A NOTE ON THE DENSITY THEOREM

BY

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In this note we prove:

THEOREM. Let R be a right primitive ring with pair-wise non-isomorphic faithful irreducible modules $M_1, M_2, \ldots, M_k$. Let $D_i = \text{End}_R M_i$. For each i, let \{v_{ij}\}_{j=1}^{n_i}$ be elements of $M_i$ linearly independent over $D_i$. For each i, let \{u_{ij}\}_{j=1}^{n_i}$ be a set of elements of $M_i$. Then there exists an element $r$ of $R$ such that $u_{ij} = v_{ij}r$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$.

Thus the statement of the density theorem generalizes from the case of a single faithful irreducible module to the case where we have a finite collection of pairwise nonisomorphic faithful irreducible modules.

I would like to thank Professors Alperin and Herstein for suggesting the above theorem.

Proof. It is enough to show that for given $(a, b) \in N \times N$ there exists an element $r_{ab} \in R$ such that $v_{ij}r_{ab} = 0$ if $(i, j) \neq (a, b)$ and $v_{ij}r_{ab} \neq 0$ if $(i, j) = (a, b)$. Without loss of generality, we consider only the case where $(a, b) = (k, n_k)$. By the Jacobson Density Theorem [1], we can choose $t \in R$ such that $v_k^j t = 0$, $j = 1, 2, \ldots, n_k - 1$, and $v_k^{n_k} t \neq 0$. Consider the external direct sum of modules

$$\sum_{i=1}^{k-1} l(M_i),$$

where $l(M_i)$ stands for the direct sum of $n_i$ copies of $M_i$. Let $\alpha = v_{11} t + v_{12} t + \cdots + v_{1n_1} t + v_{21} t + \cdots + v_{k-1n_{k-1}} t$. The relation $f: \alpha R \rightarrow M_k$ defined by $aR \mapsto v_{kn} ta$, where $a \in R$, is a nonzero module homomorphism if it is well-defined as a function. This is impossible by the Jordan–Hölder Theorem, since $M_k$ is not isomorphic to any other $M_i$. Thus there is an $s \in R$ such that $\alpha s = 0$ and $v_{kn} ts \neq 0$. Let $r_{kn} = ts$. This completes the proof.

REFERENCE


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