Some quadratic reverses of the continuous triangle inequality for the Bochner integral of vector-valued functions in Hilbert spaces are given. Applications for complex-valued functions are provided as well.

1. INTRODUCTION

Let \( f : [a, b] \to \mathbb{K}, \mathbb{K} = \mathbb{C} \text{ or } \mathbb{R} \) be a Lebesgue integrable function. The following inequality is the continuous version of the triangle inequality

\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx
\]

and plays a fundamental role in Mathematical Analysis and its applications.

It seems, see [6, p. 492], that the first reverse inequality for (1.1) was obtained by Karamata in his book from 1949, [4]:

\[
\cos \theta \int_a^b |f(x)| \, dx \leq \left| \int_a^b f(x) \, dx \right|
\]

provided

\[
|\arg f(x)| \leq \theta, \quad x \in [a, b],
\]

where \( \theta \) is a given angle in \((0, \pi/2)\).

This integral inequality is the continuous version of a reverse inequality for the generalised triangle inequality

\[
\cos \theta \sum_{i=1}^n |z_i| \leq \left| \sum_{i=1}^n z_i \right|
\]

provided

\[
a - \theta \leq \arg(z_i) \leq a + \theta, \quad \text{for } i \in \{1, \ldots, n\},
\]
where \( a \in \mathbb{R} \) and \( \theta \in (0, \pi/2) \), which, as pointed out in [6, p. 492], was first discovered by Petrovich in 1917, [7], and, subsequently rediscovered by other authors, including Karamata [4, p. 300–301], Wilf [8], and in an equivalent form by Marden [5].

The first to consider the problem for sums in the more general case of Hilbert and Banach spaces, were Diaz and Metcalf [1].

In our previous work [2], we pointed out some continuous versions of Diaz and Metcalf results providing reverses of the generalised triangle inequality in Hilbert spaces. We mention here some results from [2] which may be compared with the new ones obtained in Sections 2 and 3 below.

We recall that \( f \in L([a, b]; H) \), the space of Bochner integrable functions defined on \([a, b]\) and with values in the Hilbert space \( H \), if and only if the function \( f : [a, b] \rightarrow H \) is Bochner measurable on \([a, b]\) and \( \|f\| \) is Lebesgue integrable on \([a, b]\) (see for instance [9, pp. 132 et seq.]).

**Theorem 1.** If \( f \in L([a, b]; H) \) and there exists a constant \( K \geq 1 \) and a vector \( e \in H, \|e\| = 1 \) such that

\[
\|f(t)\| \leq K \Re \langle f(t), e \rangle \quad \text{for almost all } t \in [a, b],
\]

then we have the inequality:

\[
\int_a^b \|f(t)\| \, dt \leq K \left( \int_a^b f(t) \, dt \right).
\]

The case of equality holds in (1.5) if and only if

\[
\int_a^b f(t) \, dt = \frac{1}{K} \left( \int_a^b \|f(t)\| \, dt \right) e.
\]

As particular cases of interest that may be applied in practice, we note the following corollaries established in [2].

**Corollary 1.** Let \( e \) be a unit vector in the Hilbert space \((H; \langle \cdot, \cdot \rangle), \rho \in (0, 1)\) and \( f \in L([a, b]; H) \) so that

\[
\|f(t) - e\| \leq \rho \quad \text{for almost every } t \in [a, b].
\]

Then we have the inequality

\[
\sqrt{1 - \rho^2} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\|,
\]

with equality if and only if

\[
\int_a^b f(t) \, dt = \sqrt{1 - \rho^2} \left( \int_a^b \|f(t)\| \, dt \right) \cdot e.
\]
COROLLARY 2. Let \( e \) be a unit vector in \( H \) and \( M \geq m > 0 \). If \( f \in L([a, b]; H) \) is such that
\[
\Re \langle Me - f(t), f(t) - me \rangle \geq 0
\]
or, equivalently,
\[
\|f(t) - \frac{M + m}{2} e\| \leq \frac{1}{2}(M - m)
\]
for almost every \( t \in [a, b] \), then we have the inequality
\[
\frac{2\sqrt{mM}}{M + m} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\|
\]
or, equivalently
\[
0 \leq \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \leq \frac{(\sqrt{M} - \sqrt{m})^2}{M + m} \left\| \int_a^b f(t) \, dt \right\|.
\]
The equality holds in (1.12) (or in the second part of (1.13)) if and only if
\[
\int_a^b f(t) \, dt = \frac{2\sqrt{mM}}{M + m} \left( \int_a^b \|f(t)\| \, dt \right) e.
\]

The case of additive reverse inequalities for the continuous triangle inequality has been considered in [3].

We recall here the following general result.

THEOREM 2. If \( f \in L([a, b]; H) \) is such that there exists a vector \( e \in H \), \( \|e\| = 1 \) and \( k : [a, b] \to [0, \infty) \) a Lebesgue integrable function such that
\[
\|f(t)\| - \Re \langle f(t), e \rangle \leq k(t)
\]
for almost every \( t \in [a, b] \), then we have the inequality:
\[
(0 \leq) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \leq \int_a^b k(t) \, dt.
\]
The equality holds in (1.15) if and only if
\[
\int_a^b \|f(t)\| \, dt \geq \int_a^b k(t) \, dt
\]
and
\[
\int_a^b f(t) \, dt = \left( \int_a^b \|f(t)\| \, dt - \int_a^b k(t) \, dt \right) e.
\]
This general result has some particular cases of interest that may be easily applied [3].

**Corollary 3.** If \( f \in L([a, b]; H) \) is such that there exists a vector \( e \in H \), \( \|e\| = 1 \) and \( \rho \in (0, 1) \) such that

\[
\|f(t) - e\| \leq \rho \quad \text{for almost every } t \in [a, b],
\]

then

\[
0 \leq \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\|
\leq \frac{\rho^2}{\sqrt{1 - \rho^2}(1 + \sqrt{1 - \rho^2})} \Re \left< \int_a^b f(t) \, dt, e \right>.
\]

**Corollary 4.** If \( f \in L([a, b]; H) \) is such that there exists a vector \( e \in H \), \( \|e\| = 1 \) and \( M \geq m > 0 \) such that either

\[
\Re \left< Me - f(t), f(t) - me \right> \geq 0
\]

or, equivalently,

\[
\left\| f(t) - \frac{M + m}{2} e \right\| \leq \frac{1}{2} (M - m)
\]

for almost every \( t \in [a, b] \), then

\[
0 \leq \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\|
\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \Re \left< \int_a^b f(t) \, dt, e \right>.
\]

**Corollary 5.** If \( f \in L([a, b]; H) \) and \( r \in L_2([a, b]) \), \( e \in H \), \( \|e\| = 1 \) are such that

\[
\|f(t) - e\| \leq r(t) \quad \text{for almost every } t \in [a, b],
\]

then

\[
(0 \leq) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \leq \frac{1}{2} \int_a^b r^2(t) \, dt.
\]

The main aim of this paper is to point out some quadratic reverses for the continuous triangle inequality, namely, upper bounds for the nonnegative difference

\[
\left( \int_a^b \|f(t)\| \, dt \right)^2 - \left\| \int_a^b f(t) \, dt \right\|^2
\]

under various assumptions on the functions \( f \in L([a, b]; H) \). Some related results are also pointed out. Applications for complex-valued functions are provided as well.
2. Quadratic Reverses of the Triangle Inequality

The following lemma holds.

**Lemma 1.** Let \( f \in L([a,b]; H) \) be such that there exists a function \( k : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \), \( \Delta := \{(t,s) \mid a \leq t \leq s \leq b\} \) with the property that \( k \in L(\Delta) \) and

\[
\|f(t)\| \|f(s)\| - \text{Re} \langle f(t), f(s) \rangle \leq k(t, s),
\]

for almost every \((t, s) \in \Delta\). Then we have the following quadratic reverse of the continuous triangle inequality:

\[
\left( \int_a^b \|f(t)\| \, dt \right)^2 \leq \left\| \int_a^b f(t) \, dt \right\|^2 + 2 \int_\Delta k(t, s) \, dt \, ds.
\]

The case of equality holds in (2.2) if and only if it holds in (2.1) for almost every \((t, s) \in \Delta\).

**Proof:** We observe that the following identity holds

\[
\left( \int_a^b \|f(t)\| \, dt \right)^2 = \int_a^b \int_a^b \|f(t)\| \|f(s)\| \, dt \, ds - \left\langle \int_a^b f(t) \, dt, \int_a^b f(s) \, ds \right\rangle
\]

\[
= \int_a^b \int_a^b \|f(t)\| \|f(s)\| \, dt \, ds - \int_a^b \int_a^b \text{Re} \langle f(t), f(s) \rangle \, dt \, ds
\]

\[
= \int_a^b \int_a^b \left[ \|f(t)\| \|f(s)\| - \text{Re} \langle f(t), f(s) \rangle \right] \, dt \, ds := I.
\]

Now, observe that for any \((t, s) \in [a,b] \times [a,b]\), we have

\[
\|f(t)\| \|f(s)\| - \text{Re} \langle f(t), f(s) \rangle = \|f(s)\| \|f(t)\| - \text{Re} \langle f(s), f(t) \rangle
\]

and thus

\[
I = 2 \int_\Delta \left[ \|f(t)\| \|f(s)\| - \text{Re} \langle f(t), f(s) \rangle \right] \, dt \, ds.
\]

Using the assumption (2.1), we deduce

\[
\int_\Delta \left[ \|f(t)\| \|f(s)\| - \text{Re} \langle f(t), f(s) \rangle \right] \, dt \, ds \leq \int_\Delta k(t, s) \, dt \, ds,
\]

and, by the identities (2.3) and (2.4), we deduce the desired inequality (2.2).

The case of equality is obvious and we omit the details.

**Remark 1.** From (2.2) one may deduce a coarser inequality that can be useful in some applications. It is as follows:

\[
(0 \leq) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \leq \sqrt{2} \left( \int_\Delta k(t, s) \, dt \, ds \right)^{1/2}.
\]
**Remark 2.** If the condition (2.1) is replaced with the following refinement of the Schwarz inequality

(2.5) \( 0 \leq \langle k(t, s) \rangle \leq \|f(t)\|\|f(s)\| - \text{Re} \langle f(t), f(s) \rangle \)

for almost every \((t, s) \in \Delta\), then the following refinement of the quadratic triangle inequality is valid

(2.6) \( \left( \int_t^b \|f(t)\| \, dt \right)^2 \geq \left( \int_t^b k(t, s) \, dt \right)^2 + 2 \int_t^b k(t, s) \, dt \, ds \)

The equality holds in (2.6) if and only if the case of equality holds in (2.5) for almost every \((t, s) \in \Delta\).

The following result holds.

**Theorem 3.** Let \( f \in L([a, b]; H) \) be such that there exists \( M \geq 1 \geq m \geq 0 \) such that either

(2.7) \( \text{Re} \langle Mf(s) - f(t), f(t) - mf(s) \rangle \geq 0 \) for almost every \((t, s) \in \Delta\),
or, equivalently,

(2.8) \( \|f(t) - \frac{M + m}{2} f(s)\| \leq \frac{1}{2} (M - m) \|f(s)\| \) for almost every \((t, s) \in \Delta\).

Then we have the inequality:

(2.9) \( \left( \int_a^b \|f(t)\| \, dt \right)^2 \leq \left( \int_a^b \|f(t)\| \, dt \right)^2 + \frac{1}{2} \cdot (M - m)^2 \int_a^b (s - a) \|f(s)\|^2 \, ds. \)

The case of equality holds in (2.9) if and only if

(2.10) \( \|f(t)\|\|f(s)\| - \text{Re} \langle f(t), f(s) \rangle = \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \|f(s)\|^2 \)

for almost every \((t, s) \in \Delta\).

**Proof:** Firstly, observe that, in an inner product space \((H; \langle \cdot, \cdot \rangle)\) and for \( x, z, \) \( Z \in H \), the following statements are equivalent

(i) \( \text{Re} \langle Z - x, x - z \rangle \geq 0 \) and

(ii) \( \|x - (Z + z)/2\| \leq \|Z - z\|/2. \)

This shows that (2.7) and (2.8) are obviously equivalent.

Now, taking the square in (2.8), we get

\[ \|f(t)\|^2 + \left( \frac{M + m}{2} \right)^2 \|f(s)\|^2 \leq 2 \text{Re} \left( f(t), \frac{M + m}{2} f(s) \right) + \frac{1}{4} (M - m)^2 \|f(s)\|^2, \]
for almost every \((t, s) \in \Delta\), and obviously, since
\[
2\left(\frac{M + m}{2}\right)\|f(t)\|\|f(s)\| \leq \|f(t)\|^2 + \left(\frac{M + m}{2}\right)^2\|f(s)\|^2,
\]
we deduce that
\[
2\left(\frac{M + m}{2}\right)\|f(t)\|\|f(s)\| \leq 2\Re\left(\langle f(t), \frac{M + m}{2}f(s) \rangle + \frac{1}{4}(M - m)^2\|f(s)\|^2,
\]
giving the much simpler inequality:
\[
(2.11) \quad \|f(t)\|\|f(s)\| - \Re\langle f(t), f(s) \rangle \leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \|f(s)\|^2
\]
for almost every \((t, s) \in \Delta\).

Applying Lemma 1 for \(k(t, s) := \frac{1}{4} \cdot \frac{(M - m)^2}{M + m}\|f(s)\|^2\), we deduce
\[
(2.12) \quad \left(\int_a^b \|f(t)\| \, dt\right)^2 \leq \left(\int_a^b f(t) \, dt\right)^2 + \frac{1}{2} \cdot \frac{(M - m)^2}{M + m} \int_{\Delta} \|f(s)\|^2 \, ds
\]
with equality if and only if (2.11) holds for almost every \((t, s) \in \Delta\).

Since
\[
\int_{\Delta} \|f(s)\|^2 \, ds = \int^b_a \left(\int^b_a \|f(s)\|^2 \, dt\right) \, ds = \int^b_a (s - a)\|f(s)\|^2 \, ds,
\]
then by (2.12) we deduce the desired result (2.9).

Another result which is similar to the one above is incorporated in the following theorem.

**Theorem 4.** With the assumptions of Theorem 3, we have
\[
(2.13) \quad \left(\int_a^b \|f(t)\| \, dt\right)^2 - \left(\int_a^b f(t) \, dt\right)^2 \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{MM}} \left(\int_a^b f(t) \, dt\right)^2
\]
or, equivalently,
\[
(2.14) \quad \int_a^b \|f(t)\| \, dt \leq \left(\frac{M + m}{2\sqrt{MM}}\right)^{1/2}\left(\int_a^b f(t) \, dt\right).
\]
The case of equality holds in (2.13) or (2.14) if and only if
\[
(2.15) \quad \|f(t)\|\|f(s)\| = \frac{M + m}{2\sqrt{MM}} \Re\langle f(t), f(s) \rangle,
\]
for almost every \((t, s) \in \Delta\).
PROOF: From (2.7), we deduce
\[ \|f(t)\|^2 + Mm\|f(s)\|^2 \leq (M + m) \text{Re} \langle f(t), f(s) \rangle \]
for almost every \((t, s) \in \Delta\). Dividing by \(\sqrt{Mm} > 0\), we deduce
\[ \frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm}\|f(s)\|^2 \leq \frac{M + m}{\sqrt{Mm}} \text{Re} \langle f(t), f(s) \rangle \]
and, obviously, since
\[ 2\|f(t)\|\|f(s)\| \leq \frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm}\|f(s)\|^2, \]
hence
\[ \|f(t)\|\|f(s)\| \leq \frac{M + m}{\sqrt{Mm}} \text{Re} \langle f(t), f(s) \rangle \]
for almost every \((t, s) \in \Delta\), giving
\[ \|f(t)\|\|f(s)\| - \text{Re} \langle f(t), f(s) \rangle \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{Mm}} \text{Re} \langle f(t), f(s) \rangle. \]
Applying Lemma 1 for \(k(t, s) := (\sqrt{M} - \sqrt{m})^2/\sqrt{Mm} \text{Re} \langle f(t), f(s) \rangle\), we deduce
\[ (\int_a^b \|f(t)\| dt)^2 \leq \int_a^b \|f(t)\| dt^2 + \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{Mm}} \text{Re} \langle f(t), f(s) \rangle. \]
On the other hand, since
\[ \text{Re} \langle f(t), f(s) \rangle = \text{Re} \langle f(s), f(t) \rangle \quad \text{for any } (t, s) \in [a, b]^2, \]
hence
\[ \iint_\Delta \text{Re} \langle f(t), f(s) \rangle dt \, ds = \frac{1}{2} \int_a^b \int_a^b \text{Re} \langle f(t), f(s) \rangle dt \, ds \]
\[ = \frac{1}{2} \text{Re} \left\langle \int_a^b f(t) dt, \int_a^b f(st) ds \right\rangle \]
\[ = \frac{1}{2} \left\| \int_a^b f(t) dt \right\|^2 \]
and thus, from (2.16), we get (2.13).

The equivalence between (2.13) and (2.14) is obvious and we omit the details. \(\square\)
3. Related Results

The following result also holds.

**Theorem 5.** Let \( f \in L([a, b]; H) \) and \( \gamma, \Gamma \in \mathbb{R} \) be such that either

\[ \text{Re} \left( \Gamma f(s) - f(t), f(t) - \gamma f(s) \right) \geq 0 \text{ for almost every } (t, s) \in \Delta, \]

or, equivalently,

\[ \left\| f(t) - \frac{\Gamma + \gamma}{2} f(s) \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|f(s)\| \text{ for almost every } (t, s) \in \Delta. \]

Then we have the inequality:

\[ \int_a^b \left[ (b - s) + \gamma \Gamma (s - a) \right] \|f(s)\|^2 \, ds \leq \frac{\Gamma + \gamma}{2} \left( \int_a^b \|f(s)\|^2 \, ds \right)^2. \]

The case of equality holds in (3.3) if and only if the case of equality holds in either (3.1) or (3.2) for almost every \((t, s) \in \Delta.\)

**Proof:** The inequality (3.1) is obviously equivalent to

\[ \|f(t)\|^2 + \gamma \Gamma \|f(s)\|^2 \leq (\Gamma + \gamma) \text{Re} \left( f(t), f(s) \right) \]

for almost every \((t, s) \in \Delta.\)

Integrating (3.4) on \(\Delta,\) we deduce

\[ \int_a^b \left( \int_a^s \|f(t)\|^2 \, dt \right) \, ds + \gamma \Gamma \int_a^b \left( \int_a^s \|f(t)\|^2 \, dt \right) \, ds \]

\[ \leq (\Gamma + \gamma) \int_a^b \left( \int_a^s \text{Re} \left( f(t), f(s) \right) \, dt \right) \, ds. \]

It is easy to see, on integrating by parts, that

\[ \int_a^b \left( \int_a^s \|f(t)\|^2 \, dt \right) \, ds = s \int_a^s \|f(t)\|^2 \, dt \bigg|_a^b - \int_a^b s \|f(s)\|^2 \, ds \]

\[ = b \int_a^s \|f(s)\|^2 \, ds - \int_a^b s \|f(s)\|^2 \, ds \]

\[ = \int_a^b (b - s) \|f(s)\|^2 \, ds \]

and

\[ \int_a^b \left( \|f(s)\|^2 \int_a^s \, dt \right) \, ds = \int_a^b (s - a) \|f(s)\|^2 \, ds. \]

Since

\[ \frac{d}{ds} \left( \int_a^b f(t) \, dt \right) \|^2 = \frac{d}{ds} \left( \int_a^s f(t) \, dt, \int_a^s f(t) \, dt \right) \]

\[ = \left( f(s), \int_a^s f(t) \, dt \right) + \left( \int_a^s f(t) \, dt, f(s) \right) \]

\[ = 2 \text{Re} \left( \int_a^s f(t) \, dt, f(s) \right), \]
hence
\[
\int_a^b \left( \int_a^s \text{Re} \langle f(t), f(s) \rangle \, dt \right) \, ds = \int_a^b \text{Re} \left( \int_a^s f(t) \, dt \right) \, ds = \frac{1}{2} \int_a^b \frac{d}{ds} \left( \left\| \int_a^s f(t) \, dt \right\|^2 \right) \, ds = \frac{1}{2} \left\| \int_a^b f(t) \, dt \right\|^2.
\]
Utilising (3.5), we deduce the desired inequality (3.3).

The case of equality is obvious and we omit the details.

**REMARK 3.** Consider the function \( \varphi(s) := (b - s) + \gamma (s - a) \), \( s \in [a, b] \). Obviously,
\[
\varphi(s) = (\Gamma \gamma - 1)s + b - \gamma a.
\]
Observe that, if \( \Gamma \gamma \geq 1 \), then
\[
b - a = \varphi(a) \leq \varphi(s) \leq \varphi(b) = \gamma (b - a), \quad s \in [a, b]
\]
and, if \( \Gamma \gamma < 1 \), then
\[
\gamma (b - a) \leq \varphi(s) \leq b - a, \quad s \in [a, b].
\]
Taking into account the above remark, we may state the following corollary.

**COROLLARY 6.** Assume that \( f, \gamma, \Gamma \) are as in Theorem 5.

(a) If \( \Gamma \gamma \geq 1 \), then we have the inequality
\[
(b - a) \int_a^b \left\| f(s) \right\|^2 \, ds \leq \frac{\Gamma + \gamma}{2} \left\| \int_a^b f(s) \, ds \right\|^2.
\]
(b) If \( 0 < \Gamma \gamma < 1 \), then we have the inequality
\[
\gamma (b - a) \int_a^b \left\| f(s) \right\|^2 \, ds \leq \frac{\Gamma + \gamma}{2} \left\| \int_a^b f(s) \, ds \right\|^2.
\]

4. APPLICATIONS FOR COMPLEX-VALUED FUNCTIONS

Let \( f : [a, b] \to \mathbb{C} \) be a Lebesgue integrable function and \( M \geq 1 \geq m \geq 0 \). The condition (2.7) from Theorem 3, which plays a fundamental role in the results obtained above, can be translated in this case as
\[
(4.1) \quad \text{Re} \left[ \left( M f(s) - f(t) \right) \overline{\left( f(t) - m f(s) \right)} \right] \geq 0
\]
for almost every \( a \leq t \leq s \leq b \).
Since, obviously
\[ \Re \left[ (M f(s) - f(t)) (\overline{f(t)} - m \overline{f(s)}) \right] = \left[ (M \Re f(s) - \Re f(t)) (\Re f(t) - m \Re f(s)) \right] + \left[ (M \Im f(s) - \Im f(t)) (\Im f(t) - m \Im f(s)) \right] \]

hence a sufficient condition for the inequality in (4.1) to hold is
\[ (4.2) \quad m \Re f(s) \leq \Re f(t) \leq M \Re f(s) \quad \text{and} \quad m \Im f(s) \leq \Im f(t) \leq M \Im f(s) \]
for almost every \( a \leq t \leq s \leq b \).

Utilising Theorems 3, 4 and 5 we may state the following results incorporating quadratic reverses of the continuous triangle inequality:

**Proposition 1.** With the above assumptions for \( f, M \) and \( m \), and if (4.1) holds true, then we have the inequalities
\[
\left( \int_a^b |f(t)| \, dt \right)^2 \leq \int_a^b |f(t)|^2 \, dt + \frac{1}{2} \frac{(M - m)^2}{M + m} \int_a^b (s - a) |f(s)|^2 \, ds,
\]
\[
\int_a^b |f(t)| \, dt \leq \left( \frac{M + m}{2 \sqrt{M m}} \right)^{1/2} \int_a^b |f(t)| \, dt,
\]
and
\[
\int_a^b \left[ (b - s) + mM(s - a) \right] |f(s)|^2 \, ds \leq \frac{M + m}{2} \left| \int_a^b f(s) \, ds \right|^2.
\]

**Remark 4.** One may wonder if there are functions satisfying the condition (4.2) above. It suffices to find examples of real functions \( \varphi : [a, b] \to \mathbb{R} \) verifying the following double inequality
\[ (4.3) \quad \gamma \varphi(s) \leq \varphi(t) \leq \Gamma \varphi(s) \]
for some given \( \gamma, \Gamma \) with \( 0 \leq \gamma \leq 1 \leq \Gamma < \infty \) for almost every \( a \leq t \leq s \leq b \).

For this purpose, consider \( \psi : [a, b] \to \mathbb{R} \) a differentiable function on \( (a, b) \), continuous on \( [a, b] \) and with the property that there exists \( \Theta \geq 0 \geq \theta \) such that
\[ (4.4) \quad \theta \leq \psi'(u) \leq \Theta \text{ for any } u \in (a, b). \]

By Lagrange's mean value theorem, we have, for any \( a \leq t \leq s \leq b \)
\[ \psi(s) - \psi(t) = \psi'(\xi)(s - t) \]
with \( t \leq \xi \leq s \). Therefore, for \( a \leq t \leq s \leq b \), by (4.4), we have the inequality
\[ \theta(s - t) \leq \psi(s) - \psi(t) \leq \Theta(s - t) \leq \Theta(b - a). \]

If we choose the function \( \varphi : [a, b] \to \mathbb{R} \) given by
\[ \varphi(t) := \exp \left[ -\psi(t) \right], \quad t \in [a, b], \]
and \( \gamma := \exp \left[ \theta(b - a) \right] \leq 1, \Gamma := \exp \left[ \Theta(b - a) \right] \geq 1 \), then (4.3) holds true for any \( a \leq t \leq s \leq b \).
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