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INERTIAL SUBALGEBRAS OF ALGEBRAS POSSESSING FINITE AUTOMORPHISM GROUPS

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Abstract

Let R be a commutative ring with identity, and let A be a finitely generated R-algebra with Jacobson radical N and center C. An R-inertial subalgebra of A is a R-separable subalgebra B with the property that $B+N=A$. Suppose A is separable over C and possesses a finite group G of R-automorphisms whose restriction to C is faithful with fixed ring R. If R is an inertial subalgebra of C, necessary and sufficient conditions for the existence of an R-inertial subalgebra of A are found when the order of G is a unit in R. Under these conditions, an R-inertial subalgebra B of A is characterized as being the fixed subring of a group of R-automorphisms of A. Moreover, $A \simeq B \otimes_R C$. Analogous results are obtained when C has an R-inertial subalgebra $S \supset R$.

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1. Introduction

All rings are assumed to be associative and to possess an identity element. By an R-algebra A over a commutative ring R (an R-algebra A) we mean a ring A together with a homomorphism from R into $Z(A)$, the center of A. When we say A is a finitely generated or projective R-algebra, we mean it is finitely generated or projective as an R-module. The Jacobson radical of an R-algebra A will be denoted by $\operatorname{rad} A = N$, while the Jacobson radical of its center C will be denoted by $\operatorname{rad} C = \mathfrak{n}$. If A is separable over its center C, it will be termed a central separable C-algebra. We note that in this case $N = \mathfrak{n}A$ (DeMeyer and Ingraham (1971), p. 79). Finally, an R-inertial subalgebra B of an R-algebra A is an R-separable subalgebra B of A with the property that $B+N=A$. If $R+N=A$, then R is an R-inertial subalgebra of A. In this instance we will state simply that R is an inertial subalgebra of A.

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The Wedderburn Principal Theorem asserts, in this terminology, that if R is a field and A is a finitely generated R-algebra such that A/N is separable over R, then A contains an R-inertial subalgebra. Azumaya (1951) generalized this result to finitely generated algebras over local Hensel rings. Ingraham (1966) defined a (G, R)-algebra C to be a finitely generated, faithful, and commutative R-algebra possessing a finite group G of R-algebra automorphisms with fixed subring $C^G = R$. He found necessary and sufficient conditions for the existence of R-inertial subalgebras of connected (G, R)-algebras. When an R-inertial subalgebra exists, he showed that it is unique and, moreover, a Galois (DeMeyer and Ingraham (1971)) extension of R.

We will say that a finitely generated R-algebra is normal with group G if it possesses a finite group G of R-algebra automorphisms which restricts faithfully to the center C of A in such a way that C is a (G, R)-algebra. If, moreover, A is separable over C, then A will be said to be a normal central separable R-algebra with center C and group G. (We note that this terminology is somewhat at variance with that of Eilenberg and MacLane (1948), Pareigis (1964) and Childs (1964). In these papers G is a set, not necessarily a group, of R-automorphisms of A which restricts faithfully to a group of R-automorphisms of C with respect to which C is a Galois extension of R.) Let C be a (G, R)-algebra, and suppose B is a central separable R-algebra with the action of G on $A = B \otimes_R C$ induced by $g(b \otimes c) = b \otimes g(c)$, for each $g \in G$. If the order of G is a unit in R, we will show (Proposition 2.2) that there exists an element $x \in G$ such that $\sum_{g \in G} g(x) = 1$. In this case, $A^G = B$ by Lemma 1.4, Childs and DeMeyer (1967). Further, if R is assumed to be an inertial subalgebra of C, then $C/\mathfrak{n} = R$ and $A/\mathfrak{n}A = B$. Therefore $B + \mathfrak{n}A = A$, so that B is an R-inertial subalgebra of A.

EXAMPLE 1.1. *Let B be a central separable R-algebra and let C be a (G, R)-algebra. Then $A = B \otimes_R C$ is a normal central separable R-algebra with center C and group G. If the order of G is a unit in R, then $A^G = B$. If R is an inertial subalgebra of C, then B is an R-inertial subalgebra of A.*

The object of this paper is to show that under suitable conditions, Example 1.1 is characteristic. Namely, suppose A is a normal central separable R-algebra, and that R is an inertial subalgebra of C. We find necessary and sufficient conditions for A to have an R-inertial subalgebra B. Under these conditions, $A \simeq B \otimes_R C$ so that B may be considered to be the fixed subring of an extension of G from C to A. We obtain analogous results when C has an R-inertial subalgebra $S \supset R$.

2. Preliminaries

This section considers some of the general properties of trace algebras and inertial subalgebras which apply to algebras separable over their centers. The following result is useful for showing that certain subalgebras of such an algebra add with its Jacobson radical to give the entire algebra.

PROPOSITION 2.1. *Suppose A is a finitely generated R-algebra which is separable over its center C. If B is an R-inertial subalgebra of A with center S, then $A = B \cdot C \simeq B \otimes_S C$ as C-algebras.*

PROOF. There is a natural C-algebra homomorphism, $\mu: B \otimes_S C \to A$ induced by $\mu(b \otimes c) \to b \cdot c$. Since A is central separable over C, it follows that $N = \mathfrak{n}A$. Consequently, a straightforward application of the original Nakayama lemma yields $A = B \cdot C$. Thus μ is onto. Let U be the two-sided ideal of C given by $U = C \cap \operatorname{Ker} \mu$. Since $B \otimes_S C$ is central separable over C, it follows that $\operatorname{Ker} \mu = U \cdot (B \otimes_S C)$. Thus for $x \in U$, we have $0 = \mu(x) = \mu(x \cdot 1) = x \cdot \mu(1) = x$. Therefore $U = (0)$, whence μ is an isomorphism.

Suppose A is finitely generated R-algebra which possesses a finite group G of R-automorphisms. The fixed subring of A under G will be denoted by A^G. Let U be a (two-sided) ideal of A. If H is any subgroup of G, then the inertia subgroup of H relative to U is defined by $J(H, U) = \{g \in H \mid g(x) - x \in U \text{ for all } x \in A\}$. We denote $J(G, U)$ simply by J_U. Since $N = \operatorname{rad}(A)$ is invariant under G, the inertia subgroup J_N will be a normal subgroup of G. We denote J_N by J. For any subgroup H of G, the trace map $t_H : A \to A^H$ is defined by $t_H(a) = \sum_{g \in H} g(a)$. We denote t_G simply by t.

PROPOSITION 2.2. *Let A be a finitely generated, faithful R-algebra which possesses a finite group G of R-automorphisms. Then $t_J(A) = A^J$ if and only if the order of J is a unit of R.*

PROOF. Suppose that $n = |J|$ is a unit in R. Then $t_J(1) = \sum_{g \in J} g(1) = n$. Since $t_J(A)$ is an ideal of A^J, it follows that $t_J(A) = A^J$.

Conversely, suppose that $t_J(A) = A^J$. Then there exists an $x \in A$ such that $t_J(x) = 1$. It follows that $1 - nx = t_J(x) - nx = \sum_{g \in J} (g(x) - x)$ is an element of N. Suppose that n is not a unit in R. Then, since $nx \in \mathfrak{m}A \neq A$ for some maximal ideal $\mathfrak{m}$ of R, 1 must be an element of every maximal left ideal of A containing $\mathfrak{m}A$, a contradiction.

In Section 3, the order of J will always be assumed a unit of the inertial subalgebra of the center. So we will be able to identify A^J with $t_J(A)$. One important reason for this requirement is to be able to apply the next proposition.

PROPOSITION 2.3. *Suppose A is a finitely generated R-algebra which possesses a* *finite group G of R-automorphisms. If U is an ideal of A such that the order of J_U is a unit in R, then $t_{J_U}(A)+U = A$.*

PROOF. If n denotes $|J_U|$, then for each $a \in A$ we have

$$t_{J_U}(a) - n\cdot a = \sum_{g\in J_U}(g(a)-a) \in U.$$

The conclusion follows immediately.

COROLLARY 2.4. *Let A be a finitely generated R-algebra which is central separable over C, and suppose A possesses a finite group G of R-automorphisms. If the order of J is a unit in R, then $A = C\cdot t_J(A)$.*

PROOF. Since A is central separable over C, $N = \mathfrak{n}A$. Therefore $C\cdot t_J(A)+\mathfrak{n}A = A$ by Proposition 2.3. The result now follows by Nakayama's lemma.

3. Inertial subalgebras of normal central separable algebras

In this section we consider the problems of existence and uniqueness of inertial subalgebras of normal central separable algebras. First, to avoid confusion, we note the following definitions. A commutative (but not necessarily Noetherian) ring R is said to be semilocal if it does not contain infinitely many distinct maximal ideals. If R contains a unique maximal ideal it is said to be local. If R possesses no idempotents except 0 and 1 it is said to be connected. We now begin by citing the following lemma, which is Proposition 2.1 of Ford (1976).

LEMMA 3.1. *If A is a finitely generated R-algebra with R-inertial subalgebra B, then every central idempotent of A is contained in B.*

THEOREM 3.2. *Let R be a semilocal ring, and let A be a normal central separable R-algebra with center C and group G. Suppose R is an inertial subalgebra of C, and that $C/\mathfrak{n}$ is R-projective. If $J_{\mathfrak{n}A} = G$ and the order of G is a unit in R, then $t(A) = A^G$ is an R-inertial subalgebra of A such that $t(A)\oplus N = A$, $t(A)\otimes_R C \simeq A$, and $t(A) \simeq A\otimes_C R$.*

PROOF. That a finitely generated, commutative and semi-local ring may be decomposed into a finite direct sum of connected semilocal rings is well known. Thus we may write $C = \oplus\sum_{i=1}^n C_i$, where each C_i is connected. This in turn induces a decomposition on the C-algebra A as $A = \oplus\sum_{i=1}^n A_i$, where each A_i is central separable over C_i. Moreover, in view of Lemma 3.1, R may also be decomposed as $R = \oplus\sum_{i=1}^n R_i$, where each R_i is connected, semi-local and an inertial subalgebra of C_i.

Let e be the primitive idempotent of C_i, so that $C_i = Ce$. Define

$$G_e = \{g \in G \mid g(e) = e\}.$$

It follows from Corollary 2.4 that G_e restricts faithfully from Ae to Ce. Moreover, Ce is in fact a (G_e, Re)-algebra by Ingraham (1966), p. 84, so that Ae is a normal central separable Re-algebra with center Ce and group G_e. It is not hard to verify that all the conditions satisfied by A and G are also satisfied by Ae and G_e. Thus we may assume, without loss of generality, that C (and therefore R) is connected.

We reason as follows to show that under these conditions R must be a field and C must be local. By assumption R is an inertial subalgebra of C, so that $R+\mathfrak{n} = C$. Since $C/\mathfrak{n}$ is R-projective, it is a consequence of Lemma 2.3, Ingraham (1966), that $R \cap \mathfrak{n} = (0)$. Hence $R \simeq C/\mathfrak{n}$. This, together with the fact that C is semilocal implies that R is a finite direct sum of fields. Inasmuch as R is also connected, there can be only one such summand. A similar line of reasoning shows that C must be local. In view of this, A/N must be free and of finite rank over $C/\mathfrak{n}$.

Moreover, by Proposition 2.3, each of the basis elements for A/N over $C/\mathfrak{n}$ has a representative in $t(A)$. Since A is projective over the local ring C, a familiar argument (Bourbaki (1962), p. 43) allows us to conclude that A is in fact free (of finite rank) over C; and, moreover, that a basis $\{x_i\}_{i=1}^m$ can be chosen for A over C such that each $x_i \in t(A)$. Let $u \in t(A) \cap N$. Since $N = \mathfrak{n}A$, we can represent $u = \sum n_i x_i$, where $n_i \in \mathfrak{n}$. We note that u and each x_i is invariant under G. Necessarily then, each $n_i \in C^G = R$. Since $R \cap \mathfrak{n} = (0)$, each $n_i = 0$ so that $t(A) \cap N = (0)$. Therefore,

$$t(A) \oplus N = A,$$

and

$$t(A) \simeq t(A)/(t(A) \cap N) \simeq (t(A)+N)/N \simeq A/N \simeq A \otimes_C C/\mathfrak{n} \simeq A \otimes_C R.$$

Now A is separable over C, so that $t(A) \simeq A \otimes_C R$ is separable over R. Thus $t(A)$ is an R-inertial subalgebra of A. The center of $t(A)$, $Z(t(A))$, must therefore be separable over R. That $Z(t(A))+\mathfrak{n} = C$ follows from Theorem 1.2, Ford (1976), whence $Z(t(A))$ is an R-inertial subalgebra of C. However, by Proposition 2.6, Ingraham (1966), the R-inertial subalgebra of a finitely generated and commutative R-algebra is unique. Thus $Z(t(A)) = R$. It is now immediate by Proposition 2.1 that $t(A) \otimes_R C \simeq A$.

Suppose that R is an arbitrary commutative ring and that A is a finitely generated R-algebra the center of which is a (G, R)-algebra C. The group G will be said to be identified with a group of R-automorphisms of A provided A possesses a group of R-automorphisms whose faithful restriction to C is G. We will now investigate conditions which guarantee the existence of R-inertial subalgebras of A. We begin with a lemma which shows how the maximal ideals of R interact with $t(A)$.

LEMMA 3.3. *Let A be a normal central separable R-algebra with center C and group G, and such that R is an inertial subalgebra of C. If the order of G is a unit in R, then for each maximal ideal $\mathfrak{m}$ of R, $t(A) \cap \mathfrak{m}A = \mathfrak{m}t(A)$.*

PROOF. Let us denote $t(A)$ by B. Since $A = C \cdot B$ by Corollary 2.4, it follows that $\mathfrak{n}A = \mathfrak{n}B$. Thus each $x \in \mathfrak{n}A$ can be represented as $x = \sum n_i w_i$, where $n_i \in \mathfrak{n}$ and $w_i \in B$. It follows that $t(x) = \sum g(x) = \sum g(\sum n_i w_i) = \sum t(n_i) w_i \in (\text{rad}\, R) \cdot B$. If also $x \in B$, then $x = t(x)/|G| \in (\text{rad}\, R) \cdot B$. Consequently, $B \cap \mathfrak{n}A = (\text{rad}\, R) \cdot B$. One next observes that $\mathfrak{m}A = \mathfrak{m}BC = \mathfrak{m}B(\mathfrak{n}+R) = \mathfrak{m}B + \mathfrak{m}B\mathfrak{n} = \mathfrak{m}B + \mathfrak{n}B$. Suppose $y \in B \cap \mathfrak{m}A$. Then $y = u + v$, where $u \in \mathfrak{m}B$, and $v \in \mathfrak{n}B$. Hence

$$v = y - u \in B \cap \mathfrak{n}A = (\text{rad}\, R) \cdot B \subset \mathfrak{m}B.$$

Therefore $y \in \mathfrak{m}B$, so that $B \cap \mathfrak{m}A = \mathfrak{m}B$.

THEOREM 3.4. *Let A be a finitely generated R-algebra which is separable over its center C and with R an inertial subalgebra of C. Suppose C possesses a finite G of automorphisms whose order is a unit in R, and such that $C^G = R$. Then a necessary and sufficient condition for the existence of an R-inertial subalgebra of A is that G be identifiable as a group of automorphisms of A with the property that $J_{\mathfrak{n}A} = G$.*

PROOF. Necessity. Suppose B is an R-inertial subalgebra of A. Arguing as in Theorem 3.1, we find that $Z(B) = R$. Therefore by Proposition 2.1,

$$A = B \cdot C \simeq B \otimes_R C.$$

Accordingly each element $g \in G$ induces an R-automorphism of A via

$$g(b \otimes c) = b \otimes g(c).$$

Since the order of G is a unit in R, there exists an element $x \in C$ such that $t(x) = 1$. Consequently, $t(A) = A^G = B$ by Lemma 1.4, Childs and DeMeyer (1967). Hence for each $g \in G$ and each $a \in A$,

$$g(a) - a = g(\sum bc) - (\sum bc) = \sum b(g(c) - c) \in \mathfrak{n}A.$$

Therefore $J_{\mathfrak{n}A} = G$.

Sufficiency. Assuming that $J_{\mathfrak{n}A} = G$, we will show that $t(A)$ is an R-inertial subalgebra of A. That $t(A) + N = A$ follows from Proposition 2.3. The separability of $t(A)$ over R will follow by a well-known result of Endo–Watanabe (1967) if it can be shown that $t(A)/\mathfrak{m}t(A)$ is $R/\mathfrak{m}$-separable for every maximal ideal $\mathfrak{m}$ of R.

Let $\mathfrak{m}$ be an arbitrary maximal ideal of R. Each $g \in G$ induces, in the natural way, an $R/\mathfrak{m}$-automorphism of $C/\mathfrak{m}C$. Moreover, the invariance of $\mathfrak{m}C$ under G implies that $J_{\mathfrak{m}C}$ is a normal subgroup of G, so that $G/J_{\mathfrak{m}C}$ acts faithfully as a group of $R/\mathfrak{m}$-automorphisms of $C/\mathfrak{m}C$. Let us denote $t(A)$ by B. It follows from

Corollary 2.4 that $A = B \cdot C$. For each $j \in J_{mC}$ and each $a \in A$, we then have

$$j(a) - a = j(\sum bc) - (\sum bc) = \sum (j(c) - c)\, b \in mA.$$

This implies that $J_{mA} = J_{mC}$, so that G/J_{mA} can be identified as a group of R/m-automorphisms of A/mA. We denote the group G/J_{mA} by G_m.

Let $\{\sigma_k\}_{k=1}^s$ be a full set of coset representatives for J_{mA} in G, so that

$$J_{mA} \cup J_{mA} \cdot \sigma_2 \cup J_{mA} \cdot \sigma_3 \cup \ldots \cup J_{mA} \cdot \sigma_s = G.$$

Suppose $\bar{a} \in (A/mA)^{G_m}$ is arbitrary. There exists an element $\bar{x} \in A/mA$ such that $\bar{a} = \bar{t}(\bar{x}) = \sum_{\bar{g} \in G_m} \bar{g}(\bar{x})$. Thus $\bar{a} = b + mA$, where $b = \sum_{i=1}^s \sigma_i(x)$. Denote the order of J_{mA} by n. Then

$$n \cdot b - t(x) = n \sum_{i=1}^{s} \sigma_i(x) - \sum_{g \in G} g(x) = n \sum_{i=1}^{s} \sigma_i(x) - \sum_{i=1}^{s} \sum_{h \in J_{mA}} h\sigma_i(x)$$

$$= \sum_{h \in J_{mA}} \sum_{i=1}^{s} (\sigma_i(x) - h\sigma_i(x)) \in mA.$$

It follows that $\bar{a} = t(x)/n + mA \in (B + mA)/mA$. We have therefore shown that $(A/mA)^{G_m} = (B + mA)/mA$. Similarly, one can show that $(C/mC)^{G_m} = R/m$.

We now show that A/mA fulfills the conditions of Theorem 3.2. It has already been established that A/mA is a normal central separable algebra with group G_m and center C/mC. Since $R + m = C$, there is exactly one maximal ideal of C lying over mC, whence C/mC is local. Moreover,

$$R/m + \operatorname{rad}(C/m) = R/m + (\mathfrak{n} + mC)/mC = C/mC,$$

which shows that R/m is an inertial subalgebra of C/mC. Finally, that $J_{\operatorname{rad}(A/mA)} = G_m = J_{\operatorname{rad}(C/mC)}$ follows easily from the fact that $A^G + N = A$. Therefore by Theorem 3.2, $\bar{t}(a/mA) = (B + mA)/mA$ is an R/m-inertial subalgebra of A/mA.

We are now in a position to show that $B = t(A)$ is an R-inertial subalgebra of A. By Lemma 3.3, $B \cap mA = mB$. Hence $B/mB = B/(B \cap mA) \simeq (B + mA)/mA$, so that B/mB is R/m-separable. Clearly $B = t(A)$ is finitely generated over R. Therefore by Endo–Watanabe (1967), it follows that B is separable over R. It having been previously demonstrated that $B + N = A$, the fact that B is an R-inertial subalgebra of A is now established.

It is to be noted, in view of Proposition 2.1 and Example 1.1, that under these conditions an R-inertial subalgebra B of a normal central separable R-algebra may be characterized as being the fixed subalgebra A^{G^*} of A by some extension G^* of G from C to A. This does not, however, imply either that the inertial subalgebras are unique or even that they are isomorphic. In fact, quite the contrary is the case as we are about to show. For, although Ingraham (1965) has demonstrated that

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inertial subalgebras of commutative algebras are unique, and Malcev (1942) has proved that inertial subalgebras of algebras over fields must be unique up to an inner automorphism, it has been noted that this phenomenon does not occur generally. Indeed an extensive class of algebras which possess nonisomorphic inertial subalgebras has been exhibited, Ford (1976). We proceed to show that there are members of this class which satisfy all the conditions of Theorem 3.4. Let $R = Z(5)$ (the localization of the rational integers at the prime ideal generated by 5), and let $C = R+5Ri$, where $i^2 = -1$. We observe that R is an inertial subalgebra of C. Define G to be the group of order 2 generated by $g(r+si) = r-si$, for all $r, s \in R$. We observe that $C^G = R$, and that the order of G is a unit in R. Define $Q(C)$ to be the generalized quaternion algebra over C. That is,

$$Q(C) = C \oplus C\alpha \oplus C\beta \oplus C\alpha\beta,$$

where $\alpha^2 = -1 = \beta^2$, and $\alpha\beta = \beta\alpha$. Extend G to $Q(C)$ by defining $g(\alpha) = \alpha$ and $g(\beta) = \beta$. Denote the 2×2 matrices over R by $M_2(R)$. A straightforward calculation along the lines of p. 18, Dickson (1960) yields the following result.

EXAMPLE 3.5. *$Q(C)$ is a normal central separable R-algebra with center C and group G, and which possesses R-inertial subalgebras which are not isomorphic. One pair of nonisomorphic R-inertial subalgebras is $Q(R)$ and $M_2(R)$.*

The following theorem leads to results which tie in with the outer Galois theory of DeMeyer (1965). In his paper, DeMeyer considers R-algebras which are normal central separable R-algebras with center C a Galois extension of R. In contrast, we assume that C contains an R-inertial subalgebra S, and that $J_n = J_{nA}$ with the order of this subgroup a unit in S. When $J_n = J_{nA}$, we denote the subgroup simply by J.

THEOREM 3.6. *Let A be a normal central separable R-algebra with center C and group G, and suppose C is connected and contains an R-inertial subalgebra S. If $J_n = J_{nA}$ and the order of this group is a unit in S, then $t_J(A) \simeq t_G(A) \otimes_R S$ is an R-inertial subalgebra of A. Moreover, $t_G(A)$ is a central separable R-algebra such that $t_G(A) \otimes_R C \simeq A \simeq t_J(A) \otimes_S C$.*

PROOF. Since C is connected, it follows from Theorem 2.10, Ingraham (1966) that $C^J = S$ and, moreover, that S is a Galois extension of R with group G/J. Consequently, $t_J(A)$ is an R-inertial subalgebra of A by Theorem 3.4. Now $t_J(A)$ and $S\otimes; t_G(A)$ are both Galois over $t_G(A)$ in the sense of DeMeyer (1965). Therefore $t_J(A) \simeq S \otimes_R t_G(A)$ by Lemma 1.4, Childs and DeMeyer (1967). From this, and the fact that S is faithfully flat over R, it follows that $t_G(A)$ is separable over R. The isomorphisms

$$A \simeq t_J(A) \otimes_S C \simeq t_G(A) \otimes_R S \otimes_S C \simeq t_G(A) \otimes_R C$$

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are consequences of Proposition 2.1. These in turn imply that $Z(t_G(A)) \subset C^G = R$.

Corollary 3.7. *There exists a one–one correspondence between the R-separable subalgebras B of A which contain $t_G(A)$, and the R-separable subalgebras K of the center C of A given by*

$$\begin{array}{ccc} K & \longrightarrow & K.t_G(A) \\ Z(B) & \longleftarrow & B \end{array}$$

Proof. $t_G(A)$ is a central separable R-algebra with the property that

$$A \simeq t_G(A) \otimes_R C$$

by Theorem 3.6. Using this decomposition of A, the correspondence is established in the same way as it is in Lemma 2, DeMeyer (1965).

This theorem does not lead to a Galois correspondence between separable subalgebras of A and subgroups of G as it did in DeMeyer (1965), since this correspondence depends upon C being a Galois extension of R. In our case, that assumption forces $C = R$, by uniqueness of inertial subalgebras in commutative algebras (Ingraham (1966)). However, since S is Galois over R with group G/J, we can apply DeMeyer's theorem directly to obtain the following correspondence.

Corollary 3.8. *The subalgebra $t_J(A)$ is a Galois extension of $t_G(A)$, and there exists a one–one correspondence (the usual one) between the R-separable subalgebras of $t_J(A)$ which contain $t_G(A)$ and the subgroups of G/J.*

A number of results in Ingraham (1966) relating to inertial subalgebras of (G, R)-algebras can now be extended to the setting of normal central separable algebras.

Theorem 3.9. *Let R be a connected ring, and let A be a normal central separable R-algebra with group G and center C. Suppose the order of G is a unit in R, and that C possesses an R-inertial subalgebra S. If $J_{\mathfrak{m}} = J_{\mathfrak{m}A}$ for all maximal ideals $\mathfrak{m}$ of C, then A possesses an R-inertial subalgebra B.*

Proof. Since A is a normal R-algebra, its center C is a (G, R)-algebra. In view of the fact that R is connected, it follows from Ingraham (1966), Lemma 2.14, that C may be decomposed as a finite direct sum $C = \oplus \sum Ce$, where each Ce is connected. Furthermore, defining $G_e = \{g \in G \mid g(e) = e\}$ as in Theorem 3.2, we find that Ce will be a (G_e, Re)-algebra with Se as its Re-inertial subalgebra.

Each maximal ideal of C will exclude exactly one of the primitive idempotents of C. Thus the maximal ideals of Ce have the form $\mathfrak{m}e$, where $\mathfrak{m}$ is a maximal ideal of C which excludes e. Let $J_{\mathfrak{m}e}$ denote $J(G_e, \mathfrak{m}e)$, and let $J_{\mathfrak{m}}$ denote $J(G, \mathfrak{m})$.

For $g \in J_{\mathfrak{m}A}$, we have $g(e) - e \in \mathfrak{m}$. Since $e \notin \mathfrak{m}$ it follows that $g(e) = e$, so that $J_{\mathfrak{m}A} \subset J_{\mathfrak{m}Ae}$. Thus we have the following chain of inclusions: $J_{\mathfrak{m}e} \subset J_{\mathfrak{m}} \subset J_{\mathfrak{m}A} \subset J_{\mathfrak{m}Ae} \subset J_{\mathfrak{m}e}$. Therefore $J_{\mathfrak{m}e}$ in fact equals $J_{\mathfrak{m}Ae}$. It follows easily that $J_{\mathfrak{n}e} = J_{\mathfrak{n}Ae}$. We now show that G_e restricts faithfully from Ae to Ce, the final step in showing Ae to be a normal central Re-algebra with group G_e and center Ce. Let $g \in G_e$, and suppose $g(ce) = ce$ for all $c \in C$. Necessarily $g \in J_{\mathfrak{n}e}$, so that $g \in J_{\mathfrak{n}Ae}$. We note that $Ae = Ce \cdot t_{J_{\mathfrak{n}Ae}}(Ae)$ by Corollary 2.4. Thus for arbitrary $ae \in Ae$, it follows that $g(ae) = g(\sum ce \cdot \lambda e) = \sum g(ce) \cdot g(\lambda e) = \sum ce \cdot \lambda e = ae$. Whence G_e restricts faithfully to Ce. It now follows from Theorem 3.6 that $Be = t_{J_{\mathfrak{n}e}}(Ae)$ is an Re-inertial subalgebra of Ae. Therefore $B = \oplus \sum Be$ will be an R-inertial subalgebra of A.

COROLLARY 3.10. *Let R be a connected ring, and let A be a normal central separable R-algebra with group G and center C. Suppose that the order of G is a unit in R, and that $C/\mathfrak{n}$ is R-separable. If $J_{\mathfrak{m}'} = J_{\mathfrak{m}A}$ for all maximal ideals $\mathfrak{m}'$ and $\mathfrak{m}$ of C which exclude the same primitive idempotent, then A possesses an R-inertial subalgebra.*

PROOF. C possesses an R-inertial subalgebra by Ingraham (1966), Theorem 2.15. Therefore A will possess an R-inertial subalgebra by Theorem 3.9.

It is still a matter for conjecture as to whether the conclusion of Theorem 3.4 and the subsequent corollaries still hold without the hypothesis that the order of G is a unit. In all examples considered by the author thus far, in particular those of the type of Example 1.1, the answer is in the affirmative.

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