord} F(z) \ge m$. Thus the p_{ijl}'s satisfy the following system of equations

$$p_{01h} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{l=0}^{h} \binom{h}{l} a_{i,j,h-l} p_{ijl} = 0 , \quad h = 0, 1, \ldots, m-1 .$$

On multiplying these m equations by $q_0, q_1, \ldots, q_{m-1}$, respectively, we obtain

$$(6) \quad q_h p_{01h} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{l=0}^{h} \binom{h}{l} q_h a_{i,j,h-l} p_{ijl} = 0 , \quad h = 0, 1, \ldots, m-1 .$$

This is a system of linear equations in the unknowns $\{p_{ijl}\}$ and with rational integer coefficients. Put

$$G_h = q_h + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{l=0}^{h} \binom{h}{l} |q_h a_{i,j,h-l}| .$$

Clearly, we have

$$G_h \le (L+1)\left(2C^2\right)^h , \quad h = 0, 1, \ldots, m-1 .$$

The number of unknowns for the system of equations (6) is equal to $n > m$. So we see from Lemma 1 that this system of equations has a set of rational integer solutions $\{p_{ijl}\}$ not all zero and satisfying

$$\max_{i,j,l} |p_{ijl}| \le \left(G_0 \ldots G_{m-1}\right)^{1/(n-m)}$$
$$\le \left[(L+1)^m \left(2C^2\right)^{m(m-1)/2}\right]^{1/R} = M .$$

Since

$$P_{ij}(z) = r_i! \sum_{l=0}^{r} r!\left(r_i! l!\right)^{-1} p_{ijl} z^l ,$$

it follows that

$$\overline{|P_{ij}|} \le r_i! 2^r M , \quad i = 0, 1, \ldots, k , \quad j = 1, \ldots, n_i .$$

Because

$$\sigma_h = p_{01h} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{l=0}^{h} \binom{h}{l} a_{i,j,h-l} p_{ijl} ,$$

we have

$$|\rho_h| = (r!/h!)|\sigma_h| \le (L+1) r! (h!)^{-1} (1+C)^h M ,$$

completing the proof of the lemma.

Let

(7) $$F_0(z) = F(z) , \quad F_\tau(z) = T(z) \frac{d}{dz} F_{\tau-1}(z) , \quad \tau = 1, 2, \ldots .$$

It follows from the system of differential equations (1) and (7) that

$$F_\tau(z) = \sum_{i=0}^{k} \sum_{j=1}^{n_i} P_{ij\tau}(z) f_{ij}(z) ,$$

where $P_{ij\tau}(z)$ satisfy the following recurrence relations:

$$P_{ij0}(z) = P_{ij}(z) ,$$

$$P_{01\tau}(z) = T(z) \frac{d}{dz} P_{0,1,\tau-1}(z) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} T(z) Q_{ij0}(z) P_{i,j,\tau-1}(z) ,$$

$$P_{ij\tau}(z) = T(z) \frac{d}{dz} P_{i,j,\tau-1}(z) + \sum_{l=1}^{n_i} T(z) Q_{ilj}(z) P_{i,l,\tau-1}(z) ,$$

$$i = 1, \ldots, k , \quad j = 1, \ldots, n_i .$$

Clearly, $P_{ij\tau}(z) \in \mathbb{Z}[z]$. Further, put

$$\tilde{f}_0(z) = f_{01}(z) \equiv 1 , \quad \tilde{f}_\nu(z) = f_{ij}(z) ,$$

$$\tilde{P}_{\tau 0}(z) = P_{01\tau}(z) , \quad \tilde{P}_{\tau\nu}(z) = P_{ij\tau}(z) ,$$

where

$$\nu = \nu(i, j) = \sum_{l=0}^{i-1} n_l + j - 1 , \quad i = 1, \ldots, k , \quad j = 1, \ldots, n_i .$$

In particular, set $\nu = 0$ if $i = 0$ (of course $j = 1$). Conversely, if ν is given, then we can determine i by the following inequality

$$\sum_{l=0}^{i-1} n_l \le \nu \le \sum_{l=0}^{i} n_l - 1 . \tag{8}$$

Let $P(z)$ be the matrix

$$P(z) = \left(\tilde{P}_{\tau\nu}(z)\right)_{0 \le \tau, \nu \le L} ,$$

and put

$$\Delta(z) = \det P(z) .$$

LEMMA 3. *Let* $\{f_{ij}(z)\}$ *be a set of E-functions defined as above which satisfy the system of differential equations* (1) *and which, with* 1 , *are linear independent over* $\mathbb{C}(z)$. *Then there exists a constant*

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$$N_0 = L(L+1)(g+1)/2 + R + \sigma_1$$

such that when $r^* = \min(r_1, \ldots, r_k) > N_0$ *we have* $\Delta(z) \not\equiv 0$ *, and*

$$\Delta(z) = z^{(L+1)r-R-(L(L+1)/2)-p}\Delta_1(z) ,$$

where $\Delta_1(z) \not\equiv 0$ *,* $\Delta_1(z) \in \mathbb{Z}[z]$ *and*

$$\deg \Delta_1(z) \le t = R + L(L+1)(g+1)/2 + p .$$

Proof. Suppose that the rank of $\Delta(z)$ is $W + 1 < L + 1$. Obviously $W \ge 0$. Then there is at least one non-zero minor determinant of order $W + 1$ in $P(z)$. Without loss of generality, we assume that it is in the left upper corner, namely that it is the principal minor determinant $\Delta_0(z) \not\equiv 0$. Then there exists a set of rational functions

$$D_{\omega\nu}(z) \in \mathbb{Q}(z) , \quad \omega = 0, 1, \ldots, W , \quad \nu = W+1, \ldots, L ,$$

such that

$$\tilde{P}_{\tau\nu}(z) = \sum_{\omega=0}^{W} \tilde{P}_{\tau\omega}(z)D_{\omega\nu}(z) , \quad \tau = 0, 1, \ldots, W , \quad \nu = W+1, \ldots, L$$

(see Lemma 6 in [6]). $F_\tau(z)$ can be rewritten as

$$F_\tau(z) = \sum_{\nu=0}^{L} \tilde{P}_{\tau\nu}(z)\tilde{f}_\nu(z) ; \tag{9}$$

and we have

$$F_\tau(z) = \sum_{\nu=0}^{W} \tilde{P}_{\tau\nu}(z)u_\nu(z) , \quad \tau = 0, 1, \ldots, W , \tag{10}$$

where

$$u_\nu(z) = \tilde{f}_\nu(z) + \sum_{\omega=W+1}^{L} \tilde{f}_\omega(z)D_{\nu\omega}(z) .$$

Let $T_1(z)$ be the least common denominator of the $D_{\omega\nu}(z)$. Put

$$U_\nu(z) = T_1(z)u_\nu(z) , \quad \nu = 0, 1, \ldots, W .$$

From Lemma 6 in [6] and Lemma 4 in [5] we find that

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$$\operatorname{ord} U_\nu(z) \le \sigma_1 .$$

In view of (10), we get

$$\Delta_0(z)U_\nu(z) = \sum_{\omega=0}^{W} T_1(z)\Delta_{\omega\nu}(z)F_\omega(z) , \tag{11}$$

where $\Delta_{\omega\nu}(z)$ is the cofactor of the element $\tilde{P}_{\omega\nu}(z)$ of the matrix corresponding to $\Delta_0(z)$. It is easy to compute that

$$\begin{aligned} \operatorname{ord}\left(\Delta_0(z)U_\nu(z)\right) &\le \deg \Delta_0(z) + \operatorname{ord} U_\nu(z) \\ &\le (W+1)r + L(L+1)g/2 + \sigma_1 , \end{aligned}$$

$$\operatorname{ord} \tilde{P}_{\tau\nu}(z) > r - r_{i(\nu)} - \tau ,$$

$$\operatorname{ord} F_\tau(z) > \operatorname{ord} F_0(z) - L \ge m - L ,$$

$$\operatorname{ord} \Delta_{\omega\nu}(z) \ge Wr - \sum_{\substack{0\le\tau\le W \\ \tau\ne\nu}} r_{i(\tau)} - L(L+1)/2 ,$$

where $i = i(\tau)$ satisfies (8). It follows from (11) that

$$(W+1)r + L(L+1)g/2 + \sigma_1 > Wr - \sum_{\substack{0\le\tau\le W \\ \tau\ne\nu}} r_{i(\tau)} - L(L+1)/2 + m - L .$$

Since $W < L$ this inequality implies that there exists at least one suffix $i(\tau)$ in the interval $0 \le i(\tau) \le k$ such that

$$r_{i(\tau)} < L(L+1)(g+1)/2 + R + \sigma_1 .$$

This contradicts the assumption of the lemma. Hence we must have $W = L$, that is, $\Delta(z) \not\equiv 0$.

Without loss of generality, we suppose that

$$\operatorname{ord} \tilde{f}_{\nu_0}(z) = \operatorname{ord} f_{i_0 j_0}(z) = p$$

for some $\nu_0 = \nu\left(i_0, j_0\right)$. By (9) we obtain

$$\Delta(z)\tilde{f}_{\nu_0}(z) = \sum_{\omega=0}^{L} F_\omega(z)\Delta_{\omega\nu_0}(z) .$$

Thus

$$\begin{aligned}\operatorname{ord} \Delta(z) &\geq \min_{0\leq\omega\leq L} \left(\operatorname{ord} F_\omega(z) + \operatorname{ord} \Delta_{\omega\nu_0}(z)\right) - \operatorname{ord} \tilde{f}_{\nu_0}(z) \\ &\geq m - L + rL - \sum_{\substack{0\leq\tau\leq L \\ \tau\neq\nu_0}} r_{i(\tau)} - L(L+1)/2 - p \\ &\geq (L+1)r - R - L(L+1)/2 - p ,\end{aligned}$$

and therefore

$$\begin{aligned}\deg \Delta_1(z) &\leq \deg \Delta(z) - \operatorname{ord} \Delta(z) \\ &\leq (L+1)r + L(L+1)g/2 - (L+1)r + R + L(L+1)/2 + p \\ &= R + L(L+1)(g+1)/2 + p = t ,\end{aligned}$$

completing the proof of the lemma.

LEMMA 4. *Under the assumptions of Lemma* 3 *there exist* $L + 1$ *suffixes* $J(\tau)$ $(0 \leq \tau \leq L)$ *such that*

$$0 \leq J(0) < J(1) < \ldots < J(L) \leq L + t$$

and

$$\det\left(\tilde{P}_{J(\tau),\nu}(\alpha)\right)_{0\leq\tau,\nu\leq L} \neq 0 .$$

The proof of this lemma is similar to Lemma 7 of [6].

LEMMA 5. *Under the hypotheses of Lemma* 4 *there exist* $(L+1)^2$ *rational integers* $q_{\tau\nu}$ $(0 \leq \tau, \nu \leq L)$ *with the following properties:*

(i) $\det\left(q_{\tau\nu}\right)_{0\leq\tau,\nu\leq L} \neq 0$;

(ii) for each pair (τ, ν) *we have*

(12) $$|q_{\tau\nu}| \leq C_1 r_{i(\nu)}! ,$$

where $i = i(\nu)$ *satisfies* (8) *and*

$$C_1 = 2^r[(L+t)g+r+L]^{L+t}T^{L+t}(2H)^{r+(L+t)g}M ;$$

(iii) for $\tau = 0, 1, \ldots, L$, *we have*

(13) $$\left|\sum_{\nu=0}^{L} q_{\tau\nu}\tilde{f}_\nu(\alpha)\right| \leq C_2\left(\prod_{i=1}^{k} (r_i!)^{n_i}\right)^{-1} ,$$

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where

$$C_2 = (L+1)[(L+t)g]^{L+t}(2H)^{2(L+t)g}H^{(L+2)r}(2T)^{L+t}[(L+1)r]^{2t}(1+C)^{(L+1)r}e^{2CH}M .$$

Proof. Clearly

$$\deg \tilde{P}_{J(\tau),\nu}(z) \le r + J(\tau)g \le r + (L+t)g .$$

Put

$$q_{\tau\nu} = b^{r+(L+t)g}\tilde{P}_{J(\tau),\nu}(\alpha) .$$

Thus all the $q_{\tau\nu}$ are rational integers. It follows from Lemma 4 that

$$\det\left(q_{\tau\nu}\right)_{0\le\tau,\nu\le L} \ne 0 .$$

Now consider two power series

$$U(z) = \sum_{l=0}^{\infty} u_l z^l \quad \text{and} \quad V(z) = \sum_{l=0}^{\infty} v_l z^l .$$

$V(z)$ is said to majorize the series $U(z)$ if

$$v_l \ge 0 , \quad |u_l| \le v_l , \quad l = 0, 1, \ldots .$$

We write $U(z) \ll V(z)$.

It is not difficult to verify by induction that

$$\tilde{P}_{J(\tau),\nu}(z) \ll T^{J(\tau)}\overline{|P_{ij}|}\prod_{l=0}^{J(\tau)-1}(lg+r+L)(1+z)^{J(\tau)g+r} ,$$

where (i, j) corresponds to the suffix ν . Thus

$$\begin{aligned}|q_{\tau\nu}| &\le b^{r+(L+t)g}T^{J(\tau)}\overline{|P_{ij}|}\left[\left(J(\tau)-1\right)g+r+L\right]^{J(\tau)}(1+|\alpha|/b)^{J(\tau)g+r} \\ &\le T^{L+t}\overline{|P_{ij}|}[(L+t)g+r+L]^{L+t}(2H)^{(L+t)g+r} \\ &\le C_1 r_{i(\nu)}! .\end{aligned}$$

In view of (9), we see, again by induction, that

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$$F_{J(\tau)}(z) \ll T^{J(\tau)}(1+z)^{J(\tau)g} \prod_{l=0}^{J(\tau)-1} \big(lg+(d/dz)\big)F_0(z)$$

$$\ll T^{J(\tau)}\big(J(\tau)g\big)^{J(\tau)}(1+z)^{J(\tau)g}\big(1+(d/dz)\big)^{J(\tau)}F_0(z)$$

$$\ll (2T)^{J(\tau)}\big(J(\tau)g\big)^{J(\tau)}(1+z)^{J(\tau)g} \sum_{h=m}^{\infty} \frac{r!\,|\sigma_h|}{\big(h-J(\tau)\big)!} z^{h-J(\tau)} .$$

Hence

$$\left|\sum_{\nu=0}^{L} q_{\tau\nu}\tilde{f}_\nu(\alpha)\right|$$

$$\le b^{r+(L+t)g}\left|F_{J(\tau)}(a/b)\right|$$

$$\le r!(2T)^{J(\tau)}\big(J(\tau)g\big)^{J(\tau)}b^{r+(L+t)g}(1+|a|/b)^{J(\tau)g} \sum_{h=m}^{\infty} \frac{|\sigma_h|}{\big(h-J(\tau)\big)!}(|a|/b)^{h-J(\tau)}$$

$$\le (L+1)r!(2T)^{L+t}[(L+t)g]^{L+t}(1+C)^m b^{r+(L+t)g}(1+|a|/b)^{(L+t)g}$$

$$\cdot\ (|a|/b)^m(|a|/b)^{-J(\tau)}e^{2C|a|/b}\big[\big(m-J(\tau)\big)!\big]^{-1}M$$

$$\le (L+1)r![(L+t)g]^{L+t}(2H)^{2(L+t)g}H^{(L+2)r}(2T)^{L+t}$$

$$\cdot\ e^{2CH}(1+C)^{(L+1)r}M \cdot \big[\big(m-J(\tau)\big)!\big]^{-1} .$$

On the other hand, we have

$$\big(m-J(\tau)\big)! \ge [m-(L+t)]! \ge \left(\sum_{i=0}^{k} n_i r_i - 2t\right)!$$

$$\ge \left(\sum_{i=0}^{k} n_i r_i\right)![(L+1)r]^{-2t} .$$

Therefore

$$\left|\sum_{\nu=0}^{L} q_{\tau\nu}\tilde{f}_\nu(\alpha)\right| \le (L+1)[(L+t)g]^{L+t}(2H)^{2(L+t)g}H^{(L+2)r}(2T)^{L+t}$$

$$\cdot\ [(L+1)r]^{2t}(1+C)^{(L+1)r}e^{2CH}M\left(\prod_{i=1}^{k}(r_i!)^{n_i}\right)^{-1}$$

$$\le c_2\left(\prod_{i=1}^{k}(r_i!)^{n_i}\right)^{-1} ,$$

completing the proof.

LEMMA 6. *Let*

$$L_i(X) = \sum_{j=1}^{k} \alpha_{ij}x_j , \quad i = 1, \ldots, k ,$$

be k *linearly independent linear forms, and let*

$$M_i(Y) = \sum_{j=1}^{k} \beta_{ij}y_j , \quad i = 1, \ldots, k ,$$

be a further k *linear forms. Suppose that*

$$\sum_{i=1}^{k} L_i(X)M_i(Y) = XY$$

holds identically for $X, Y \in \mathbb{R}^k$. *Let* $\lambda_1, \ldots, \lambda_k$ *denote the successive minima of the parellelepiped defined by* $|L_i(X)| \le 1$ $(1 \le i \le k)$. *Denote by* $\nu_1, \ldots, \nu_k$ *the successive minima of the parallelepiped defined by* $|M_i(Y)| \le 1$ $(1 \le i \le k)$. *Then we have*

$$\lambda_i \nu_{k+1-i} \ge 1/k , \quad i = 1, \ldots, k .$$

See Lemma 1 of [10] for the proof of this lemma.

4. Proof of Theorem 1

Let r be a positive integer satisfying (2). From Stirling's formula

$$r! = (2\pi r)^{\frac{1}{2}} r^r e^{-r+\rho(r)} , \quad 0 < \rho(r) < 1/(12r) ,$$

we obtain that

$$\log g(r)/r = \log r - 2(L+1)^2(g+\sigma+1)(\log B)^{\frac{1}{2}}(\log r)^{\frac{1}{2}} - 1 + \sigma(r) ,$$

where

$$\sigma(r) = \frac{\log r}{2r} + \frac{\log 2\pi}{2r} + \frac{\rho(r)}{r} .$$

It is easily verified that $0 < \sigma(r) < 1$ for $r \ge 2$. Hence

$$\log r - 2(L+1)^2(g+\sigma+1)(\log B)^{\frac{1}{2}}(\log r)^{\frac{1}{2}} - 1 < \log g(r)/r$$
$$< \log r - 2(L+1)^2(g+\sigma+1)(\log B)^{\frac{1}{2}}(\log r)^{\frac{1}{2}} .$$

From the definition of $g(r)$ and this inequality, we can immediately verify that

$$g(1) = 1 \ ; \quad g(r) < 1 \quad \text{if} \quad 2 \le r \le B^{4(L+1)^4(g+\sigma+1)^2} .$$

Because $c(r)$ is a strictly increasing function of r (when $r \ge 2$) and $g(r) > x \ge 1$, it follows that r must satisfy

$$r \ge B^{4(L+1)^4(g+\sigma+1)^2} + 1 ,$$

thus the inequality (3) holds. By the definition of r , we also have

(14) $$(r-1)! \le e^{2c(r)}x < r! .$$

Similarly, define the integers $r_1, \ldots, r_k$ by the inequalities

(15) $$\left(r_i-1\right)! \le e^{2c(r)}\bar{x}_i \le r_i! \ , \quad i = 1, \ldots, k .$$

Clearly, the inequalities (14), (15) imply $r = \max\left(r_1, \ldots, r_k\right)$. Now write

(16) $$R = \left[(L+1)r(\log B/\log r)^{\frac{1}{2}}\right] + 1 ,$$

where $[y]$ denotes the integer part of y . Because of the inequality (3), and noting that $r/(\log r)^{\frac{1}{2}}$ is an increasing function of r , we easily verify that

$$L < R \le \sum_{i=0}^{k} n_i r_i + L .$$

We now show that $r^* = \min\left(r_1, \ldots, r_k\right) > N_0$. We have

$$\log r > 4(L+1)^4(g+\sigma+1)^2 \log B ,$$

$$R > (L+1)r(\log B/\log r)^{\frac{1}{2}} ,$$

by (3) and (16). So $2R > N_0$. If we were to assume that some $r_i \le N_0$, then we have the inequality

$$\log\left(r_i!\right) < r_i \log r_i < 2R \log 2R \le 2c(r) ,$$

and so $r_i! < e^{2c(r)}$. This is contrary to the definition of r_i , hence

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certainly $r^* > N_0$. Thus we have verified that $r, r_1, \ldots, r_k$ and R satisfy the conditions of Lemmas 2 and 3.

By Lemma 5, we have obtained $(L+1)^2$ integers $q_{\tau\nu}$ $(0 \le \tau, \nu \le L)$ satisfying $\det(q_{\tau\nu}) \ne 0$. Further let $\{x_{ij}\}$ $(1 \le i \le k, 1 \le j \le n_i)$ be a set of integers satisfying the hypotheses of Theorem 1, and let b be any integer. Then we can form a $(L+1) \times (L+1)$ determinant which does not vanish; without loss of generality, we may assume that, say,

$$D = \begin{vmatrix} b & x_{11} & \cdots & x_{kn_k} \\ q_{10} & q_{11} & \cdots & q_{1L} \\ \cdots & \cdots & \cdots & \cdots \\ q_{L0} & q_{L1} & \cdots & q_{LL} \end{vmatrix} \ne 0 .$$

Let

$$L_0 = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} f_{ij}(\alpha) ,$$

$$L_\tau = \sum_{\nu=0}^{L} q_{\tau\nu} \tilde{f}_\nu(\alpha) , \quad \tau = 1, \ldots, L .$$

Thus D can be rewritten as

$$D = \begin{vmatrix} L_0 & x_{11} & \cdots & x_{kn_k} \\ L_1 & q_{11} & \cdots & q_{1L} \\ \cdots & \cdots & \cdots & \cdots \\ L_L & q_{L1} & \cdots & q_{LL} \end{vmatrix} .$$

Decomposing this determinant according to the first column, we obtain

$$D = L_0 M_0 + L_1 M_1 + \ldots + L_L M_L ,$$

where M_i is the cofactor of L_i , $i = 0, 1, \ldots, L$. By (12) and (13) of Lemma 5, we have

$$|M_0| \le L!C_1^L \prod_{i=1}^{k} (r_i!)^{n_i} ,$$

$$|M_\tau| \le (L-1)!C_1^{L-1} \prod_{i=1}^{k} (r_i!)^{n_i} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{|x_{ij}|}{r_i!}$$

$$\le (L-1)!C_1^{L-1} \prod_{i=1}^{k} (r_i!)^{n_i} \sum_{i=1}^{k} \frac{n_i\bar{x}_i}{r_i!}$$

$$|L_\tau| \le C_2\left(\prod_{i=1}^{k} (r_i!)^{n_i}\right)^{-1} , \quad \tau = 1, \ldots, L .$$

But D is a non-zero integer, so $|D| \ge 1$. Therefore we have

$$(17) \qquad 1 \le |D| \le L!C_1^L \prod_{i=1}^{k} (r_i!)^{n_i}|L_0| + L!C_1^{L-1}C_2 \sum_{i=1}^{k} \frac{n_i\bar{x}_i}{r_i}$$

$$= U + V .$$

According to the definition of r_i , we have

$$\prod_{i=1}^{k} (r_i!)^{n_i} \le r^L e^{2Lc(r)} \prod_{i=1}^{k} \bar{x}_i^{n_i} ,$$

$$\sum_{i=1}^{k} \frac{n_i\bar{x}_i}{r_i!} \le Le^{-2c(r)} .$$

Thus

$$(18) \qquad 2U \le 2L!C_1^L r^L e^{2Lc(r)}|L_0| \prod_{i=1}^{k} \bar{x}_i^{n_i} ,$$

$$(19) \qquad 2V \le 2(L+1)!C_1^{L-1}C_2 e^{-2c(r)} .$$

We shall next establish upper estimates for (18) and (19). By using (3) and the definition of R , we can easily obtain the following inequalities:

$$L + t = L + R + L(L+1)(g+1)/2 + p \le R + L(L+1)(g+1) + \sigma ,$$

$$(L+t)g + r + L = g[R+L(L+1)(g+1)+\sigma] + r + L \le (g+2)r ,$$

$$(L+t)g \le (g+1)r .$$

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Moreover

$$C_1 \le 2^r T^{R+L(L+1)(g+1)+\sigma}[(g+2)r]^{R+L(L+1)(g+1)+\sigma}$$

$$\cdot (2H)^{r+g[R+L(L+1)(g+1)+\sigma]}(L+1)^{(L+1)r/R}(2C^2)^{(L+1)^2r^2/(2R)} .$$

It follows that

$$2L!C_1^L r^L \le (2C^2)^{L(L+1)^2r^2/(2R)} r^{L(R-1)} \cdot L^L \cdot (2T)^{Lr} \cdot T^{L^2(L+1)(g+1)+L\sigma}$$

$$\cdot (2H)^{Lr+Lg[R+L(L+1)(g+1)+\sigma]} \cdot (g+2)^{L[R+L(L+1)(g+1)+\sigma]}$$

$$\cdot r^{L^2(L+1)(g+1)+L(\sigma+2)} \cdot (L+1)^{L(L+1)r/R} .$$

Clearly, we have, by (16),

$$(2C^2)^{L(L+1)^2r^2/(2R)} r^{L(R-1)} \le e^{(3/2)c(r)} .$$

Since $B \ge 4$, $L \ge 1$, $g \ge 0$, $\sigma \ge 0$, we have $r \ge B^{64} \ge 2^{128}$ by (3). It follows that

$$(\log r)/r \le 128(\log 2)/2^{128} < 2^{-120} ,$$

since $(\log r)/r$ is a strictly decreasing function of r (when $r \ge 2$). Thus we can obtain the following inequalities by simple calculation:

$$\frac{L\log L}{c(r)} \le \frac{L^2}{2(L+1)^4 r\log B} < 2^{-7} ;$$

$$\frac{Lr\log 2T}{c(r)} \le \frac{Lr\log B}{2(L+1)^4 r\log B} < 2^{-4} ;$$

$$\frac{[L^2(L+1)(g+1)+L\sigma]\log T}{c(r)} \le \frac{L(L+1)^2(g+1+\sigma)\log B}{2(L+1)^4(g+\sigma+1)^2 r\log B} < 2^{-7} ;$$

$$\frac{\{Lr+Lg[R+L(L+1)(g+1)+\sigma]\}\log 2H}{c(r)}$$

$$\le \frac{Lr\log B}{2(L+1)^4 r\log B} + \frac{Lg(L+1)\log B}{(L+1)^2(g+\sigma+1)\log r} + \frac{Lg\log B}{(L+1)^2(\log B)^{\frac12} r(\log r)^{\frac12}}$$

$$+ \frac{[L^2g(L+1)^2(g+1)+Lg\sigma]\log B}{2(L+1)^4(g+\sigma+1)^2 r\log B}$$

$$\le 2^{-4} + 2^{-6} + 2^{-132} + 2^{-130} \le 2^{-3} ;$$

$$\frac{L[R+L(L+1)(g+1)+\sigma]\log(g+2)}{c(r)} \le \frac{LR\log(g+2)}{c(r)} + \frac{L(L+1)^2(g+\sigma+1)\log(g+2)}{c(r)}$$

$$\le \frac{LR(g+1)}{c(r)} + \frac{L(L+1)^2(g+\sigma+1)^2}{c(r)}$$

$$\le 1/(\log r) + 1/\left(2(L+1)^3 r\right) + 1/\left(2(L+1)r\right)$$

$$\le 2^{-7} + 2^{-132} + 2^{-130} \le 2^{-6} \ ;$$

$$\frac{L[(L+1)^2(g+1)+\sigma+2]\log r}{c(r)} \le \frac{L(L+1)^2(g+\sigma+1)\log r}{c(r)} + \frac{2L\log r}{c(r)}$$

$$\le \frac{L}{2(L+1)^2\log B}\ \frac{\log r}{r} + \frac{2L}{2(L+1)^4\log B}\ \frac{\log r}{r}$$

$$\le 2^{-132} + 2^{-123} \le 2^{-7} \ ;$$

$$\frac{L(L+1)r\log(L+1)}{Rc(r)} \le 1/r \le 2^{-7} \ .$$

It follows from the above relations that

$$2L!C_1^L r^L \le e^{\left((3/2)+2^{-7}+2^{-4}+2^{-7}+2^{-3}+2^{-6}+2^{-7}+2^{-7}\right)c(r)} \le e^{2c(r)} \ .$$

Substituting this inequality in (18), we obtain

$$(20) \qquad 2U < e^{2(L+1)c(r)} \prod_{i=1}^{k} \bar{x}_i^{n_i} |L_0| \ .$$

Similarly, we can also deduce that

$$2(L+1)!C_1^{L-1}C_2 \le r^{LR+2R-(L+2)}\left(2C^2\right)^{L(L+1)^2r^2/(2R)}$$

$$\cdot (L+1)^{2[R+L(L+1)(g+1)+\sigma]+L+2+L(L+1)r/R} \cdot (4HTC)^{(2L+1)r}$$

$$\cdot (2T)^{L[L(L+1)(g+1)+\sigma]} \cdot (2H)^{(L+1)g[R+L(L+1)(g+1)+\sigma]}$$

$$\cdot (g+2)^{LR} \cdot [(g+2)r]^{L[L(L+1)(g+1)+\sigma]} \cdot e^{B}$$

$$\cdot r^{2[\sigma+L(L+1)(g+1)]+L+2} \ .$$

Much as in the above calculations, we can also obtain the following inequalities:

$$r^{(L+2)(R-1)}\left(2C^2\right)^{L(L+1)^2r^2/(2R)} \le e^{(7/4)c(r)} \quad (\text{since } \ L \ge 1) \ ;$$

$$\frac{\{2[R+L(L+1)(g+1)+\sigma]+L+2+L(L+1)r/R\}\log(L+1)}{c(r)}$$

$$\le \frac{2R\log(L+1)}{c(r)} + \frac{\{2[L(L+1)(g+1)+\sigma]+L+2\}\log(L+1)}{c(r)} + \frac{L(L+1)r\log(L+1)}{Rc(r)}$$

$$\le 2/\log r + 1/(8r) + 3/(4r) + (\log r)^{\frac{1}{2}}/(8r)$$

$$\le 2^{-6} + 2^{-131} + 2^{-128} + 2^{-187} \le 2^{-5} ;$$

$$\frac{(2L+1)r\log(4HCT)}{c(r)} \le (L+1)^{-3} \le 2^{-3} ;$$

$$\frac{L[L(L+1)(g+1)+\sigma]\log 2T}{c(r)} \le \frac{1}{2(L+1)r} \le 2^{-7} ;$$

$$\frac{(L+1)g[R+L(L+1)(g+1)+\sigma]\log 2H}{c(r)}$$

$$\le \frac{\log B}{\log r} + \frac{1}{2(L+1)^3 r} + \frac{1}{2(L+1)r} \le 2^{-6} + 2^{-132} + 2^{-130} < 2^{-5} ;$$

$$\frac{LR\log(g+2)}{c(r)} \le \frac{1}{\log r} + \frac{1}{2(L+1)^3 r} \le 2^{-6} ;$$

$$\frac{L[L(L+1)(g+1)+\sigma]\log[(g+2)r]}{c(r)}$$

$$\le 1/\big(2(L+1)r\big) + (\log r)/\big(2(L+1)r\big) \le (\log r)/r < 2^{-7} ;$$

$$\frac{B}{c(r)} \le \frac{B}{2(L+1)^4 r\log B} \le \frac{B}{2^5 B^{64}} < 2^{-7} ;$$

$$\frac{\{2[\sigma+L(L+1)(g+1)]+L+2\}\log r}{c(r)} \le \frac{\log r}{r} < 2^{-7} .$$

Substituting these inequalities in (19), we obtain

$$(21) \qquad 2V < e^{\left((7/4)+2^{-5}+2^{-3}+2^{-7}+2^{-5}+2^{-6}+2^{-7}+2^{-7}+2^{-7}-2\right)c(r)} < 1 .$$

From (17), (20) and (21), we deduce that

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} |L_0| > e^{-2(L+1)c(r)} .$$

Since b is any integer, it follows that

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} \left\| \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} f_{ij}(\alpha) \right\| > e^{-2(L+1)c(r)} .$$

Thus Theorem 1 is proved.

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The proof of Corollary 1.2 is quite similar to the proof of the corollary to Theorem 1 in [1]. Corollaries 1.2 and 1.3 plainly follow.

5. Proof of Theorem 2

From the hypotheses of Theorem 2 we know that the $f_i(z)$ $(i = 1, \ldots, k)$ belong to an irreducible set of functions so $\sigma = q$. As in the proof of Theorem 1, the integer r must satisfy

$$r \geq B^{4(k+1)^{4(g+q+1)^2}} + 1 ,$$

namely, the inequality (4) holds. We shall use induction to prove the inequality (5).

Before commencing our induction, we introduce the following notation:

$$g_l(r) = e^{-2c_l(r)} r! ;$$

$$c_l(r) = (l+1)^2 (g_l+q_l+1) (\log B_l)^{\frac{1}{2}} r (\log r)^{\frac{1}{2}} ;$$

$$H_l = 2(l+2) , \quad l = 1, 2, \ldots, k .$$

In particular, $g = g_k$, $q = q_k$, $B = B_k$, $c(r) = c_k(r)$, $g(r) = g_k(r)$.

When $k = 1$, it is clear that the inequality (5) follows from Corollary 1.2. We suppose (5) is true for $k - 1$ and prove it for k $(k \geq 2)$. Put

$$\mu_i = \|yf_i(\alpha)\| e^{H_k(k+1)^{-1}c(r)} , \quad i = 1, 2, \ldots, k ,$$

$$\mu_{k+1} = (\mu_1 \ldots \mu_k)^{-1} .$$

Clearly all $\mu_i > 0$, $i = 1, \ldots, k$. We consider two separate cases.

(i) There exists some μ_j $(1 \leq j \leq k)$ satisfying $\mu_j \geq 1$. Without loss of generality, we assume $\mu_k \geq 1$. According to the induction hypotheses, the inequality (5) is true for $k - 1$, namely

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$$y\|yf_1(\alpha)\| \ldots \|yf_{k-1}(\alpha)\| > e^{-H_{k-1}c_{k-1}(r')}, \tag{22}$$

where r' is a positive integer satisfying the condition

$$g_{k-1}(r') \le y < g_{k-1}(r') .$$

It is obvious that $g_l \le g_{l+1}$, $q_l \le q_{l+1}$, $B_l \le B_{l+1}$ $(l = 1, \ldots, k)$ from the definitions. Hence

$$c_{k-1}(r) < (k+1)^2(g_k+q_k+1)(\log B_k)^{\frac{1}{2}}r(\log r)^{\frac{1}{2}} = c_k(r) .$$

It follows that $g_{k-1}(r) > g_k(r) = g(r)$. This implies $r' < r$. Because $c(r)$ is an increasing function of r , we have

$$c_{k-1}(r') < c_{k-1}(r) \le k^2(k+1)^{-2}c_k(r) .$$

Since $\mu_k \ge 1$ and

$$k^2(k+1)^{-1}H_{k-1}/H_k + (k+1)^{-1} < 1 \quad (\text{when } k \ge 2),$$

we obtain by (22) that

$$\begin{aligned} y\|yf_1(\alpha)\| \ldots \|yf_k(\alpha)\| &= y\|yf_1(\alpha)\| \ldots \|yf_{k-1}(\alpha)\|\mu_k \exp\left\{-H_k(k+1)^{-1}c(r)\right\} \\ &> \exp\left\{-k^2(k+1)^{-2}H_{k-1}c_k(r)-H_k(k+1)^{-1}c_k(r)\right\} \\ &= \exp\left\{-\left[k^2(k+1)^{-2}H_{k-1}/H_k+(k+1)^{-1}\right]H_kc_k(r)\right\} \\ &> \exp\{-H_kc_k(r)\} = e^{-2(k+2)c(r)} . \end{aligned}$$

Thus the inequality (5) is also true for k .

(ii) All the μ_i $(i = 1, \ldots, k)$ satisfy $0 < \mu_i < 1$. We suppose that the inequality (5) does not hold (when $k \ge 2$). Then there exists an integer $y \ge 2$ such that the following inequality holds:

$$y\|yf_1(\alpha)\| \ldots \|yf_k(\alpha)\| \le e^{-H_kc(r)} . \tag{23}$$

Now let us consider a set of linear forms

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$$M_i(X) = \mu_i^{-1}\left(x_i - f_i(\alpha)x_{k+1}\right) , \quad i = 1, \ldots, k ,$$

$$M_{k+1}(X) = \mu_{k+1}^{-1}x_{k+1} .$$

Denote by ν_1 the first successive minimum of the parallelepiped defined by

$$|M_i(X)| \le 1 , \quad 1 \le i \le k+1 .$$

Further let $y_1, \ldots, y_k$ be a set of integers satisfying the following equalities:

$$|y_i - yf_i(\alpha)| = \|yf_i(\alpha)\| , \quad i = 1, \ldots, k .$$

Since $y \ge 2$, $\{y, y_1, \ldots, y_k\}$ is a set of integers not all zero. By the definitions of μ_i's and the inequality (23), we have

$$\left|\mu_i^{-1}\left(y_i - yf_i(\alpha)\right)\right| = \mu_i^{-1}\|yf_i(\alpha)\| = e^{-H_k(k+1)^{-1}c(r)} , \quad 1 \le i \le k ,$$

$$\left|\mu_{k+1}^{-1}y\right| = y\|yf_1(\alpha)\| \ldots \|yf_k(\alpha)\|e^{k(k+1)^{-1}H_kc(r)}$$

$$\le e^{-H_k(k+1)^{-1}c(r)} .$$

According to the definition of successive minima, we have

$$\nu_1 \le e^{-H_k(k+1)^{-1}c(r)} . \tag{24}$$

Let us consider the further set of linear forms

$$L_i(X) = \mu_i x_i , \quad i = 1, \ldots, k ,$$

$$L_{k+1}(X) = \mu_{k+1}\left(x_1f_1(\alpha) + \ldots + x_kf_k(\alpha) + x_{k+1}\right) .$$

Without loss of generality, we can suppose that

$$\mu_1^{-1} = \max\left(\mu_1^{-1}, \ldots, \mu_k^{-1}\right) > 1 .$$

Henceforth we suppose that s is a positive integer such that

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$g(s-1) \le \mu_1^{-1} < g(s)$. Likewise, s must satisfy the following inequalities

$$s \ge B^{4(k+1)^4(g+q+1)^2} + 1$$

and

$$(s-1)! \le e^{2c(r)}\mu_1^{-1} < s! \ .$$

Similarly, define the integers $s_2, \ldots, s_k$ by the inequalities

$$(s_i-1)! \le e^{2c(r)}\mu_i^{-1} < s_i! \ , \quad i = 2, \ldots, k \ .$$

It is clear that $s = \max(s, s_2, \ldots, s_k)$. Finally, let

$$R(s) = \left[(k+1)s(\log B/\log s)^{\frac{1}{2}}\right] + 1 \ .$$

Much as in the proof of Theorem 1, we can verify that the integers $s, s_2, \ldots, s_k$ and $R(s)$ satisfy all the hypotheses concerning $r, r_1, \ldots, r_k$ and R in Lemmas 3, 4 and 5, respectively. Thus, according to Lemma 5, we can obtain $k + 1$ linearly independent integer points

$$\left(q_{i0}, q_{i1}, \ldots, q_{ik}\right) \ , \quad i = 0, 1, \ldots, k \ ,$$

such that, for $i = 0, 1, \ldots, k$,

$$\left|q_{ij}\right| \le C_1 s_j! \le C_1 s e^{2c(s)}\mu_j^{-1} \ , \quad j = 0, 1, \ldots, k \ ;$$

and

$$(25) \qquad \left|\sum_{j=0}^{k} q_{ij}f_j(\alpha)\right| \le C_2\left(s!s_2! \ldots s_k!\right)^{-1} \le C_2 e^{-2kc(s)}\mu_{k+1}^{-1} \ ,$$

where C_1 and C_2 are the $(k+1)$th successive minimum of the parallelepiped defined by

$$\left|L_i(X)\right| \le 1 \ , \quad i = 1, \ldots, k+1 \ .$$

Then λ_{k+1} satisfies

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$$\lambda_{k+1} \le \max\left\{C_1 s e^{2c(s)},\ C_2 e^{-2kc(s)}\right\}$$

by (25). We shall prove that $C_1 s e^{2c(s)} > C_2 e^{-2kc(s)}$. As in the calculations in the proof of Theorem 1, we can obtain the following inequalities:

$$\left(2C^2H\right)^{(k+1)s} < e^{(k+1)^{-3}c(s)} ;$$

$$(2H)^{(k+t)g} < e^{(k+1)^{-2}c(s)} ;$$

$$s^{2t-1} < e^{\left[(k+1)^{-2}+2/(k+1)\right]c(s)} ;$$

$$(k+1)^{2t+1} < e^{\left[(k+1)^{-4}+2^{-50}(k+1)^{-2}\right]c(s)} .$$

Thus we may deduce that

$$\frac{C_1 s e^{2c(s)}}{C_2 e^{-2kc(s)}}$$

$$\begin{aligned}
&= e^{2(k+1)c(s)} C_1 s/C_2 \\
&\ge e^{2(k+1)c(s)} \cdot \left(2C^2H\right)^{-(k+1)s} \cdot (2H)^{-(k+t)g} \cdot (k+1)^{-2t-1} \cdot s^{-2t+1} \\
&> \exp\left\{\left[2(k+1)-(k+1)^{-3}-2(k+1)^{-2}-2(k+1)^{-1}-(k+1)^{-4}-2^{-50}(k+1)^{-2}\right]c(s)\right\} \\
&> 1 .
\end{aligned}$$

Hence $C_1 s e^{2c(s)} > C_2 e^{-2kc(s)}$. Thus

$$\lambda_{k+1} \le C_1 s e^{2c(s)} .$$

We have, by Lemma 6,

$$\nu_1 \ge (k+1)^{-1}\lambda_{k+1}^{-1} \ge \left[(k+1)C_1 s e^{2c(s)}\right]^{-1} .$$

As in the proof of Theorem 1, we have

$$\left(2C^2\right)^{(k+1)^2 s^2/(2R(s))} s^{R(s)-1} \le e^{(3/2)(k+1)^{-1}c(s)} ,$$

$$[(g+2)T]^{R(s)+k(K+1)(g+1)+q+1} \le e^{2^{-2}(k+1)^{-1}c(s)} ,$$

$$(2H)^{s+g[R(s)+k(k+1)(g+1)+q]} \le e^{2^{-3}(k+1)^{-1}c(s)} ,$$

$$s^{k(k+1)(g+1)+q+2} \le e^{2^{-4}(k+1)^{-1}c(s)} ,$$

$$(k+1)^{(k+1)s/R(s)+1} \le e^{2^{-4}(k+1)^{-1}c(s)} .$$

We can deduce from the above relations that

$$\begin{aligned}(k+1)C_1 se^{2c(s)} &\le e^{2c(s)} \cdot \left(2C^2\right)^{(k+1)^2 s^2/\left(2R(s)\right)} s^{R(s)-1} \\ &\quad \cdot [(g+2)T]^{R(s)+k(k+1)(g+1)+q+1} \cdot (2H)^{s+g[R(s)+k(k+1)(g+1)+q]} \\ &\quad \cdot s^{k(k+1)(g+1)+q+2} \cdot (k+1)^{(k+1)s/R(s)+1} \\ &< e^{2c(s)} \cdot \exp\left\{\left((3/2)+2^{-2}+2^{-3}+2^{-4}+2^{-4}\right)(k+1)^{-1}c(s)\right\} \\ &= e^{\left(2+2(k+1)^{-1}\right)c(s)} = e^{H_k(k+1)^{-1}c(s)} .\end{aligned}$$

Hence

$$\nu_1 > e^{-H_k(k+1)^{-1}c(s)} .$$

We shall show that

$$c(s) \le c(r) . \tag{26}$$

By the definition of μ_1 , we have

$$y\mu_1 = y\|yf_1(\alpha)\|e^{H_k(k+1)^{-1}c(r)} .$$

By the definition of $c_1(r)$, we see that

$$\begin{aligned}c_1(r) &= 2^2\left(g_1+q_1+1\right)\left(\log B_1\right)^{\frac{1}{2}} r(\log r)^{\frac{1}{2}} \\ &\le 4(k+1)^{-2}c(r) < c(r) \quad (k \ge 2) ;\end{aligned}$$

hence

$$g_1(r) = e^{-2c_1(r)} r! > e^{-2c(r)} r! = g(r) .$$

Denote by r'' a positive integer satisfying

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$$g_1(r''-1) \le y < g_1(r'') .$$

Since $g_1(r) > g(r)$, it follows that $r'' < r$. This implies

$$c_1(r'') \le c_1(r) \le 4(k+1)^{-2}c(r) . \tag{27}$$

By the conclusion of Corollary 1.2 $(k = 1)$ and (27), we obtain

$$\begin{aligned} y\mu_1 = y\|yf_1(\alpha)\|e^{H_k(k+1)^{-1}c(r)} &> e^{-4c_1(r'')+H_k(k+1)^{-1}c(r)} \\ &\ge \exp\{[-16(k+1)^{-2}+2(k+2)(k+1)^{-1}]c(r)\} > 1 \quad (k \ge 2) , \end{aligned}$$

hence $\mu_1^{-1} < y$. It follows that $s \le r$ by the definitions of s and r . Thus (26) is true. Finally, we obtain

$$\nu_1 > e^{-H_k(k+1)^{-1}c(r)} .$$

This is contrary to inequality (24), hence the assumption (23) is not valid. This proves Theorem 2.

The proof of Corollary 1.2 is quite similar to the proof of the corollary to Theorem 2 of [*1*].

6. Remarks

If $\mathbb{K}$ were an imaginary quadratic field $\left(\mathbb{K} = \mathbb{Q}(\sqrt{-d})\right)$ in Theorem 1 and its corollaries, then we could use Lemma 31 of Schneider [*4*] in place of Lemma 1 here to construct the auxiliary polynomials in Lemma 2. Further we note that the conjugate to β $(\beta \in \mathbb{K})$ is its complex conjugate $\bar{\beta}$, so $|\beta| = |\bar{\beta}|$. Thus all the details of the proofs of the theorems and corollaries are as in the case $\mathbb{K} = \mathbb{Q}$. Only the parameter B depends on d . Therefore, if $\mathbb{K}$ were an imaginary quadratic field, we would also obtain Theorem 1 and its corollaries.

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