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A NEW PROOF OF THE MACDONALD IDENTITIES FOR A_{n-1}

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Abstract

A new, elementary proof of the Macdonald identities for A_{n-1} using induction on n is given. Specifically, the Macdonald identity for A_n is deduced by multiplying the Macdonald identity for A_{n-1} and n Jacobi triple product identities together.

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1. Introduction

Throughout this paper q is any complex number satisfying $|q| < 1$. We will use the following notation for infinite products. Let

$$(x; q)_\infty = \prod_{m=0}^{\infty}(1 - xq^m)$$

and $$(x_1, x_2, \dots, x_n; q)_\infty = (x_1; q)_\infty (x_2; q)_\infty \dots (x_n; q)_\infty.$$

The Jacobi triple product identity is

$$(1.1) \qquad (x, qx^{-1}, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n,$$

where $\binom{n}{2} = n(n-1)/2$.

For proofs of this identity, see Andrews [2], Andrews [3, pp. 63–64], Gasper and Rahman [9, p. 12] or Hardy and Wright [10, pp. 282–283].

The Macdonald identities are multivariate generalizations of the Jacobi triple product identity. These identities were formulated in terms of affine root systems

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and proved by Macdonald [13], although some instances of these identities were discovered earlier by Dyson [7, Section 2] and others (see also [13, p. 94]).

The Macdonald identity for A_{n-1} is

$$(1.2)\qquad (q;q)_\infty^{n-1} \prod_{1\le i<j\le n} (x_i x_j^{-1};q)_\infty (q x_i^{-1} x_j;q)_\infty = \sum_m^{(n)} q^{\frac{n}{2}(m_1^2+\cdots+m_n^2)+m_1+2m_2+\cdots+nm_n} x_1^{nm_1}\cdots x_n^{nm_n} \prod_{1\le i<j\le n}\left(1-\frac{x_i q^{m_i}}{x_j q^{m_j}}\right).$$

Here $\sum_m^{(n)}$ means sum over all integers $m_1, \cdots, m_n$ satisfying $m_1+\cdots+m_n=0$, and this notation will be used throughout. The aim of this paper is to give a simple proof of this identity using induction on n. The inductive step consists of writing

$$\prod_{1\le i<j\le n+1} (x_i x_j^{-1};q)_\infty (q x_i^{-1} x_j;q)_\infty = \prod_{1\le i<j\le n} (x_i x_j^{-1};q)_\infty (q x_i^{-1} x_j;q)_\infty \prod_{j=1}^{n} (x_j x_{n+1}^{-1};q)_\infty (q x_j^{-1} x_{n+1};q)_\infty .$$

Now use (1.2) as the inductive hypothesis to expand the product

$$\prod_{1\le i<j\le n} (x_i x_j^{-1};q)_\infty (q x_i^{-1} x_j;q)_\infty$$

and use the Jacobi triple product identity (1.1) to expand each product

$$(x_j x_{n+1}^{-1};q)_\infty (q x_j^{-1} x_{n+1};q)_\infty .$$

The truth of the Macdonald identity for A_n then follows after some calculation and simplification.

The idea of multiplying identities together has been used before.

1. Carlitz and Subbarao [4] proved the quintuple product identity (the Macdonald identity for BC_1) by multiplying two Jacobi triple product identities together. Hirschhorn [12] obtained a generalization using a similar method.
2. Carlitz and Subbarao [5] gave a proof of Winquist's identity (the Macdonald identity for B_2), and a generalization, by multiplying four Jacobi triple product identities together. A different generalization was obtained by Hirschhorn [11], who also multiplied four Jacobi triple product identities together.
3. Garvan [8, Theorem 3.5] proved the Macdonald identity for A_2 by multiplying three Jacobi triple product identities together.
4. Cooper [6, Chapter 2] gave new proofs of the Macdonald identities for G_2 and $G_2^\vee$ and obtained some generalizations by multiplying two Macdonald identities for A_2 together.

A knowledge of root systems is not necessary to read this paper. There are only two prerequisites. The first is the Jacobi triple product identity. The second is the Vandermonde determinant identity, which will be stated below in Lemma 4.1.

The reader can consult Milne [14] for another proof of the Macdonald identities for A_n, Stanton [15] for the Macdonald identities for the other infinite families and Macdonald [13] for the general case. Milne's paper [14] mentions connections with other topics and contains a large number of references.

2. The initial cases $n = 1$ and $n = 2$

Let

$$F_n(x_1, \dots, x_n; q) = \prod_{1 \le i < j \le n} (x_i x_j^{-1}; q)_\infty (q x_i^{-1} x_j; q)_\infty. \tag{2.1}$$

The Macdonald identity for A_{n-1} is an explicit formula for the Laurent series expansion of F_n in powers of $x_1, \dots, x_n$. Denote the multivariable Laurent expansion of F_n by

$$F_n(x_1, \dots, x_n; q) = \sum c_n(\alpha_1, \dots, \alpha_n; q) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{2.2}$$

where the summation is over integers $-\infty < \alpha_1, \alpha_2, \dots, \alpha_n < \infty$. Since F_n is homogeneous of degree zero, that is,

$$F_n(\lambda x_1, \dots, \lambda x_n; q) = F_n(x_1, \dots, x_n; q),$$

for any non-zero complex number λ, the summation in (2.2) is actually over all integers satisfying $\alpha_1 + \cdots + \alpha_n = 0$. The remainder of this paper is devoted to showing that the multivariate Laurent series expansion of F_n is given by the following formula.

THEOREM 2.3 (Macdonald identity for A_{n-1}).

$$\begin{aligned} &F_n(x_1, \dots, x_n; q) \\ &\quad = c_n(q) \sum_m^{(n)} q^{\frac{n}{2}(m_1^2 + \cdots + m_n^2) + m_1 + 2m_2 + \cdots + nm_n} x_1^{nm_1} \cdots x_n^{nm_n} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i q^{m_i}}{x_j q^{m_j}}\right), \end{aligned} \tag{2.4}$$

where the summation is over all integers $-\infty < m_1, \dots, m_n < \infty$ *satisfying* $m_1 + m_2 + \cdots + m_n = 0$, *and*

$$c_n(q) = \frac{1}{(q; q)_\infty^{n-1}}. \tag{2.5}$$

The case $n = 1$ is trivial: it claims $1 = 1$. Observe that the case $q = 0$ is also trivial: in this case, the only non-zero term on the right-hand side is the one corresponding to $m_1 = m_2 = \cdots = m_n = 0$, and so both sides reduce to $\prod_{1\le i<j\le n}(1 - x_i x_j^{-1})$. From now on we will assume $0 < |q| < 1$.

The case $n = 2$ (Macdonald identity for A_1) is

$$\begin{aligned}(2.6)\qquad &\left(\frac{x_1}{x_2};q\right)_\infty \left(\frac{qx_2}{x_1};q\right)_\infty \\ &= \frac{1}{(q;q)_\infty} \sum_{m_1+m_2=0} q^{m_1^2+m_2^2+m_1+2m_2} x_1^{2m_1} x_2^{2m_2} \left(1 - \frac{x_1 q^{m_1}}{x_2 q^{m_2}}\right).\end{aligned}$$

This is equivalent to the Jacobi triple product identity. To see this, put $x = x_1 x_2^{-1}$ in (1.1) to get

$$\begin{aligned}(x_1x_2^{-1}, qx_1^{-1}x_2, q; q)_\infty &= \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{m}{2}} \left(\frac{x_1}{x_2}\right)^m \\ &= \sum_{m=-\infty}^{\infty} q^{2m^2-m} \left(\frac{x_1}{x_2}\right)^{2m} - \sum_{m=-\infty}^{\infty} q^{2m^2+m} \left(\frac{x_1}{x_2}\right)^{2m+1} \\ &= \sum_{m=-\infty}^{\infty} q^{2m^2-m} x_1^{2m} x_2^{-2m} \left(1 - \frac{q^{2m}x_1}{x_2}\right).\end{aligned}$$

Now let $m_1 = m$ and $m_2 = -m$ and divide both sides by $(q;q)_\infty$. The result is (2.6), and thus the Macdonald identity for A_1 is equivalent to the Jacobi triple product identity. This completes the easy part of the induction. The cases $n = 1$ and $n = 2$ have been verified.

3. The inductive step: outline

Suppose that the Macdonald identity for A_{n-1} is true. That is, take as the inductive hypothesis the statement that equations (2.4) and (2.5) are true for some value of n. Now consider the function F_{n+1}. We want to show that F_{n+1} has a Laurent series expansion of the same form as (2.4) but with n replaced with $n+1$. We proceed in four steps.

1. Use the Macdonald identity for A_{n-1} together with the Jacobi triple product identity to find an explicit formula for $c_{n+1}(\alpha_1, \dots, \alpha_{n+1}; q)$ as an infinite product. In particular, this formula gives as a special case the value of the constant term $c_{n+1}(0, \dots, 0; q)$ (cf. Stanton [15], where a different procedure is used to compute the constant term).
2. Use the result of step 1 to deduce which coefficients are zero and which are non-zero.

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3. Use the results of steps 1 and 2 to express any non-zero coefficient $c_{n+1}(\alpha_1, \dots, \alpha_{n+1}; q)$ as a multiple of one of a finite set of $(n+1)!$ coefficients. This set of coefficients is called the *orbit* of the constant term under the action of the symmetric group S_{n+1}.
4. Show that each coefficient in the orbit of the constant term is equal to $\pm c_{n+1}(0, \dots, 0; q)$.

These four steps will complete the inductive step and a summary will be given in the last section.

4. Step 1: an infinite product formula for the coefficients

In this section we will obtain a formula for the coefficient $c_{n+1}(\alpha_1, \dots, \alpha_{n+1}; q)$ as an infinite product. In the course of the calculation we shall encounter the Vandermonde determinant identity.

LEMMA 4.1 (Vandermonde determinant identity). *Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be a permutation of $(1, 2, \dots, n)$. Let S_n be the set of all permutations of $(1, 2, \dots, n)$. Let $\operatorname{sgn}(\sigma) = +1$ or -1, depending on whether σ is an even or odd permutation, respectively. Then*

$$\sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) a_1^{\sigma_1 - 1} a_2^{\sigma_2 - 2} \cdots a_n^{\sigma_n - n} = \prod_{1 \le i < j \le n} \left(1 - \frac{a_i}{a_j}\right). \tag{4.2}$$

For an outline of a proof of this identity, see [1, pp. 41–42, exercises 10 and 11].

The main result of this section is the following.

LEMMA 4.3. *Let $\alpha_1, \cdots, \alpha_{n+1}$ be integers satisfying $\alpha_1 + \cdots + \alpha_{n+1} = 0$. Then*

$$\begin{aligned} c_{n+1}(\alpha_1, \dots, \alpha_{n+1}; q) = & (-1)^{\alpha_{n+1}} q^{\binom{\alpha_1}{2} + \cdots + \binom{\alpha_n}{2}} \frac{(q^{n+1}; q^{n+1})_\infty^{n-1}}{(q; q)_\infty^{2n-1}} \\ & \times \prod_{1 \le i < j \le n} (q^{(\alpha_j + j) - (\alpha_i + i)}, q^{n+1+(\alpha_i+i)-(\alpha_j+j)}; q^{n+1})_\infty. \end{aligned} \tag{4.4}$$

PROOF. Write

$$F_{n+1}(x_1, \dots, x_{n+1}; q) = F_n(x_1, \dots, x_n; q) \prod_{i=1}^{n} (x_i x_{n+1}^{-1}, q x_i^{-1} x_{n+1}; q)_\infty.$$

The idea is to use the inductive hypothesis to expand F_n as a series, use the Jacobi triple product identity to expand each product $(x_i x_{n+1}^{-1}; q)_\infty (q x_i^{-1} x_{n+1}; q)_\infty$ and then extract the coefficient of $x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}$. The resulting series can be summed by the inductive hypothesis to give the required result. The details are as follows.

By (2.1), (2.4), (4.2) and the Jacobi triple product identity (1.1) applied n times, we have

$$\begin{aligned}
&F_{n+1}(x_1, \dots, x_{n+1}; q) \\
&\quad = F_n(x_1, \dots, x_n; q) \prod_{i=1}^{n} (x_i x_{n+1}^{-1}; q)_\infty (q x_i^{-1} x_{n+1}; q)_\infty \\
&\quad = \frac{1}{(q;q)_\infty^{n-1}} \sum_m^{(n)} q^{\frac{n}{2}(m_1^2+\cdots+m_n^2)+m_1+2m_2+\cdots+nm_n} x_1^{nm_1} \cdots x_n^{nm_n} \\
&\qquad \times \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma)(x_1 q^{m_1})^{\sigma_1-1} \cdots (x_n q^{m_n})^{\sigma_n - n} \\
&\qquad \times \frac{1}{(q;q)_\infty^n} \sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_n=-\infty}^{\infty} (-1)^{i_1+\cdots+i_n} q^{\binom{i_1}{2}+\cdots+\binom{i_n}{2}} x_1^{i_1} \cdots x_n^{i_n} x_{n+1}^{-(i_1+\cdots+i_n)}. \\
&\quad = \frac{1}{(q;q)_\infty^{2n-1}} \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) \sum_m^{(n)} \sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_n=-\infty}^{\infty} (-1)^{i_1+\cdots+i_n} \\
&\qquad \times q^{\frac{n}{2}(m_1^2+\cdots+m_n^2)+m_1+2m_2+\cdots+nm_n+\binom{i_1}{2}+\cdots+\binom{i_n}{2}+m_1(\sigma_1-1)+\cdots+m_n(\sigma_n-n)} \\
&\qquad \times x_1^{nm_1+(\sigma_1-1)+i_1} \cdots x_n^{nm_n+(\sigma_n-n)+i_n} x_{n+1}^{-i_1-\cdots-i_n}.
\end{aligned}$$

Put

$$\begin{aligned}
nm_1 + (\sigma_1 - 1) + i_1 &= \alpha_1, \\
&\vdots \\
nm_n + (\sigma_n - n) + i_n &= \alpha_n, \\
-i_1 - \cdots - i_n &= \alpha_{n+1}.
\end{aligned}$$

Then $\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} = 0$ and

$$\begin{aligned}
\text{(4.5)} \qquad & c_{n+1}(\alpha_1, \cdots, \alpha_{n+1}; q) \\
&\quad = \frac{(-1)^{\alpha_{n+1}}}{(q;q)_\infty^{2n-1}} \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) \sum_m^{(n)} q^{\frac{n}{2}(m_1^2+\cdots+m_n^2)+m_1+\cdots+nm_n} \\
&\qquad \times q^{\binom{\alpha_1 - nm_1 - (\sigma_1-1)}{2}+\cdots+\binom{\alpha_n - nm_n - (\sigma_n - n)}{2}+m_1(\sigma_1-1)+\cdots+m_n(\sigma_n-n)}.
\end{aligned}$$

Now,

$$\begin{aligned}
\binom{a-b-c}{2} &= \frac{(a-b-c)(a-b-c-1)}{2} \\
&= \binom{a}{2} + \frac{b^2}{2} - ab + (b-a)c + \frac{c^2}{2} + \frac{b}{2} + \frac{c}{2},
\end{aligned}$$

so

$$\begin{aligned}\binom{\alpha_k - nm_k - (\sigma_k - k)}{2} &= \binom{\alpha_k}{2} + \frac{n^2 m_k^2}{2} - n\alpha_k m_k + (nm_k - \alpha_k)(\sigma_k - k) \\ &\quad + \frac{(\sigma_k - k)^2}{2} + \frac{nm_k}{2} + \frac{\sigma_k - k}{2}.\end{aligned}$$

Thus we find

$$\begin{aligned}\sum_{k=1}^{n}\binom{\alpha_k - nm_k - (\sigma_k - k)}{2} &= \sum_{k=1}^{n}\binom{\alpha_k}{2} + \frac{n^2}{2}\sum_{k=1}^{n} m_k^2 - n\sum_{k=1}^{n}\alpha_k m_k \\ &\quad + \sum_{k=1}^{n}(nm_k - \alpha_k)(\sigma_k - k) + \frac{1}{2}\sum_{k=1}^{n}(\sigma_k - k)^2 + 0 + 0\end{aligned}$$

while

$$\begin{aligned}\sum_{k=1}^{n}(\sigma_k - k)^2 &= \sum_{k=1}^{n}\sigma_k^2 - 2\sum_{k=1}^{n}k(\sigma_k - k) - \sum_{k=1}^{n}k^2 \\ &= -2\sum_{k=1}^{n}k(\sigma_k - k).\end{aligned}$$

Substitute these back into equation (4.5) to obtain

$$\begin{aligned}&c_{n+1}(\alpha_1, \cdots, \alpha_{n+1}; q) \\ &\quad = \frac{(-1)^{\alpha_{n+1}}}{(q;q)_\infty^{2n-1}} q^{\binom{\alpha_1}{2}+\cdots+\binom{\alpha_n}{2}} \sum_{\sigma\in S_n}(\operatorname{sgn}\sigma) \\ &\qquad \times \sum_m{}^{(n)} q^{\frac{n(n+1)}{2}(m_1^2+\cdots+m_n^2)+m_1+\cdots+nm_n-n(\alpha_1 m_1+\cdots+\alpha_n m_n)} \\ &\qquad \times q^{(\sigma_1-1)((n+1)m_1-(\alpha_1+1))+\cdots+(\sigma_n-n)((n+1)m_n-(\alpha_n+n))} \\ &\quad = \frac{(-1)^{\alpha_{n+1}}}{(q;q)_\infty^{2n-1}} q^{\binom{\alpha_1}{2}+\cdots+\binom{\alpha_n}{2}} \\ &\qquad \times \sum_m{}^{(n)} q^{\frac{n(n+1)}{2}(m_1^2+\cdots+m_n^2)+(n+1)(m_1+\cdots+nm_n)-n((\alpha_1+1)m_1+\cdots+(\alpha_n+n)m_n)} \\ &\qquad \times \sum_{\sigma\in S_n}(\operatorname{sgn}\sigma) q^{(\sigma_1-1)((n+1)m_1-(\alpha_1+1))+\cdots+(\sigma_n-n)((n+1)m_n-(\alpha_n+n))} \\ &\quad = \frac{(-1)^{\alpha_{n+1}}}{(q;q)_\infty^{2n-1}} q^{\binom{\alpha_1}{2}+\cdots+\binom{\alpha_n}{2}} \\ &\qquad \times \sum_m{}^{(n)} q^{\frac{n(n+1)}{2}(m_1^2+\cdots+m_n^2)+(n+1)(m_1+\cdots+nm_n)-n((\alpha_1+1)m_1+\cdots+(\alpha_n+n)m_n)}\end{aligned}$$

$$\times \prod_{1\le i<j\le n} \left(1 - \frac{q^{(n+1)m_i-(\alpha_i+i)}}{q^{(n+1)m_j-(\alpha_j+j)}}\right)$$
$$= \frac{(-1)^{\alpha_{n+1}}}{(q;q)_\infty^{2n-1}} q^{\binom{\alpha_1}{2}+\cdots+\binom{\alpha_n}{2}} (q^{n+1};q^{n+1})_\infty^{n-1}$$
$$\times \prod_{1\le i<j\le n} (q^{(\alpha_j+j)-(\alpha_i+i)}, q^{n+1+(\alpha_i+i)-(\alpha_j+j)}; q^{n+1})_\infty,$$

where in the last step we have used the induction hypothesis (2.5) with q^{n+1} for q and $x_i = q^{-(\alpha_i+i)}$. This proves Lemma 4.3.

COROLLARY 4.6.

$$c_{n+1}(0,\ldots,0;q) = \frac{1}{(q;q)_\infty^n}. \tag{4.7}$$

PROOF. Taking $\alpha_1 = \cdots = \alpha_{n+1} = 0$ in Lemma 4.3 gives

$$c_{n+1}(0,\cdots,0;q) = \frac{(q^{n+1};q^{n+1})_\infty^{n-1}}{(q;q)_\infty^{2n-1}} \prod_{1\le i<j\le n} (q^{j-i};q^{n+1})_\infty (q^{n+1+i-j};q^{n+1})_\infty$$
$$= \frac{(q^{n+1};q^{n+1})_\infty^{n-1}}{(q;q)_\infty^{2n-1}} \prod_{i=1}^{n} (q^i;q^{n+1})_\infty^{n-1}$$
$$= \frac{(q;q)_\infty^{n-1}}{(q;q)_\infty^{2n-1}} = \frac{1}{(q;q)_\infty^n}.$$

5. Step 2: the nonzero coefficients

By lemma 4.3, we have for any integers $\beta_1,\ldots,\beta_{n+1}$ satisfying

$$(\beta_1 - 1) + \cdots + (\beta_{n+1} - (n+1)) = 0, \tag{*}$$

that

$$c_{n+1}(\beta_1 - 1, \beta_2 - 2, \ldots, \beta_{n+1} - (n+1); q) \tag{5.1}$$
$$= (-1)^{\beta_{n+1}-(n+1)} q^{\binom{\beta_1-1}{2}+\cdots+\binom{\beta_n-n}{2}} \frac{(q^{n+1};q^{n+1})_\infty^{n-1}}{(q;q)_\infty^{2n-1}}$$
$$\times \prod_{1\le i<j\le n} (q^{\beta_j-\beta_i}, q^{n+1+\beta_i-\beta_j}; q^{n+1})_\infty.$$

The quantity $(q^{\beta_j-\beta_i}, q^{n+1+\beta_i-\beta_j}; q^{n+1})_\infty$ is zero if and only if $\beta_j - \beta_i \equiv 0 \pmod{n+1}$. Therefore the product in (5.1) is non-zero if and only if $\beta_j - \beta_i \not\equiv 0 \pmod{n+1}$

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for all i, j with $1 \le i < j \le n$, that is, $\beta_1, \dots, \beta_n$ all lie in distinct residue classes $(\bmod\, n+1)$, or equivalently, $\beta_k \equiv \sigma_k \ (\bmod\, n+1)$, $k = 1, \dots, n$, for some permutation $\sigma = (\sigma_1, \cdots, \sigma_{n+1})$ of $(1, \cdots, n+1)$. The condition (*) then forces $\beta_{n+1} \equiv \sigma_{n+1}$ $(\bmod\, n+1)$.

Thus we can write $\beta_k = (n+1)m_k + \sigma_k$, $k = 1, \cdots, n+1$, and the condition (*) is equivalent to $m_1 + \cdots + m_{n+1} = 0$.

We summarize this in the following lemma.

LEMMA 5.2. *A coefficient is non-zero if and only if it is of the form*

$$c_{n+1}((n+1)m_1 + \sigma_1 - 1,\ (n+1)m_2 + \sigma_2 - 2,\ \dots,\ (n+1)m_{n+1} + \sigma_{n+1} - (n+1);\, q)$$

for some permutation $\sigma = (\sigma_1, \dots, \sigma_{n+1})$ *and some integers* $m_1, \dots, m_{n+1}$ *satisfying* $m_1 + \cdots + m_{n+1} = 0$.

6. Step 3: a relation between the non-zero coefficients

The main result in this section is an expression for

$$c_{n+1}((n+1)m_1 + \sigma_1 - 1, \dots, (n+1)m_{n+1} + \sigma_{n+1} - (n+1);\, q)$$

in terms of $c_{n+1}(\sigma_1 - 1, \dots, \sigma_{n+1} - (n+1);\, q)$. Before getting to this, let us record some results that will be needed.

LEMMA 6.1. *If k is an integer, then*

$$\frac{(xq^k, q^{1-k}x^{-1}; q)_\infty}{(x, qx^{-1}; q)_\infty} = \frac{(-1)^k}{x^k q^{\binom{k}{2}}}.$$

PROOF. If $k \ge 0$ then

$$\begin{aligned}\frac{(xq^k, q^{1-k}x^{-1}; q)_\infty}{(x, qx^{-1}; q)_\infty} &= \frac{(1-x^{-1})(1-q^{-1}x^{-1})\cdots(1-q^{1-k}x^{-1})}{(1-x)(1-qx)\cdots(1-q^{k-1}x)} \\ &= (-x^{-1})(-q^{-1}x^{-1})\cdots(-q^{1-k}x^{-1}) \\ &= \frac{(-1)^k}{x^k q^{\binom{k}{2}}}.\end{aligned}$$

If $k < 0$, let $j = -k$. Then

$$\frac{(xq^k, q^{1-k}x^{-1}; q)_\infty}{(x, qx^{-1}; q)_\infty} = \frac{(xq^{-j}, q^{j+1}x^{-1}; q)_\infty}{(x, qx^{-1}; q)_\infty}$$

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$$
\begin{aligned}
&= \frac{(1-xq^{-j})\cdots(1-xq^{-1})}{(1-x^{-1}q)\cdots(1-x^{-1}q^{j})} \\
&= (-xq^{-j})\cdots(-xq^{-1}) \\
&= (-1)^j x^j q^{-\frac{j(j+1)}{2}} \\
&= \frac{(-1)^k}{x^k q^{\binom{k}{2}}}.
\end{aligned}
$$

LEMMA 6.2. *If* $\sum_{k=1}^{n+1} m_k = 0$ *and* $(\sigma_1, \cdots, \sigma_{n+1})$ *is a permutation of* $(1, \cdots, n+1)$ *then*

(a)
$$
\begin{aligned}
&\sum_{k=1}^{n}\left(\binom{(n+1)m_k+\sigma_k-k}{2}-\binom{\sigma_k-k}{2}\right) \\
&\qquad = \frac{(n+1)^2}{2}\sum_{k=1}^{n} m_k^2 + (n+1)\sum_{k=1}^{n}\sigma_k m_k - (n+1)\sum_{k=1}^{n} k m_k + \frac{n+1}{2} m_{n+1}.
\end{aligned}
$$

(b)
$$
\begin{aligned}
&\sum_{1\le i<j\le n} (\sigma_j-\sigma_i)(m_j-m_i) \\
&\qquad = n\sum_{k=1}^{n}\sigma_k m_k - \sigma_{n+1}m_{n+1} + \frac{(n+1)(n+2)}{2} m_{n+1}.
\end{aligned}
$$

(c)
$$
\sum_{1\le i<j\le n}\binom{m_j-m_i}{2} = \frac{n}{2}\sum_{k=1}^{n} m_k^2 - \sum_{k=1}^{n} k m_k - \frac{1}{2}m_{n+1}^2 - \frac{n+1}{2} m_{n+1}.
$$

PROOF. (a) $\binom{a+b-c}{2}-\binom{b-c}{2} = \frac{a^2}{2}-\frac{a}{2}+ab-ac$, so

$$
\begin{aligned}
&\binom{(n+1)m_k+\sigma_k-k}{2}-\binom{\sigma_k-k}{2} \\
&\qquad = \frac{(n+1)^2}{2}m_k^2 - \frac{n+1}{2}m_k + (n+1)\sigma_k m_k - (n+1)k m_k,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{n}\left(\binom{(n+1)m_k+\sigma_k-k}{2}-\binom{\sigma_k-k}{2}\right) \\
&\qquad = \frac{(n+1)^2}{2}\sum_{k=1}^{n} m_k^2 - \frac{n+1}{2}\sum_{k=1}^{n} m_k + (n+1)\sum_{k=1}^{n}\sigma_k m_k - (n+1)\sum_{k=1}^{n} k m_k \\
&\qquad = \frac{(n+1)^2}{2}\sum_{k=1}^{n} m_k^2 + (n+1)\sum_{k=1}^{n}\sigma_k m_k - (n+1)\sum_{k=1}^{n} k m_k + \frac{n+1}{2} m_{n+1}.
\end{aligned}
$$

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(b)
$$\sum_{1\le i<j\le n}(\sigma_j-\sigma_i)(m_j-m_i)$$
$$=\sum_{1\le i<j\le n}(\sigma_i m_i+\sigma_j m_j)-\sum_{1\le i<j\le n}(\sigma_i m_j+\sigma_j m_i)$$
$$=(n-1)\sum_{k=1}^{n}\sigma_k m_k-\sum_{i\neq j}\sigma_i m_j$$
$$=n\sum_{k=1}^{n}\sigma_k m_k-\sum_{k=1}^{n}\sigma_k\sum_{k=1}^{n}m_k$$
$$=n\sum_{k=1}^{n}\sigma_k m_k-\left(\frac{(n+1)(n+2)}{2}-\sigma_{n+1}\right)(-m_{n+1})$$
$$=n\sum_{k=1}^{n}\sigma_k m_k-\sigma_{n+1}m_{n+1}+\frac{(n+1)(n+2)}{2}m_{n+1}.$$

(c)
$$\sum_{1\le i<j\le n}\binom{m_j-m_i}{2}$$
$$=\sum_{1\le i<j\le n}\frac{(m_j-m_i)(m_j-m_i-1)}{2}$$
$$=\sum_{1\le i<j\le n}\left(\frac{m_j^2}{2}-m_i m_j+\frac{m_i^2}{2}-\frac{m_j}{2}+\frac{m_i}{2}\right)$$
$$=\frac{1}{2}\sum_{k=1}^{n}(k-1)m_k^2-\frac{1}{2}\sum_{i\neq j}m_i m_j+\frac{1}{2}\sum_{k=1}^{n}(n-k)m_k^2$$
$$-\frac{1}{2}\sum_{k=1}^{n}(k-1)m_k+\frac{1}{2}\sum_{k=1}^{n}(n-k)m_k$$
$$=\frac{n-1}{2}\sum_{k=1}^{n}m_k^2-\frac{1}{2}\left(\left(\sum_{k=1}^{n}m_k\right)^2-\sum_{k=1}^{n}m_k^2\right)+\frac{1}{2}\sum_{k=1}^{n}(n-2k+1)m_k$$
$$=\frac{n}{2}\sum_{k=1}^{n}m_k^2-\frac{1}{2}(-m_{n+1})^2+\frac{n+1}{2}\sum_{k=1}^{n}m_k-\sum_{k=1}^{n}km_k$$
$$=\frac{n}{2}\sum_{k=1}^{n}m_k^2-\sum_{k=1}^{n}km_k-\frac{1}{2}m_{n+1}^2-\frac{n+1}{2}m_{n+1}.$$

LEMMA 6.3. *Let* $(\sigma_1, \dots, \sigma_{n+1})$ *be a permutation of* $(1, 2, \dots, n+1)$, *and let* $m_1, \dots, m_{n+1}$ *be integers satisfying* $m_1 + \cdots + m_{n+1} = 0$. *Then*

$$c_{n+1}((n+1)m_1 + \sigma_1 - 1, \ \dots, (n+1)m_{n+1} + \sigma_{n+1} - (n+1); q)$$
$$= q^{\frac{n+1}{2}(m_1^2 + \cdots + m_{n+1}^2) + m_1\sigma_1 + \cdots + m_{n+1}\sigma_{n+1}} c_{n+1}(\sigma_1 - 1, \ \dots, \sigma_{n+1} - (n+1); q).$$

PROOF. By Lemma 4.3 we have

$$\frac{c_{n+1}((n+1)m_1 + \sigma_1 - 1, \ \dots, (n+1)m_{n+1} + \sigma_{n+1} - (n+1); q)}{c_{n+1}(\sigma_1 - 1, \ \dots, \sigma_{n+1} - (n+1); q)}$$
$$= (-1)^{(n+1)m_{n+1}} q^{\left(\binom{(n+1)m_1+\sigma_1-1}{2} - \binom{\sigma_1-1}{2}\right) + \cdots + \left(\binom{(n+1)m_n+\sigma_n-n}{2} - \binom{\sigma_n-n}{2}\right)}$$
$$\times \prod_{1 \le i < j \le n} \frac{(q^{(n+1)(m_j-m_i)+(\sigma_j-\sigma_i)}, q^{(n+1)-(n+1)(m_j-m_i)-(\sigma_j-\sigma_i)}; q^{n+1})_\infty}{(q^{\sigma_j-\sigma_i}, q^{n+1-(\sigma_j-\sigma_i)}; q^{n+1})_\infty}$$

By lemma 6.1, with q^{n+1} for q, $x_i = q^{\sigma_j - \sigma_i}$ and $k = m_j - m_i$, this is equal to

$$(6.4) \qquad (-1)^{(n+1)m_{n+1}} q^{\left(\binom{(n+1)m_1+(\sigma_1-1)}{2} - \binom{\sigma_1-1}{2}\right) + \cdots + \left(\binom{(n+1)m_n+(\sigma_n-n)}{2} - \binom{\sigma_n-n}{2}\right)}$$
$$\times \prod_{1 \le i < j \le n} \frac{(-1)^{m_j-m_i}}{q^{(\sigma_j-\sigma_i)(m_j-m_i)+(n+1)\binom{m_j-m_i}{2}}}.$$

The power of -1 in this expression is, modulo 2,

$$-(n+1)m_{n+1} + \sum_{1 \le i < j \le n} (m_j - m_i)$$
$$= -(n+1)m_{n+1} + \sum_{k=1}^{n} (2k - n - 1)m_k$$
$$\equiv -(n+1) \sum_{k=1}^{n+1} m_k \equiv 0.$$

The power of q is

$$\sum_{k=1}^{n} \left[\binom{(n+1)m_k + \sigma_k - k}{2} - \binom{\sigma_k - k}{2} \right] - \sum_{1 \le i < j \le n} (\sigma_j - \sigma_i)(m_j - m_i)$$
$$- (n+1) \sum_{1 \le i < j \le n} \binom{m_j - m_i}{2},$$

which by Lemma 6.2 becomes simply

$$\frac{n+1}{2}(m_1^2 + \cdots + m_{n+1}^2) + m_1\sigma_1 + \cdots + m_{n+1}\sigma_{n+1}$$

and this proves the lemma.

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7. Step 4: the orbit of the constant term

In this section we will show that the $(n+1)!$ coefficients in the orbit of the constant term are in fact all equal to the constant term, up to a multiplicative factor of ± 1.

LEMMA 7.1. *Let* $\sigma = (\sigma_1, \dots, \sigma_{n+1})$ *be a permutation of* $(1, 2, \dots, n+1)$. *Then*

$$c_{n+1}(\sigma_1 - 1, \sigma_2 - 2, \dots, \sigma_{n+1} - (n+1)\,;q) = (\operatorname{sgn}\sigma) c_{n+1}(0, 0, \dots, 0; q).$$

PROOF. We will show that for $1 \le i \le n$,

$$\begin{aligned}(7.2)\qquad & c_{n+1}(\sigma_1 - 1, \dots, \sigma_i - i, \sigma_{i+1} - (i+1), \dots, \sigma_{n+1} - (n+1)\,; q) \\ &\quad = -c_{n+1}(\sigma_1 - 1, \dots, \sigma_{i+1} - i, \sigma_i - (i+1), \dots, \sigma_{n+1} - (n+1)\,; q).\end{aligned}$$

Since any coefficient $c_{n+1}(\sigma_1 - 1, \dots, \sigma_{n+1} - (n+1); q)$ can be obtained from $c_{n+1}(0, \dots, 0; q)$ by repeated applications of (7.2), (that is, the symmetric group is generated by transpositions), this is enough to prove the lemma.

CASE 1. By Lemma 4.3, we have for $1 \le i \le n-1$,

$$\begin{aligned}&\frac{c_{n+1}(\sigma_1 - 1, \dots, \sigma_i - i, \sigma_{i+1} - (i+1), \dots, \sigma_{n+1} - (n+1)\,; q)}{c_{n+1}(\sigma_1 - 1, \dots, \sigma_{i+1} - i, \sigma_i - (i+1), \dots, \sigma_{n+1} - (n+1)\,; q)} \\ &\quad = q^{\binom{\sigma_i - i}{2} + \binom{\sigma_{i+1} - (i+1)}{2} - \binom{\sigma_{i+1} - i}{2} - \binom{\sigma_i - (i+1)}{2}} \frac{(q^{\sigma_{i+1} - \sigma_i}, q^{n+1+\sigma_i - \sigma_{i+1}}; q^{n+1})_\infty}{(q^{\sigma_i - \sigma_{i+1}}, q^{n+1+\sigma_{i+1} - \sigma_i}; q^{n+1})_\infty} \\ &\quad = q^{\binom{\sigma_i - i}{2} + \binom{\sigma_{i+1} - (i+1)}{2} - \binom{\sigma_{i+1} - i}{2} - \binom{\sigma_i - (i+1)}{2}} \frac{(1 - q^{\sigma_{i+1} - \sigma_i})}{(1 - q^{\sigma_i - \sigma_{i+1}})} \\ &\quad = -1\end{aligned}$$

after simplification.

CASE 2.

$$\begin{aligned}&\frac{c_{n+1}(\sigma_1 - 1, \dots, \sigma_n - n, \sigma_{n+1} - (n+1); q)}{c_{n+1}(\sigma_1 - 1, \dots, \sigma_{n+1} - n, \sigma_n - (n+1); q)} \\ &\quad = \frac{(-1)^{\sigma_{n+1} - (n+1)}}{(-1)^{\sigma_n - (n+1)}} q^{\binom{\sigma_n - n}{2} - \binom{\sigma_{n+1} - n}{2}} \prod_{i=1}^{n-1} \frac{(q^{\sigma_n - \sigma_i}, q^{n+1+\sigma_i - \sigma_n}; q^{n+1})_\infty}{(q^{\sigma_{n+1} - \sigma_i}, q^{n+1+\sigma_i - \sigma_{n+1}}; q^{n+1})_\infty} \\ &\quad = (-1)^{\sigma_{n+1} - \sigma_n} q^{\binom{\sigma_n - n}{2} - \binom{\sigma_{n+1} - n}{2}} \frac{(q^{\sigma_{n+1} - \sigma_n}, q^{n+1+\sigma_n - \sigma_{n+1}}; q^{n+1})_\infty}{(q^{\sigma_n - \sigma_{n+1}}, q^{n+1+\sigma_{n+1} - \sigma_n}; q^{n+1})_\infty} \\ &\qquad \times \frac{\prod_{i \ne n} (q^{\sigma_n - \sigma_i}, q^{n+1+\sigma_i - \sigma_n}; q^{n+1})_\infty}{\prod_{i \ne n+1} (q^{\sigma_{n+1} - \sigma_i}, q^{n+1+\sigma_i - \sigma_{n+1}}; q^{n+1})_\infty}\end{aligned}$$

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$$
\begin{aligned}
&= (-1)^{\sigma_{n+1}-\sigma_n} q^{\binom{\sigma_n - n}{2} - \binom{\sigma_{n+1}-n}{2}} \frac{(1-q^{\sigma_{n+1}-\sigma_n})}{(1-q^{\sigma_n - \sigma_{n+1}})} \\
&\quad \times \frac{\prod_{1\le k\le n+1, k\ne\sigma_n} (q^{\sigma_n - k}, q^{n+1+k-\sigma_n}; q^{n+1})_\infty}{\prod_{1\le k\le n+1, k\ne\sigma_{n+1}} (q^{\sigma_{n+1} - k}, q^{n+1+k-\sigma_{n+1}}; q^{n+1})_\infty} \\
&= -q^{\binom{\sigma_n - n}{2} - \binom{\sigma_{n+1}-n}{2} + \sigma_{n+1} - \sigma_n} \\
&\quad \times \frac{\prod_{1\le k<\sigma_n} (q^{\sigma_n - k}, q^{n+1+k-\sigma_n}; q^{n+1})_\infty}{\prod_{1\le k<\sigma_{n+1}} (q^{\sigma_{n+1} - k}, q^{n+1+k-\sigma_{n+1}}; q^{n+1})_\infty} \\
&\quad \times \frac{q^{-\binom{n+2-\sigma_n}{2}} \prod_{\sigma_n<k\le n+1} (q^{k-\sigma_n}, q^{n+1+\sigma_n - k}; q^{n+1})_\infty}{q^{-\binom{n+2-\sigma_{n+1}}{2}} \prod_{\sigma_{n+1}<k\le n+1} (q^{k-\sigma_{n+1}}, q^{n+1+\sigma_{n+1} - k}; q^{n+1})_\infty} \\
&= -q^{\binom{\sigma_n - n}{2} - \binom{\sigma_{n+1}-n}{2} + \sigma_{n+1} - \sigma_n - \binom{n+2-\sigma_n}{2} + \binom{n+2-\sigma_{n+1}}{2}} \\
&= -1
\end{aligned}
$$

after simplification, since the product in both the numerator and denominator is

$$
\prod_{k=1}^{n} (q^k, q^{n+1-k}; q^{n+1})_\infty = \prod_{k=1}^{n} (q^k; q^{n+1})_\infty^2 = \frac{(q;q)_\infty^2}{(q^{n+1}; q^{n+1})_\infty^2}.
$$

This completes the proof of the lemma.

8. Summary

It is now just a matter of putting the results of the previous sections together to complete the induction. By Lemmas 5.2, 6.3 and 7.1, in that order, followed by Corollary 4.6 and Lemma 4.1, we obtain

$$
\begin{aligned}
F_{n+1}(x_1, \dots, x_{n+1}; q) &= \sum_{\alpha_1 + \cdots + \alpha_{n+1} = 0} c_{n+1}(\alpha_1, \dots, \alpha_{n+1}; q) x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}} \\
&= \sum_m^{(n+1)} \sum_{\sigma \in S_{n+1}} c_{n+1}((n+1)m_1 + \sigma_1 - 1, \dots, (n+1)m_{n+1} + \sigma_{n+1} - (n+1); q) \\
&\quad \times x_1^{(n+1)m_1+\sigma_1-1} \cdots x_{n+1}^{(n+1)m_{n+1}+\sigma_{n+1}-(n+1)} \\
&= \sum_m^{(n+1)} q^{\frac{n+1}{2}(m_1^2 + \cdots + m_{n+1}^2)} x_1^{(n+1)m_1} \cdots x_{n+1}^{(n+1)m_{n+1}} \\
&\quad \times \sum_{\sigma \in S_{n+1}} c_{n+1}(\sigma_1 - 1, \dots, \sigma_{n+1} - (n+1); q) q^{m_1\sigma_1 + \cdots + m_{n+1}\sigma_{n+1}} \\
&\quad \times x_1^{\sigma_1 - 1} \cdots x_{n+1}^{\sigma_{n+1}-(n+1)} \\
&= c_{n+1}(0, \cdots, 0; q) \sum_m^{(n+1)} q^{\frac{n+1}{2}(m_1^2 + \cdots + m_{n+1}^2) + m_1 + 2m_2 + \cdots + (n+1)m_{n+1}}
\end{aligned}
$$

$$
\begin{aligned}
&\times x_1^{(n+1)m_1}\cdots x_{n+1}^{(n+1)m_{n+1}} \\
&\times \sum_{\sigma\in S_{n+1}} (\operatorname{sgn}\sigma)(x_1q^{m_1})^{\sigma_1-1}\cdots(x_{n+1}q^{m_{n+1}})^{\sigma_{n+1}-(n+1)} \\
= &\frac{1}{(q;q)^n}\sum_m^{(n+1)} q^{\frac{n+1}{2}(m_1^2+\cdots+m_{n+1}^2)+m_1+2m_2+\cdots+(n+1)m_{n+1}} \\
&\times x_1^{(n+1)m_1}\cdots x_{n+1}^{(n+1)m_{n+1}} \prod_{1\le i<j\le n+1}\left(1-\frac{x_iq^{m_i}}{x_jq^{m_j}}\right).
\end{aligned}
$$

This completes the inductive step and so the Macdonald identities for A_{n-1} have been shown to be true by induction on n.

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