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ON AN EQUIVALENT CLASS OF NORMS FOR BMO

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Abstract

The BMO norm of f is equivalent to

$$\sup_{(x,t)\in R_{+}^{n+1}} \left(\int |f(y) - u(x,t)|^p P_t(x-y)\,dy \right)^{1/p},$$

where P_t is the Poisson kernel. In this note, we show that P_t can be replaced by a nonnegative radial function h, which is positive in a neighbourhood of 0, with $\int_{R^n} h(x)\,dx = 1$ and $\int_1^\infty r^{n-1}(\ln r)^p \tilde{h}(r)\,dr < \infty$, where $\tilde{h}$ is the least decreasing radial majorant of h.

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1. Introduction

Let f be a locally integrable function on R^n. For any sphere S in R^n, let

$$m_S(f) = \frac{1}{|S|} \int_S f(x)\,dx$$

where $|S|$ denote the volume of S. The function f is said to have *Bounded Mean Oscillation* (BMO) if

$$\sup_S \frac{1}{|S|} \int_S |f - m_S(f)| = \|f\|_* < \infty.$$

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It follows from the John-Nirenberg Theorem that for each fixed $1 \le p < \infty$, $\|f\|_*$ is equivalent to

$$\sup_S \left(\frac{1}{|S|}\int_S |f - m_S(f)|^p\right)^{1/p} \quad (= \|f\|_{(p,*)}); \tag{1.1}$$

see [3]. Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \qquad t > 0,$$

be the Poisson kernel and let $u(x,t) = (f * P_t)(x)$. It is also well known that the above $\|f\|_{(p,*)}$ is equivalent to

$$\sup_{(x,t)\in R_+^{n+1}} \left(\int |f(y) - u(x,t)|^p P_t(x-y)\,dy\right)^{1/p} \tag{1.2}$$

where $R_+^{n+1} = R^n \times R^+$.

In this note we will extend the equivalence (1.2) to a larger class of kernels. Let h be a nonnegative radial function on R^n (that is, $h(x) = h(y)$ if $|x| = |y|$), define h_t, $m_{h_t}(f)$ by

$$h_t(x) = t^{-n}h(x/t), \qquad t > 0; \qquad m_{h_t}(f) = \int_{R^n} f(x)h_t(x)\,dx,$$

and the least decreasing radial majorant of h by

$$\tilde{h}(r) = \sup_{|y|\ge r} h(y), \qquad r \ge 0.$$

The notation $\simeq$ will denote the equivalence of the norms, and $\mathscr{S}_0$ will denote the family of spheres centered at 0.

THEOREM 1.1. *Suppose* $1 \le p < \infty$ *and that* h *is a nonnegative radial function satisfying*

(i) $\int_{R^n} h(x)\,dx = 1$ *and* $\int_1^\infty r^{n-1}(\ln r)^p \tilde{h}(r)\,dr < \infty$.

(ii) *there exists* $r_0 > 0$, $\delta > 0$ *such that* $h(r) \ge \delta$ *for* $0 \le r \le r_0$.

Then

$$\sup_{S\in\mathscr{S}_0} \left(\frac{1}{|S|}\int_S |f - m_S(f)|^p\right)^{1/p} \simeq \sup_{t>0}\left(\int |f - m_{h_t}(f)|^p h_t\right)^{1/p}.$$

By translating f and by the equivalence of the BMO norm mentioned in (1.1), Theorem 1.1 implies

COROLLARY 1.2. *Let* $1 \le p < \infty$, *and let* h *be as in Theorem* 1.1. *Then for any locally integrable* f *on* R^n,

$$\|f\|_* \simeq \sup_{(x,t)} \in R_+^{n+1} \left(\int |f(y) - m_{h_t}(f_x)|^p h_t(x-y)\,dy\right)^{1/p}.$$

where $f_x(y) = f(x-y)$, $x, y \in R^n$.

Note that Theorem 1.1 localizes (1.2) at $x = 0$. The special case where $n = 1$ and h equals the Poisson kernel $1/(\pi(x^2+1))$ is used in [2] to study the class of functions of bounded mean oscillations with respect to 0, that is,

$$\sup_{T>0} \frac{1}{2T} \int_{-T}^{T} |f - m_T(f)|]^p < \infty.$$

This class plays the same role to

$$M^p = \left\{ f \colon \|f\| = \sup_{T\geq 1} \left(\frac{1}{2T} \int_{-T}^{T} |f|^p \right)^{1/p} < \infty \right\}$$

(functions with bounded averages) as BMO to L^∞. Another special case is $h(x) = 2\sin^2 x/\pi x^2$, which is used in [1] and [4] to consider the integrated Fourier transformation of functions in M^p. Theorem 1.1 also extends Theorems 4.5, 4.6 in [4] where the theorems were proved in connection with the Wiener's Tauberian Theorem for limit supremum.

We remark that unlike the BMO case, the expressions in Theorem 1.1 are not equivalent for different values of p.

The proofs of Theorem 1.1 and the remark are given in Section 2.

2. The proofs

Let $\Sigma = \{x \in R^n \colon |x| = 1\}$ denote the surface of the unit sphere of R^n, $|\Sigma|$ its surface area, σ the elements of Σ, and $d\sigma$ the elements of the surface area of Σ. Also S_r will denote the sphere in $\mathscr{S}_0$ with radius r.

LEMMA 2.1. *Let $h\colon R^n \to R^+$, $\phi\colon R^1 \to R^+$ be such that*

(i) $\lim_{r\to 0^+} r^n \phi(r)\tilde{h}(r) = a$,

and

(ii) $(r^n\phi(r))'$ *exists a.e. and* $\int \tilde{h}(r)(r^n\phi(r))'dr < \infty$.

Then there exists a constant C (depending only on ϕ and h) such that

$$\int_{R^n} F(x,T)h(x)\,dx \leq C, \quad \textit{for all } T > 0,$$

for any measurable $F\colon R^n \times R^+ \to R^+$ satisfying

$$\frac{1}{|S_t|} \int_{S_t} F(x,T)\,dx \leq \phi(t), \quad \textit{for all } t > 0,\ T > 0. \tag{2.1}$$

PROOF. The technique is to use integration by parts. For $\alpha > 0$, let $S_\alpha \in \mathscr{S}_0$, and for $T > 0$, let

$$F_T(r) = r^{n-1} \int_\Sigma F(r_\sigma, T)\, d\sigma.$$

Then

$$\begin{aligned}
\int_{S_\alpha} F(x,T)h(x)\,dx &= \int_0^\alpha \int_\Sigma F(r_\sigma,T)h(r_\sigma)r^{n-1}\,d\sigma\,dr \\
&\le \int_0^\alpha \tilde{h}(r)F_T(r)\,dr = \int_0^\alpha \tilde{h}(r)d\left(\int_0^r F_T(t)\,dt\right) \\
&= \tilde{h}(\alpha)\int_0^\alpha F_T(t)\,dt - \int_0^\alpha \left(\int_0^r F_T(t)\,dt\right) d\tilde{h}(r) \\
&\le \tilde{h}(\alpha)(\alpha^n|\Sigma|\phi(\alpha)) - \int_0^\alpha (r^n|\Sigma|)\phi(r)\,d\tilde{h}(r) \\
&\qquad\qquad \text{(by (2.1) and since } \tilde{h} \text{ is decreasing)} \\
&= |\Sigma|\left(a + \int_0^\alpha \tilde{h}(r)(r^n\phi(r))'dr\right).
\end{aligned}$$

Letting $\alpha \to \infty$, the result follows from (ii).

Let

$$U_p = \left\{ f \in L^1_{\mathrm{loc}}(R^n) \colon \sup_{S\in\mathscr{S}_0} \frac{1}{|S|}\int_S |f - m_S(f)|^p \le 1 \right\}.$$

LEMMA 2.2. *For $f \in U_p$, $S \in \mathscr{S}_0$ and for $\alpha > 0$,*

$$|m_{\alpha S}(f) - m_S(f)| \le \psi(\alpha),$$

where $\psi = \alpha^{-n/p}\chi_{[0,1]} + 2^{n/p}(1 + 2\ln\alpha)\chi_{(1,\infty)}$.

PROOF. If $0 < \alpha \le 1$, then

$$\begin{aligned}
|m_{\alpha S}(f) - m_S(f)| &\le \left(\frac{1}{|\alpha S|}\int_{\alpha S} |f - m_s(f)|^p\right)^{1/p} \\
&\le \left(\frac{1}{|\alpha S|}\int_S |f - m_S(f)|^p\right)^{1/p} \le \alpha^{-n/p}.
\end{aligned}$$

If $\alpha > 1$, then there exists k such that $2^k < \alpha \le 2^{k+1}$ and

$$
\begin{aligned}
|m_{\alpha S}(f) - m_s(f)| &\le |m_{\alpha S}(f) - m_{2^k S}(f)| + \sum_{j=0}^{k-1} |m_{2^{j+1}S}(f) - m_{2^j S}(f)| \\
&\le \left(\frac{1}{|2^k S|} \int_{2^k S} |f - m_{\alpha S}(f)|^p \right)^{1/p} \\
&\quad + \sum_{j=0}^{k-1} \left(\frac{1}{|2^j S|} \int_{2^j S} |f - m_{2^{j+1}S}(f)|^p \right)^{1/p} \\
&\le (\alpha/2^k)^{n/p} \left(\frac{1}{|\alpha S|} \int_{\alpha S} |f - m_{\alpha S}(f)|^p \right)^{1/p} \\
&\quad + 2^{n/p} \sum_{j=0}^{k-1} \left(\frac{1}{|2^{j+1} S|} \int_{2^{j+1} S} |f - m_{2^{j+1}S}(f)|^p \right)^{1/p} \\
&\le 2^{n/p}(1 + \log_2 \alpha) \\
&\le 2^{n/p}(1 + 2\ln \alpha).
\end{aligned}
$$

PROOF OF THEOREM 1.1. We first show that

$$
\sup_{T>0} \int_{R^n} |f - m_{h_T}(f)|^p h_T \le C_1 \sup_{S \in \mathscr{S}_0} \frac{1}{|S|} \int_S |f - m_S(f)|^p \tag{2.2}
$$

for some $C_1 > 0$. Without loss of generality we assume that $f \in U_p$. For α, $T > 0$ let $S_\alpha \in \mathscr{S}_0$ be the sphere of radius α. Then

$$
\begin{aligned}
&\left(\frac{1}{|S_\alpha|} \int_{S_\alpha} |f(Tx) - m_{S_T}(f)|^p \, dx \right)^{1/p} \\
&\quad \le \left(\frac{1}{|S_{\alpha T}|} \int_{S_{\alpha T}} |f(x) - m_{S_{\alpha T}}(f)|^p \, dx \right)^{1/p} + |m_{S_{\alpha T}}(f) - m_{S_T}(f)| \\
&\quad \le 1 + \psi(\alpha),
\end{aligned}
$$

where $\psi(\alpha)$ is defined as in Lemma 2.2. By letting

$$
F(x,T) = |f(Tx) - m_{S_T}(f)|^p,
$$

we have

$$
\frac{1}{|S_\alpha|} \int_{S_\alpha} F(x,T) \, dx \le (1 + \psi(\alpha))^p \le 2^p(1 + \psi^p(\alpha)).
$$

Let $\phi(\alpha) = 2^p(1 + \psi^p(\alpha))$. Then

$$
(\alpha^n \phi(\alpha))' = \begin{cases} n 2^p \alpha^{n-1}, & 0 \le x \le 1, \\ 2^p \alpha^{n-1}[n + 2^n(1 + 2\ln\alpha)^{p-1}(n + 2n\ln\alpha + 2p)], & x > 1. \end{cases}
$$

The assumption that $\int_1^\infty r^{n-1} (\ln r)^p \tilde{h}(r) \, dr < \infty$ implies that

$$
\int_0^\infty (\alpha^n \phi(\alpha))' \tilde{h}(\alpha) \, d\alpha < \infty.
$$

Hence by Lemma 2.1,

$$\int_{R^n} |f(Tx) - m_{S_T}(f)|^p h(x)\, dx = \int_{R^n} F(x,T)h(x)\, dx \le C$$

for some $C > 0$. Inequality (2.2) now follows from below

$$\begin{aligned}
&\left(\int_{R^n} |f(x) - m_{h_T}(f)|^p h_T(x)\, dx\right)^{1/p} \\
&\quad = \left(\int_{R^n} |f(Tx) - m_{h_T}(f)|^p h(x)\, dx\right)^{1/p} \\
&\quad \le \left(\int_{R^n} |f(Tx) - m_{S_T}(f)|^p h(x)\, dx\right)^{1/p} + |m_{h_T}(f) - m_{S_T}(f)| \\
&\quad \le 2\left(\int_{R^n} |f(Tx) - m_{S_T}(f)|^p h(x)\, dx\right)^{1/p} \le 2C.
\end{aligned}$$

For the reverse inequality of the equivalence, we let $c = 2(\delta|S_{r_0}|)^{-1/p}$, and for any $T > 0$, let $T' = Tr_0$. It follows that

$$\begin{aligned}
&\left(\frac{1}{|S_{T'}|}\int_{S_{T'}} |f(x) - m_{S_{T'}}(f)|^p\, dx\right)^{1/p} \\
&\quad \le \left(\frac{1}{|S_{T'}|}\int_{S_{T'}} |f(x) - m_{h_T}(f)|^p\, dx\right)^{1/p} + |m_{h_T}(f) - m_{S_{T'}}(f)| \\
&\quad \le 2\left(\frac{1}{|S_{T'}|}\int_{S_{T'}} |f(x) - m_{h_T}(f)|^p\, dx\right)^{1/p} \\
&\quad \le 2\left(\frac{1}{|S_{r_0}|}\int_{s_{r_0}} |f(Tx) - m_{h_T}(f)|^p\, dx\right)^{1/p} \\
&\quad \le c\left(\int_{S_{r_0}} |f(Tx) - m_{h_T}(f)|^p h(x)\, dx\right)^{1/p} \\
&\qquad\qquad (\text{since } h(x) = h_0(r) \ge \delta \text{ for } 0 < r \le r_0, |x| = r) \\
&\quad \le c\left(\int_{R^n} |f(Tx) - m_{h_T}(f)|^p h(x)\, dx\right)^{1/p} \\
&\quad \le c\left(\int_{R^n} |f(x) - m_{h_T}(f)|^p h_T(x)\, dx\right)^{1/p}.
\end{aligned}$$

To justify the remark in Section 1, we let $1 \le p < q < \infty$, and let

$$A_j = \{x \in R^1 : 2^j \le |x| < 2^j + 2^{j/2}\}.$$

Define

$$f(t) = \sum_{j=0}^{\infty} 2^{j/2p} \chi_{A_j}(t)\,\mathrm{sgn}(t), \qquad t \in R.$$

Since f is an odd function, $m_{[-T,T]}(f) = 0$, for all $T > 0$. For $2^k \leq T < 2^{k+1}$,

$$\frac{1}{2T}\int_{-T}^{T} |f - m_{[-T,T]}(f)|^p \leq \frac{1}{2^{k+1}} \sum_{j=0}^{k} \int_{A_j} |2^{j/2p}|^p \, dt \leq 2.$$

On the other hand, for $T = 2^k$,

$$\frac{1}{2T}\int_{-T}^{T} |f - m_{[-T,T]}(f)|^q \geq \frac{2^{kq/2p}}{2^{k/2}},$$

which tends to ∞ as $h \to \infty$.

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