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NEW TAUBERIAN THEOREMS FROM OLD

Dedicated to the memory of Professor B. Kuttner

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ABSTRACT. A new and very general and simple, yet powerful approach is introduced for obtaining new Tauberian theorems for a summability method V from known Tauberian conditions for V, where V is merely assumed to be linear and conservative. The technique yields the known theorems on the weakening of Tauberian conditions due to Meyer-König and Tietz and others and also improves many of them. Several new results are also obtained, even for classical methods of summability, including analogues of Tauber's second theorem for the Borel and logarithmic methods. The approach yields also new Tauberian conditions for the passage from summability by a method V to summability by a method V', as well as to more general methods of summability like absolute summability or summability in abstract spaces; the present paper however confines itself to ordinary summability.

1. **Introduction.** A classical theorem of Tauber states that an Abel-summable sequence s_n [or series $\sum a_n$] is indeed convergent if $(T1)$: $na_n := n(s_n - s_{n-1}) = o(1)$. We describe this by saying that $(T1)$ is a "Tauberian condition" for the Abel method A. Tauber proved further that the weaker condition $(T2)$: $\delta_n := (n+1)^{-1}\sum_{k=0}^{n} ka_k = o(1)$ is also a Tauberian condition for A. In 1967, Meyer-König and Tietz [13] focused attention on the result that as a matter of fact for *any* regular and additive method V, if $(T1)$ is a Tauberian condition, so is $(T2)$. Indeed this result is only the special case of a (slightly more general!) Tauberian theorem for additive regular methods proved by me some years earlier ([18], special case of Theorem 1, with $c = \alpha = 0$). Meyer-König and Tietz's announcement was followed by a series of papers devoted to proving that for any linear regular [or conservative] method V, if a certain condition T is Tauberian for V, so is a certain other condition T' ([12], [14], [15], [20] and others). The technique of proof in these papers depended on the following facts: (i) under the hypotheses [of any theorem considered], the conditions T and T' were in terms of certain transformations T_1s and T_2s of the sequence s; and (ii) if T' was satisfied, then it was possible to obtain an expression for $s = \{s_n\}$ in terms of T_2s and T_1s. (See for instance Leviatan's theorem given as Corollary 3.3 below and also the remarks in ([15], p. 181). Thus this approach cannot yield proof of theorems like the following:

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THEOREM 1 (GÖES [3]). *If $T_1\colon \sum(-1)^n na_n$ converges is a Tauberian condition for a linear conservative method A, then the [weaker] condition $T_2\colon na_n = o(1)$ is also Tauberian for A.*

THEOREM 2. *Let V be a linear conservative method with $T_1\colon \sqrt{n}a_n = o(1)$ as a Tauberian condition. Then the condition*

$$T_2\colon a_n = o(1) \quad \text{and} \quad e^{-\sqrt{n}} \sum_{k=0}^{n} e^{\sqrt{k}} a_k + e^{\sqrt{n+1}} \sum_{n+1}^{\infty} e^{-\sqrt{n+1}} a_k = o(1)$$

is also a Tauberian condition for V.

We now present a new approach which is free of the shortcomings noted above; besides being very general and simple, it yields all the results in the papers quoted above, as well as several new results, even for well known methods. In particular, it yields a result (Theorem 14.1 below) which is more general than Theorem 1 and which can be proved neither by the approach used by Meyer-König and Tietz and others nor by the method of proof adopted by Göes.

2. **The Main Theorem and its special form.**

THEOREM 3 [MAIN THEOREM]. *Let M^*, τ_1, τ_2 and κ be classes of sequences (of numbers) and P a sequence-to-sequence transformation such that*

(3.1) *(i) κ is additive: $x, y \in \kappa \Rightarrow x + y \in \kappa$;*

(3.2) *(ii) $x, y \in M^* \Rightarrow x - y \in M^*$;*

(3.3) *(iii) $M^* \cap \tau_1 \subset \kappa$;*

(3.4) *(iv) $s \in M^* \cap \tau_2 \Rightarrow$ (a) $Ps \in \tau_1$ and (b) $(I - P)s \in M^* \cap \kappa$.*

(3.5) *Then $M^* \cap \tau_2 \subset \kappa$.*

REMARK. If we take M^* to be the "summability field" (M) [in some sense] of a linear method M and take κ to be the "summability field" of the identity transformation ($= c$, the space of convergent sequences if M is an ordinary summability method applicable to some sequences), then the hypothesis (3.3) says that τ_1 is a Tauberian class for the method M and the conclusion (3.5) says that τ_2 is also a Tauberian class for M.

PROOF. Let (3.1)–(3.4) hold and let $s \in M^* \cap \tau_2$. Then $(I - P)s \in M^* \cap \kappa$ by (3.4). Since $s \in M^*$, by (3.2) we get $s - (I - P)s = Ps \in M^*$. But $Ps \in \tau_1$ by (3.4). Hence by (3.3), $Ps \in M^* \cap \tau_1 \subset \kappa$. Since $(I - P)s \in \kappa$ we see by (1) that $s = Ps + (I - P)s \in \kappa$. This proves the theorem.

COROLLARY 3.1. *Let M^*, κ be linear spaces of sequences, T_2 an operator, B a sequence to sequence transformation and let β, τ_1 be classes of sequences such that*

(i) $M^* \cap \tau_1 \subset \kappa$ and

(ii) $s \in M^*$ and $T_2 s \in \beta \Rightarrow B(T_2 s) \in M^* \cap \kappa$ and $(I - BT_2)s \in \tau_1$.

Then $s \in M^*$ and $T_2 s \in \beta \Rightarrow s \in \kappa$.

PROOF. This is immediate from Theorem 3 on setting $P = I - BT_2$ and $\tau_2 = \{s : T_2 s \in \beta\}$.

COROLLARY 3.2. *Let α, β be classes of sequences, M^*, κ classes of sequences satisfying (3.1) and (3.2) and let T_1, T_2, A and B be operators such that*

(i) $T_1 s \in \alpha$ and $s \in M^*$ imply that $s \in \kappa$;

(ii) $T_2 s \in \beta$ and $s \in M^*$ imply that

(3.6) *(a) $T_1 s = A(T_2 s) + T_1[B(T_2 s)]$ and*

(3.7) *(b) $B(T_2 s) \in M^* \cap \kappa$ and $A(T_2 s) \in \alpha$.*

(3.8) *Then $T_2 s \in \beta$ and $s \in M^*$ imply that $s \in \kappa$.*

PROOF. We apply Theorem 3 with $\tau_1 = \{s : T_1 s \in \alpha\}, \tau_2 = \{s : T_2 s \in \beta\}$ and $P = I - B.T_2$. Now, if $s \in M^* \cap \tau_2$, then by (3.6) and (3.7), $[T_1 - T_1(B.T_2)]s = A(T_2 s) \in \alpha$; that is, $T_1[(I - BT_2)s] = T_1 Ps \in \alpha$; hence $Ps \in \tau_1$. But also by (3.7), $(I-P)s = (BT_2)s \in M^* \cap \kappa$. Thus the hypotheses of Theorem 3 hold and hence $M^* \cap \tau_2 \subset \kappa$; that is, (3.8) holds.

REMARK. Corollary 3.2 is stronger and more general than the following theorem due to Leviatan which describes in abstract terms the hypotheses and techniques used by Meyer-König and Tietz and Stieglitz ([15], 20), so that their results are obtainable from that theorem.

COROLLARY 3.3 (LEVIATAN [12]). *Let V be a regular additive summability method with convergence field (V) and let α, β be any one of c_0, c and m. Suppose that T_1 is an operator which has a right inverse T_1^{-1}, that $\tau_1 = \{s : T_1 a \in \alpha\}$, where $a_n = s_n - s_{n-1}$ and that*

(L1.1): $T_1(T_1^{-1} a) = a$ for each $a \in \alpha$,

(L1.2): τ_1 is a Tauberian class for V; that is, $\tau_1 \cap (V) \subset c$.

Suppose also that A and B are sequence to sequence transformations such that

(L2): $a \in \beta \Rightarrow Ba \in C$ and $Aa \in \alpha$

and let T_2 satisfy the relation

(L3): $s_n = \sum_{k=0}^{n} \big(T_1^{-1}[A(T_2 a)] + B(T_2 a)\big)_k$ for all $s \in (V) \cap \tau_2$, where $\tau_2 = \{s : T_2 a \in \beta\}$.

Then $\tau_2 \cap (V) \subset c$.

PROOF. This follows at once from Corollary 3.2 whose hypotheses (i), (ii)a and (ii)b are implied by the conditions (L1.2), (L3) and (L2) respectively.

REMARK. Theorem 3 and Corollary 3.2 are strikingly more general. We note some of the significant differences: (1) Leviatan considers only Tauberian classes of the type $\tau = \{s : Ts \in \alpha\}$ where $\alpha = c_0$, c or m and where T is an operator; in Corollary 3.2, the Tauberian class can be of much more general type (for instance τ can be the class of slowly decreasing sequences), and in Theorem 3 it need not also correspond to any operator T; (2) Leviatan takes the known Tauberian class as given by an operator which has a right inverse; no such assumption is made in Corollary 3.2; (3) The possibility of solving for $s \in (V) \cap \tau_2$ in terms of $T_2 s$ (making use of T_1^{-1} *etc.*) is an essential part of the technique used by Meyer-König and Tietz and by Leviatan as well; (4) Leviatan's proof makes use of the fact that (V) is the convergence field of a conservative method,

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whereas in Corollary 3.2 it need not be convergence field of any method, whether conservative or not; (5) κ is not restricted to be c (see for instance Theorem 7 below); and (6) it is important to note that Theorem 3 can be applied to situations other than ordinary summability of number sequences: to absolute, or strong summability and to summability of sequences in groups or topological vector spaces. (In the present paper we confine ourselves to ordinary summability.)

A practical difficulty in attempting to use Corollary 3.2 is in finding the operators A and B which will satisfy the hypotheses; similarly the difficulty in trying to use Theorem 3 is in finding the sequence-to-sequence transformation P of the theorem, and sometimes in verifying that $Ps \in \tau_1$. It is therefore remarkable that the following special case of Theorem 3, with $P = I - T_2$, not only covers the most familiar and interesting cases in an elegant manner but also enables us to prove some new theorems, including $V \longrightarrow V'$ Tauberian theorems (where one passes from summability by a method V to summability by a method V'), where V' is not equal to convergence.

THEOREM 4. *(a) Let (M) be the convergence field of a linear conservative method M and E a subset of (M) and F a set of sequences. Let T_1, T_2 be sequence to sequence transformations such that*

(4.1) *(i) $T_1 s \in E$, $s \in (M)$ imply that $s \in F$;*

(4.2) *(ii) $T_2 s \in F$, $s \in (M)$ imply that $T_1(I - T_2)s \in E$.*

(4.3) *Then $T_2 s \in F$, $s \in (M)$ imply that $s \in F$.*

(b) In (4.2) above we may replace $T_2 s$ by $T_2^* s$ where T_2^* is any transformation such that $T_2 s \in F$ implies that $T_2^* s \in F$.

{In the most common examples, $E = c_0$ or c, and $F = c_0$, c or m.}

PROOF. Part (a): We apply Theorem 3 with $\tau_1 = \{s : T_1 s \in E\}$, $\kappa = F$, $\tau_2 = \{s : T_2 s \in F\}$ and $P = I - T_2$. Part (b): This follows from part (a).

3. **Examples, applications and new Tauberian theorems.**

NOTATION. Throughout this section, let V denote a linear conservative method for sequences and (V) the summability field of V. (However, sometimes we may not need the full hypotheses on V.) The symbols s_n and a_n are related by the equations $a_n = s_n - s_{n-1}$ ($s_{-1} = 0$) for $n = 0, 1, \ldots$. The symbols $\{p_n\}$, $\{q_n\}$ will always denote sequences of positive numbers. (We assume this property in order to avoid the phraseology required to deal with the case where finitely many p_n or q_n may be 0.) When the symbol u represents a sequence given by a complicated expression, we shall write $u|_n$ or $(u)_n$ or $[u]_n$ for the n-th term u_n of $u = \{u_n\}$. We shall also find it convenient to write $u = o(1)$ $[O(1), O_L(1)]$ for $u_n = o(1)$ $[O(1), O_L(1)]$; the symbols $u = o(f_n)$ *etc.* for a function f_n are also to be interpreted similarly. The symbol δ will always denote the sequence $\{\delta_n\}$ where $\delta_n := (n+1)^{-1} \sum_{k=0}^{n} k a_k$; we also define $L_n := \sum_{k=0}^{n} 1/(k+1)$ for $n = 0, 1, \ldots$. We say that a condition $Ts \in E$ is a TC (= Tauberian condition) for a method V if $s \in c$ whenever $s \in (V)$ and $Ts \in E$.

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Unless otherwise stated, the transformations T_1, T_2, T_2^* are defined by the relations $T_1 s|_n = p_n a_n$, $T_2 s|_n = q_n^{-1} \sum_{k=0}^{n} q_k a_k$ and $T_2^* s|_n = q_{n+1}^{-1} \sum_{k=0}^{n} q_k a_k$ for $n = 0, 1, \ldots$. We also have then

$$(4.4)\quad \begin{aligned} D_n := T_1(I - T_2^*)s|_n &= p_n a_n - p_n\Big[q_{n+1}^{-1}\sum_{k=0}^{n} q_k a_k - q_n^{-1}\sum_{k=1}^{n-1} q_k a_k\Big] \\ &= p_n a_n - p_n\Big[\Big(\sum_{k=0}^{n} q_k a_k\Big)(q_{n+1}^{-1} - q_n^{-1}) + a_n\Big] \\ &= p_n(q_{n+1} - q_n)\Big(\sum_{k=0}^{n} q_k a_k\Big)\Big/(q_n q_{n+1}) \\ &= \big(p_n(q_{n+1} - q_n)/q_{n+1}\big)(T_2 s|_n) \\ &= F(n)(T_2 s|_n) \quad \text{(say).} \end{aligned}$$

Equally,

$$(4.5)\quad \begin{aligned} D_n &= \big(p_n(q_{n+1} - q_n)/q_n\big)(T_2^* s|_n) \\ &= F^*(n)(T_2^* s|_n) \quad \text{(say).} \end{aligned}$$

THEOREM 5. *(a) Let E, F be (not necessarily linear) classes of sequences. Let $T_1 s \in E$ be a* TC *for V and let $s \in (V)$ be such that $T_2^* s \in F$. If $(I - T_2^*)s \in (V)$ and $\{D_n\} \in E$, then $(I - T_2^*)s \in c$.*

In particular, if $T_2^* s \in c$ and $\{D_n\} \in E$, then $s \in c$.

(b) If (i) $\{q_n\}$ is nondecreasing, (ii) $F(n) = O(1)$ and (iii) $T_1 s = o(1)$ is a TC *for V, then the condition $T_2 s = o(1)$ is also a* TC *for V.*

(c) If (i) $F(n) = O(1)$ and (ii) $T_1 s = o(1)$ is a TC *for V, then the condition $T_2^* s = o(1)$ is also a* TC *for V.*

(d) If $\{F(n)\} \in c$ and $T_1 s \in c$ is a TC *for V, then the condition $T_2^* s \in c$ is also a* TC *for V.*

PROOF. (a): $\{D_n\} \in E$ means that $T_1[(I - T_2^*)s] \in E$. Applying Theorem 4 with $(M) = (V)$ and with T_2^* instead of T_2, we get the required result.

(b), (c): The hypotheses in each case ensure that $T_2^* s = o(1)$ and that $\{D_n\} \in c_0$. The results then follow from part (a), with $E = c_0$.

(d): Now $\{D_n\} = T_1[(I - T_2^*)s] \in c$, since $T_2^* s \in c$; hence $(I - T_2^*)s \in c$ and the result follows.

COROLLARY 5.1. *Let $E = c_0$ or c. If $\{na_n\} \in E$ is a* TC *for V, then so is the condition $\{\delta_n\} \in F$, provided that (i) $E = F = c_0$, or (ii) $E = F = c$ or (iii) $E = c_0$, $F = c$ and V-$\lim \delta = 0$.*

PROOF. Let $T_1 s|_n = na_n$ and $T_2^* s|_n = \delta_n$ and apply Theorem 5(d) with $p_n = q_n = n$.

{Parameswaran [18] proved cases (i) and (iii); Meyer-König and Tietz [13] proved case (i). The further special case V = Abel's method and $E = F = c_0$ correspond to Tauber's classical Tauberian theorems.}

We note also that case (i) of Corollary 5.1 can be expressed in either of the following [equivalent] forms: If $na_n = o(1)$ is a TC for V, then for any sequence s in (V),

(5.1) $$s \in c \quad \text{if and only if } n(t_n - t_{n-1}) = o(1), \text{ where } t = C_1 s;$$

(5.2) $$s \in c \quad \text{if and only if } (I - C_1)s = o(1).$$

(5.3) $$s \in c \quad \text{if and only if } C_1(\{na_n\}) = o(1).$$

COROLLARY 5.2. *If $T_1 s|_n := n^{1/2} a_n = o(1)$ is a* TC *for V, then so is the condition $T_2^* s \in c$, where $T_2^* s|_n = (n+1)^{-1/2} \sum_{k=1}^n k^{1/2} a_k$.*

PROOF. Let $p_n = q_n = n^{1/2}$ and $T_2^* s \in c$. Then the relations (4.4), (4.5) give $T_1(I - T_2^*)s|_n = D_n = [(n+1)^{1/2} - n^{1/2}]T_2^* s|_n = o(1)$. The result then follows from Theorem 5(a) with $E = c_0$.

{Corollary 5.2 is sharper than each of two similar results given by Meyer-König and Tietz ([15], p. 182).}

COROLLARY 5.3. *If $T_1 s := \{n^{1/2} a_n\} \in E$ is a* TC *for V, where $E = c_0$ or c, then so is the condition $T_2 s \in E$, where $T_2 s|_n = e^{-\sqrt{n}} \sum_{k=0}^n e^{\sqrt{k}} a_k$.*

PROOF. Apply Theorem 5 with $p_n = q_n = e^{\sqrt{n}}$. (Or, we could appeal directly to Theorem 4, with $F = E$.) A little calculation shows that if $T_2 s \in E$, then $T_1(I - T_2)s = \{r_n T_2 s|_n\} \in E$ (since $\lim r_n = 1$).

{Meyer-König and Tietz ([15], p. 182, Satz 2.10 et seq.) proved this for $E = c_0$. The result for $V =$ the Borel method is due to Karamata [6].}

COROLLARY 5.4. *Let $E = c_0$ or m, and $p_n = O(n \log n)$. Let $T_1 s|_n := p_n a_n \in E$ be a* TC *for V. Then the condition $\{\delta_n \log n\} \in E$ is also a* TC *for V.*

PROOF. We apply Theorem 5(a) with $T_2^* s|_n = \delta_n$ and $F = E$. Since $\{\delta_n\} = (I - C_1)s \in c$, the result follows.

{We remark that the typical methods with $na_n \log n = O(1)$ as TC are the logarithmic methods l and L (19). The special case of Corollary 5.4 where $V = L$ was proved in ([17], Theorem 2).}

COROLLARY 5.5. *Theorem 2 stated above.*

{This will follow by a straightforward application of Theorem 4. We omit the details.}

THEOREM 6. *Let $T_1 s = \{na_n \log n\} \in E$ be a* TC *for V, where $E = c_0$ or c and let $T_2 s|_n := (1/\log n) \sum_{k=0}^n a_k \log(k+1)$. Then for any squence $s \in (V)$ we have: $s \in c$ if and only if $T_2 s \in E$. [Indeed then $T_2 s = o(1)$.]*

PROOF (SUFFICIENCY). We apply Theorem 5(a) with $F = E$, $p_n = n \log n$ and $q_n = \log(n+1)$. Then $D_n = T_1(I - T_2^*)s|_n = r_n T_2 s|_n$ where $r_n = n/(n+h)$ for some $h = h(n)$ between 0 and 1. Hence $\{D_n\} \in E$ and the result follows from Theorem 5(a).

(NECESSITY). Let $s \in c$. Then $t = ls \in c$, and, following Ishiguro ([4], p. 157), we see that $(I - l)s|_n = s_n - t_n = lu|_n + o(1) = T_2 s|_n + o(1)$, where $u_k = (k+1)a_k \log(k+1)$. But $(I - l)s|_n = o(1)$, by regularity of the logarithmic method l and hence $T_2 s = o(1)$.

{The case when $E = c_0$ was proved by Meyer-König and Tietz ([14], p. 211); the case when $E = c_0$ and $V = L$ was proved by Kaufman ([7], Theorem 2b).}

We now show how Theorem 3 enables us to obtain $V \longrightarrow V'$ Tauberian theorems also.

THEOREM 7. *Let $p_n = O(n)$, $T_1 s|_n = p_n a_n$ and $T_2 s|_n = \delta_n$.*

(a) Let V be linear (not necessarily conservative) with $V \subset V.C_1$ and let $T_1 s|_n \in E = O(1)$ $[O_L(1)]$ be a $V \longrightarrow C_1$ Tauberian condition. Then $T_2 s|_n = \delta_n \in E$ is also a $V \longrightarrow C_1$ Tauberian condition.

(b) Let V be linear and conservative and let $T_1 s|_n \in E = c_0$ be a TC *for V. Then the condition $T_2 s|_n = \delta_n \in E$ is also a Tauberian condition for V.*

(c) Let V be linear and conservative, $T_1 s|_n \in E = c$ a TC *for V and $\{p_n/n\} \in E$. Then the condition $T_2 s|_n = \delta_n \in E$ is also a Tauberian condition for V.*

(d) Let $V \subset V.C_1$ and let $T_1 s|_n = o(1)$ be a $V \longrightarrow C_1$ Tauberian condition. Then $T_2 s|_n = \delta_n \in O(1)$ is also a $V \longrightarrow C_1$ Tauberian condition.

PROOF. (a), (d): We apply Theorem 3 with $\kappa = (C_1)$, $P = I - T_2$ and $M^* = (V)$. Let $s \in (V)$ and $\delta_n \in E$. Then

(7.1) $$D_n = T_1(I - T_2)s|_n = [p_n/(n+1)](T_2 s|_n) \in E, \text{ since } p_n = O(n).$$

Hence it follows from Theorem 3 that $s \in \kappa = (C_1)$.

(b), (c): We apply Theorem 3 with $\kappa = c$, $P = I - T_2$ and $M^* = (V)$. Let $s \in (V)$ and $\delta_n \in E$. Then (7.1) holds if $E = c_0$ or if $E = c$ and $\{p_n/n\} \in c$. Also, $T_2 s \in c$ implies, by the linearity of V that $(I - T_2)s \in (V)$. Then $(I - T_2)s \in c$, by (7.1). It follows that $s \in c$.

{The result (b) was proved by Meyer-König and Tietz ([14] Satz 2.1, Satz 2.3) under additional hypotheses. The result in the special case of (b) with $V = B$ (Borel's method) and $p_n = n^{1/2}$ was proved by Jakimovski ([5]; the result in the special case of (b) with $V =$ Abel's method and $p_n = n$ is the classical Tauber's theorem; the special case of (a) with $V =$ Abel's method was given by O. Szasz [21]. The results in Theorem 7 are closely related to Theorem 1 of [18].}

Analogous to Tauber's second theorem for the Abel method and its generalization given above as Corollary 5.1, case (i) and (5.1), we have the following result applicable to the logarithmic methods in particular.

THEOREM 8. *(a) Let l be the logarithmic method and V a linear conservative method with $V \subset V.l$ and let $E = c_0$ or c. Let*

(8.1) $$T_1 s|_n := (n+1)\log(n+1)a_n \in E$$

be a TC *for V. Then for any sequence $s \in (V)$:*

(8.2) $$s \in c \quad \textit{if and only if } T_2 s|_n := (n+1)\log(n+1)(t_n - t_{n-1}) \in E$$

[and then $T_2 s = o(1)$].

(b) In part (a) we may take $E = m$ and replace (8.2) by

(8.3) $$t \in c \quad \textit{if and only if } T_2 s|_n := (n+1)\log(n+1)(t_n - t_{n-1}) \in E.$$

PROOF. Let $T_2's|_n := (n+1)\log n\,(t_n - t_{n-1})$. Then $T_2 s \in E \Leftrightarrow T_2's \in E$. Also,

(8.4)

$$
\begin{aligned}
(I - T_2')s|_n &= s_n - (n+1)\log n\Big\{\big(1/\log(n+1)\big)\sum_{k=0}^{n} s_k/(k+1) - (1/\log n)\sum_{k=0}^{n-1} s_k/(k+1)\Big\}\\
&= s_n - (n+1)\log n\Big\{\Big(\sum_{k=0}^{n} s_k/(k+1)\Big)\Big(\big(1/\log(n+1)\big) - (1/\log n)\Big)\\
&\qquad + s_n/[(n+1)\log n]\Big\}\\
&= (n+1)[\log(n+1) - \log n]\, ls|_n\\
&= (n+1)t_n/(n+h_n) = r_n t_n \text{ where } \lim r_n = 1.
\end{aligned}
$$

Suppose now that $s \in c$. Then by the regularity of l, $\lim t_n = \lim s_n$ exists; hence by (8.4), $(I - T_2')s \in c$ and therefore $\lim T_2 s|_n = \lim T_2's|_n = \lim s_n - \lim r_n t_n = 0$; that is, (8.2) holds with $E = c_0$.

Conversely, suppose that $s \in (V)$ and $T_2 s \in E$, so that also $T_2's \in E$. From (8.2), $T_2 s = T_1(ls)$ and $ls = t \in (V)$, since $V \subset V.l$. Hence $t \in c$. By (8.4), $(I - T_2')s \in c$ and therefore $(I - T_2)s \in c$. But $T_2 s \in c$ by (8.2) and hence $s \in c$. This proves part (a). Part (b) follows from the fact that $ls = t \in (V)$, since t is given to satisfy the TC for V.

COROLLARY 8.1. *For the logarithmic method L, the relation (8.2) holds with $E = c_0$ or c and the relation (8.3) holds with $E = m$.*

PROOF. For $L \subset L.l$ (by [10], Lemma 3) and (8.1) is a TC for L.

{The result appears to be new even in this particular case.}

THEOREM 9. *Let $V \subset V.C_1$ and let $T_1 s|_n := na_n \log n = O_L(1)$ be a* TC *for V. Then*

$$T_2^* s|_n := \delta_n = (n+1)^{-1}\sum_{k=0}^{n} ka_k = O_L(1/\log n) \tag{9.1}$$

is a $V \longrightarrow C_1$ Tauberian condition.

PROOF. Let $s \in (V)$ satisfy (9.1). We shall apply Theorem 5(a) with $p_n = n\log n$, $q_n = n$ and $E = \{s : na_n \log n = O_L(1)\}$. Then $D_n = T_1.(I - T_2^*)s|_n = (n\log n)(1/n)O_L(1/\log n) = O_L(1)$. Hence by Theorem 5(a), $(I - T_2^*)s = C_1 s \in c$.

{In particular, (9.1) is an $L \longrightarrow C_1$ Tauberian condition, where L is the logarithmic method, as proved in ([17], Theorem 2(a)).}

THEOREM 10. *Let p_n, m_n, q_n be positive sequences such that $\{q_n\}$ is nondecreasing and let*

(10.1) *(i)* $m_n = O(q_n)$

and

(10.2) *(ii)* $D_n = p_n(q_{n+1} - q_n)/q_{n+1}$ *is either* $O(q_n/m_n)$ *or* $O(1)$.

Let $T_1 s|_n := p_n a_n = o(1)$ be a TC *for V. Then the condition*

$$T_2's|_n := (1/m_n)\sum_{k=0}^{n} q_k a_k = o(1) \tag{10.3}$$

is also a TC *for* V.

PROOF. Let $s \in (V)$ satisfy (10.3). Then $T_2 s|_n = (1/q_n)\sum_{k=0}^{n} q_k a_k = (m_n/q_n)T_2' s|_n = O(1)o(1) = o(1)$, and $T_2^* s|_n := (1/q_{n+1})\sum_{k=0}^{n} q_k a_k = (m_n/q_{n+1})T_2' s|_n = O(1)o(1) = o(1)$. Since V is conservative, it follows that $(I - T_2^*)s \in (V)$. Now $T_1(I - T_2^*)s|_n = D_n(T_2 s|_n) = (m_n/q_n).(T_2' s|_n).p_n(q_{n+1} - q_n)/q_{n+1} = o(1)$, by (10.1) and (10.2). Hence $(I - T_2^*)s \in c$. But $T_2^* s = o(1)$ and hence $s \in c$.

{The special case where $m_n = n$ and $p_n = q_n$ was given by Meyer-König and Tietz ([14], Satz 2.3).}

COROLLARY 10.1. *Let* $E = c_0$ *or* m *and let* $T_1 s|_n := n a_n \log n \in E$ *be a* TC *for* V. *Then*

$$T_2' s|_n := (n+1)^{-1}\sum_{k=0}^{n}(k+1)a_k \log(k+1) = o(1) \tag{10.4}$$

is also a TC *for* V.

PROOF. Apply Theorem 10 with $p_n = n\log n$, $m_n = n+1$ and $q_n = (n+1)\log(n+1)$.

{This result has been given also by Tietz ([23], p. 74). Kaufman's result that (10.4) is a TC for the method L ([7]) is, by Corollary 10.1, deducible from Rangachari and Sitaraman's result that $T_1 s = O(1)$ is a TC for L ([19], Theorem I(L)). It is to be noted that (10.4) is sufficient, but not necessary for the convergence of s, as is seen from the example where $a_n = (-1)^n/\log n$. Our next two theorems give necessary and sufficient conditions for a V-summable sequence to be convergent, where V is as in Corollary 10.1.}

THEOREM 11. *Let* $E = c_0$ *or* c, *and let* $T_1 s|_n := n a_n \log n \in E$ *be a* TC *for* V. *Let* M *be the regular transformation defined by* $t = Ms$ *where* $t_n := L_n^{-1}\sum_{k=0}^{n} s_k/(k+1)$ *and* $L_n = 1 + 1/2 + \cdots + 1/(n+1)$. *Let* $T_2 s|_n := (n+1)L_n(t_n - t_{n-1})$. *Then for any sequence* $s \in (V)$:

$$s \in c \Leftrightarrow T_2 s \in E \Leftrightarrow (I - M)s \in E \tag{11.1}$$

[and then $T_2 s = o(1)$ *and* $(I - M)s = o(1)$*].*

PROOF. Let $T_2^* s|_n := (n+1)L_{n-1}(t_n - t_{n-1})$. Then $T_2 s \in E \Leftrightarrow T_2^* s \in E$. For $n > 1$ we have

$$\begin{aligned} T_2^* s|_n &= (n+1)L_{n-1}\Big[L_n^{-1}\Big(\sum_{k=0}^{n} s_k/(k+1)\Big) - L_{n-1}^{-1}\Big(\sum_{k=0}^{n-1} s_k/(k+1)\Big)\Big] \\ &= (n+1)L_{n-1}\Big[\Big(\sum_{k=0}^{n} s_k/(k+1)\Big)(L_n^{-1} - L_{n-1}^{-1}) + s_n/\big((n+1)L_{n-1}\big)\Big] \\ &= s_n + (n+1)L_{n-1}\Big(\sum_{k=0}^{n} s_k/(k+1)\Big)(-1)/[(n+1)L_n L_{n-1}] \\ &= s_n - t_n = (I - M)s|_n. \end{aligned} \tag{11.2}$$

Suppose now that $s \in c$. Then $s_n - t_n = o(1)$ by the regularity of M; hence, $s \in c \Rightarrow T_2^* s \in c_0 \Leftrightarrow T_2 s \in c_0 \Leftrightarrow (I - M)s \in c_0$.

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Conversely, suppose that $s \in (V)$ and $T_2 s \in E$. Then $T_2^* s \in E$, and hence $(I - T_2^*)s \in (V)$. Also $T_2 s \in E$ implies that $T_1(I - T_2^*)s|_n = T_1 t|_n = (n \log n)(t_n - t_{n-1}) \in E$. Hence $(I - T_2^*)s \in c$. But $T_2^* s \in c$, so that also $s \in c$.

{Theorem 11 may be called *Tauber's second theorem* for the method V; note the similarity in form between this theorem and the classical Tauber's second theorem expressed in the form of Corollary 5.1 above, as expressed by (5.1) and (5.2).}

COROLLARY 11.1 (KWEE [10], THEOREM 2). *If $V = L$, the logarithmic method, then (11.1) holds.*

COROLLARY 11.2. *Let $E = O(1)$ or $O_L(1)$, and let $T_1 s|_n := na_n \log n \in E$ be a* TC *for V. Suppose also that $V \subset V.M$. Then each of the conditions*

(11.3) *(i)* $T_2 s|_n := nL_n(t_n - t_{n-1}) \in E$

and

(11.4) *(ii)* $T_2^* s := (I - M)s \in E$

is a $V \longrightarrow M$ Tauberian condition.

PROOF. As in the proof of Theorem 11, $T_2 s \in E \Leftrightarrow T_2^* s = (I - M)s \in E$. Now if $s \in (V)$, then so are Ms and $(I - M)s$. Suppose that one of (11.3), (11.4) holds. Then $T_1(I - T_2^*)s = T_1(Ms) = T_2^* s \in E$ and it follows that $Ms \in c$.

{For the special case where $V = L$, part (ii) of Corollary 11.2 was proved by Kwee ([10], Theorem 5).}

THEOREM 12. *Let $E = c_0$ or c, and let $T_1 s|_n := na_n \log n \in E$ be a* TC *for V. Then for any sequence $s \in (V)$: $s \in c \Leftrightarrow T_2 s|_n := L_n^{-1} \sum_{k=0}^{n} a_k L_k \in E$. (Then $T_2 s \in c_0$.)*

PROOF (SUFFICIENCY). We note that $T_1 s \in E \Leftrightarrow T_1^* s|_n := nL_n a_n \in E$ and also

$$
\begin{aligned}
T_1^*(I - T_2)s|_n &= nL_n a_n - nL_n\Big[L_n^{-1} \sum_{k=0}^{n} a_k L_k - L_{n-1}^{-1} \sum_{k=0}^{n-1} a_k L_k\Big] \\
&= nL_n a_n - nL_n\Big[\Big(\sum_{k=0}^{n-1} a_k L_k\Big)\Big(L_n^{-1} - L_{n-1}^{-1}\Big) + a_n\Big] \\
&= n(L_n - L_{n-1})L_n^{-1} \sum_{k=0}^{n-1} a_k L_k \\
&= n(n+1)^{-1} T_2 s|_{n-1} \in E \quad \text{if and only if } T_2 s \in E.
\end{aligned} \tag{12.1}
$$

So, if $T_2 s \in E$ and $s \in (V)$, then $(I - T_2)s \in (V)$ and from (12.1) it follows that $(I - T_2)s \in c$ and hence also that $s \in c$.

(NECESSITY). Suppose that $s \in c$. Then by the regularity of the linear method M,

$$s_n - Ms|_n = o(1) \quad \text{and} \quad M(\{a_k\}) = o(1). \tag{12.2}$$

But

$$
\begin{aligned}
s_n - Ms|_n & \\
&= s_n - L_n^{-1}\sum_{k=0}^{n} s_k/(k+1)\\
&= L_n^{-1}[s_n\{1+1/2+\cdots+1/(n+1)\} - \{s_0+(1/2)s_1+\cdots+s_n/(n+1)\}]\\
&= L_n^{-1}[(s_n-s_0)+(1/2)(s_n-s_1)+\cdots+(1/n)(s_n-s_{n-1})]\\
&= L_n^{-1}[a_1+a_2+\cdots+a_n+(1/2)(a_2+a_3+\cdots+a_n)+\cdots+(1/n)a_n]\\
&= L_n^{-1}[a_1+a_2(1+1/2)+a_3(1+1/2+1/3)+\cdots+a_n(1+1/2+\cdots+1/n)]\\
&= L_n^{-1}\big[(a_1L_1+a_2L_2+a_3L_3+\cdots+a_nL_n) - \big(a_1/2+a_2/3+\cdots+a_n/(n+1)\big)\big]\\
&= L_n^{-1}\sum_{k=1}^{n} a_kL_k - L_n^{-1}\sum_{k=1}^{n} a_k/(k+1)\\
&= L_n^{-1}\sum_{k=0}^{n} a_kL_k + o(1) + L_n^{-1}\sum_{k=0}^{n} a_k/(k+1) + o(1)\\
&= T_2s|_n + o(1) + M(\{a_k\})|_n + o(1)\\
&= T_2s|_n + o(1), \quad \text{by (12.2).}
\end{aligned}
$$

That is, $T_2s|_n = s_n - Ms|_n + o(1)$. It follows from (12.2) that $T_2s|_n = o(1)$.

This completes the proof of the theorem.

$\{$We note that T_2s in Theorem 12 is the l-transform of the sequence $\{(n+1)a_nL_n\}$, parallel to the situation where $\{\delta_n\}$ in Tauber's second theorem for the Abel method A is the C_1-transform of the sequence $\{na_n\}$; and that $na_n = o(1)$ and $(n+1)a_nL_n = o(1)$ are Tauberian conditions for A and V respectively. Thus Theorem 12 may be considered to be one form of "Tauber's second theorem" for the method V. $\}$

THEOREM 13. *(a) Let E_δ be a regular Euler method and let V be such that*

$$T_1s|_n := n^{1/2}a_n \in E \quad \text{is a TC for } V \tag{13.1}$$

where $E = c_0$, and let E_δ be a regular Euler method. Then for any sequence $s \in (V)$:

$$T_2s := (I - E_\delta)s \in c_0 \Rightarrow s \in c \Rightarrow (I - E_\delta)s \in c_0. \tag{13.2}$$

(b) If (13.1) holds with $E = m$ and if $V \subset V.E_\delta$, then for any sequence $s \in (V)$:

$$T_2s := (I - E_\delta)s \in m \Rightarrow s \in m \quad \text{and} \quad Es \in c \tag{13.3}$$

where E is any regular Euler method or other method that is equivalent to E_δ for bounded sequences.

PROOF. We apply Theorem 4 with $T_2 = I - E_\delta$. Then

$$T_1(I - T_2)s = T_1.E_\delta s \in E \tag{13.4}$$

by Lemma 2 of [8], directly in case $E = m$ and by a slight modification of the proof in case $E = c_0$. Let $s \in (V)$ and $T_2s \in E$; then $E_\delta s \in (V)$, whether we are considering part (a) or part (b) of the theorem. Hence by (13.1) and (13.4), $E_\delta s \in c$. This yields (13.2) if $E = c_0$ and (13.3) if $E = m$.

{Part (a) of the theorem may be viewed as a form of "Tauber's second theorem" for V and other Borel-like methods, since (13.2) is the exact analogue of the classical Tauber's second theorem which is given by (5.2). The special case of part (a) with $V = B$, the Borel method was proved in [8]; the particular case of part (b) stated in the theorem was proved in [16].}

THEOREM 14.1 (GENERALIZATION OF THEOREM 1). *Let V be a linear conservative method and $\{p_n\}$ a sequence of numbers such that*

$$p_{n+1} = O(p_n + p_{n+1}) \quad \textit{and} \quad \inf |p_n + p_{n+1}| > 0. \tag{14.1}$$

Let the condition

$$\sum_{k=1}^{\infty} (-1)^k p_k a_k = 0 \tag{14.2}$$

be a TC *for V. Then the condition*

$$p_n a_n = o(1) \tag{14.3}$$

is also a TC *for V.*

PROOF. We shall apply Theorem 3 with $M^* = (V)$, $\kappa = c$, $\tau_1 = \{s : \sum_{k=1}^{\infty} (-1)^k p_k a_k = 0\}$, $\tau_2 = \{s : p_n a_n = o(1)\}$ and P, the transformation defined by the equation $Ps|_n = \sum_{k=1}^{n} (a_k + b_k)$ for $n = 1, 2, \ldots$, where $b_k = p_{k-1} a_{k-1}/(p_{k-1} + p_k) - p_k a_k/(p_k + p_{k+1})$ for $k \geq 1$ (and with $a_0 = 0$). The the conditions (3.1)–(3.3) of Theorem 3 are satisfied. Now suppose that $s \in (V)$ and let (14.3) hold. Then $(I - P)s|_n = -\sum_{k=1}^{n} b_k = p_n a_n/(p_n + p_{n+1}) \in c_0$, by (14.1). Thus, part (b) of condition (iv) of Theorem 3 is satisfied. Also,

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^k p_k (a_k + b_k) &= \sum_{k=1}^{n} (-1)^k [p_{k-1} a_{k-1}/(p_{k-1} + p_k) + p_{k+1} a_k/(p_k + p_{k+1})] \\
&= (-1)^n p_n p_{n+1} a_n/(p_n + p_{n+1}) \\
&= o(1), \quad \text{by (14.1) and (14.3).}
\end{aligned}$$

This means that $Ps \in \tau_1$, and all the conditions of Theorem 3 are satisfied. Hence, $(V) \cap \tau_2 \subset \kappa = c$ and the present theorem is proved.

REMARKS. The theorem of Göes given as Theorem 1 above is the case $p_n = n$ of Theorem 14.1. His method of proof makes use of a relation of the type

$$\sum_{k=1}^{\infty} p_k x_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} x_k \text{ converges} \tag{14.4}$$

(which holds for the case $p_k = k$ $(k \geq 1)$ considered by him). But if we take $p_{2n-1} = 1$ and $p_{2n} = 2$ and $x_{2n-1} = 1/n$, $x_{2n} = (-1/2)(1/n)$ for $n \geq 1$, or take $p_{2n-1} = \sqrt{n}$ and $p_{2n} = 2\sqrt{n}$ and $\{x_n\}$ as above, then in each of these instances Theorem 14.1 will apply but not Göes' method of proof, since (14.4) fails. Thus Theorem 14.1 is a significantly distinct result which contains Göes' theorem as a special case.

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THEOREM 14.2. *Let $\{p_n\}$ be a nondecreasing sequence with $p_n \longrightarrow \infty$, with $m = O(p_m)$ and $p_n(p_{n+1} - p_n) = O(p_{n+1})$. Then the following are equivalent for any linear method V:*

(a) $p_n a_n = o(1)$ *is a* TC *for* V;

(b) $u_n := p_n^{-1} \sum_{k=0}^{n} p_k a_k = o(1)$ *is a* TC *for* V;

(c) $v_n := \sum_{k=0}^{n} (-1)^k p_k a_k = o(1)$ *is a* TC *for* V;

(d) $w_n := \sum_{k=0}^{n} (-1)^k u_k = o(1)$ *is a* TC *for* V.

PROOF. It is trivial that (b) $\Rightarrow$ (d) and that (a) $\Rightarrow$ (c). By Theorem 14.1, (c) $\Rightarrow$ (a); and taking $q_n = p_n$ in Theorem 5, we see that (a) $\Rightarrow$ (b). It is then enough to prove that (d) $\Rightarrow$ (c) and hence enough to prove that $v_n = o(1)$ implies that $w_n = o(1)$. This can be shown to be the case, by a slight adaptation of the proof given by Tietz ([22], p. 49) for the special case of the theorem where $p_n = n$.

THEOREM 15. *Let V be such that*

(15.1) $$T_1 s|_n := s_n = O(n^{1/2} \log n) \quad \text{is a } V \longrightarrow l \text{ Tauberian condition.}$$

Then the condition

(15.2) $$T_2 s := (I - M)s = O(n^{1/2} \log n)$$

is also a $V \longrightarrow l$ Tauberian condition.

PROOF. Let $s \in (V)$ and let (15.2) hold. By (15.1) it is enough to prove that $s_n = O(n^{1/2} \log n)$, or, equivalently (in view of (15.2)), that $Ms = O(n^{1/2} \log n)$.

Now $Ms|_n = L_n^{-1} \sum_{k=0}^{n} s_k/(k+1)$. If we set $V_n = \sum_{k=0}^{n} a_k L_k$, then

(15.3) $$Ms|_n = s_n - L_n^{-1} V_n$$

and

(15.4) $$V_n = O(n^{1/2} \log^2 n).$$

Since $a_n = (V_n - V_{n-1})/L_{n-1}$ for $n = 1, 2, \ldots$ we have

(15.5)
$$\begin{aligned} Ms|_n &= a_0 + L_{n-1} \sum_{i=1}^{n} L_{k-1}^{-1} (V_k - V_{k-1}) \sum_{k=i}^{n} 1/(k+1) \\ &= a_0 + V_n/[L_n L_{n-1}(n+1)] + L_n^{-1} \sum_{i=1}^{n-1} V_i \Big(L_{i-1}^{-1} \sum_{k=1}^{n} 1/(k+1) - L_i^{-1} \sum_{k=i+1}^{n} 1/(k+1) \Big) \\ &= O(1) + O\Big[n^{1/2} \log^2 n \big/ \big((\log^2 n)(n+1) \big) \Big] + L_n^{-1} \sum_{i=1}^{n-1} V_i (L_n - L_i) \big/ \big(L_{i-1} L_i (i+1) \big) \\ &= O(1) + O(n^{1/2} \log n) = O(n^{1/2} \log n). \end{aligned}$$

It follows from (15.2) and (15.5) that $s_n = O(n^{1/2} \log n)$. Then $s \in (l)$, by (15.1).

THEOREM 16. *Let V be such that $V \subset V.M$ and let $T_1 s|_n := n^{1/2} a_n = O(1)$ [$O_L(1)$ respectively] be a* TC *for V. Then the condition*

$$T_2 s := (I - M)s = O(n^{1/2} \log n) \quad [O_L(n^{1/2} \log n) \text{ respectively}] \tag{16.1}$$

is a $V \longrightarrow l$ Tauberian condition.

PROOF. Let $s \in (V)$ [so that also $t = Ms \in (V)$] and let (16.1) hold. By (11.2) we have then $(n+1)L_{n-1}(t_n - t_{n-1}) = (I - M)s = O(n^{1/2} \log n)$ [$O_L(n^{1/2} \log n)$] and hence $n^{1/2}(t_n - t_{n-1}) = O(1)$ [$O_L(1)$]. Hence $t = Ms \in c$, but M and l are equivalent.

COROLLARY 16.1. *If V is a regular Euler method E_δ, or more generally, a Borel-type method $B(\alpha, \beta)$, then the condition $(I - M)s = O_L(n^{1/2} \log n)$ is a $V \longrightarrow l$ Tauberian condition.*

PROOF. For, $E_\delta \subset E_\delta.M$ by [9], and $B(\alpha, \beta) \subset B(\alpha, \beta).M$ by ([1], Lemma 5). Since $n^{1/2} a_n = O_L(1)$ is a TC for E_δ (a classical result) and also for $B(\alpha, \beta)$ ([2], Theorem 1), the required result follows from Theorem 16.

REMARK. The result in Corollary 16.1 is a significant improvement on results of earlier authors who proved the stronger condition $s_n = O(n^{1/2} \log n)$ to be a $V \longrightarrow l$ Tauberian condition when $V = E_\delta$, or B (Borel's method), or $B(\alpha, \beta)$ (see [9], [11] and [1] respecitvely).

Now let the iterations $M^{(k)}$ of the method M be defined as usual by the equations $M^{(1)} = M$, $M^{(k+1)} = M.M^{(k)}$ for $k = 1, 2, \ldots$. Then for the particular case where $V = B(\alpha, \beta)$ we have the following result which is a generalization of both Theorem 15 above and a theorem of Borwein ([1], Theorem).

THEOREM 17. *Let $s \in (V)$ where $V = B(\alpha, \beta)$ and let*

$$(I - M)s|_n = O\big((n^{1/2} \log n)^p\big), \tag{17.1}$$

where p is a positive integer. Then s is summable by the method $\big(R, \log(n+1), p\big)$ [and hence by the method $M^{(p)}$ equivalent to it].

PROOF. First we note that the methods l and $\big(R, \log(n+1), 1\big)$ are the same and the theorem for $p = 1$ is covered by Corollary 16.1. Let $p > 1$ and let s satisfy the conditions of the theorem. Then $M^{(k)} s \in (V)$ for $k = 1, 2, \ldots, p-1$ by [1] and since $(I - M)s \in (V)$ by (17.1), we see from Lemma 5 of [1] that $M^{(p-1)}(I - M)s|_n = O(n^{1/2} \log n)$. Then by Corollary 16.1, $M^{(p-1)} s \in l = (M)$ and hence $M^{(p)} s \in c$. But the methods $M^{(k)}$ and $\big(R, \log(n+1), k\big)$ are known to be equivalent for $k = 1, 2, \ldots$ ([11], Lemma 4) and this proves the theorem.

{Borwein [1] proved the result under the stronger hypothesis that $s_n = O\big((n^{1/2} \log n)^p\big)$.}

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