# A FORMALISM FOR DIFFERENTIAL ROTATION

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## I. INTRODUCTION

One of the approaches for dealing with the differential rotation of the Sun is to separate the flow in the convection zone into an axisymmetric steady part (differential rotation plus meridional circulation) and a "turbulent" part comprising all smaller scales of motion. In the equations of motion and energy the effect of the small scales is then represented by a turbulent viscosity and conductivity. Theories of this type at present belong to either of two categories:

a) Theories stressing the role of an anisotropic turbulent viscosity. Due to the presence of gravity as a preferred direction the velocity distribution in the turbulent flow is different in the horizontal and vertical directions. This implies that the turbulent viscosity is also anisotropic. Biermann (1951) showed that as a result, the convection zone cannot rotate as a solid body. Theories based on this idea were developed by Kippenhahn (1963) and Köhler (1970).

b) Theories using a latitude dependence of the efficiency of turbulent transport of heat. In the deeper layers of the convection zone the influence of rotation on convection is strong because the convective turnover time is comparable to the rotation period. Since the angle between gravity and the rotation axis varies with latitude, this influence must also vary with latitude. This produces a temperature variation between the pole and the equator which drives a meridional circulation. Since the Coriolis force dominates over the viscous force in most of the convection zone, there is a strong differential rotation associated with this circulation (quasi geostrophic flow). This type of theory has been developed by Weiss (1965), Durney and Roxburgh (1971) and Belvedere and Paterno (1977).

In both types of theories the turbulent transport coefficients are assumed to be given. In case a) the degree of anisotropy of the viscosity is a fixed parameter the value of which is not justified other than



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by agreement of the results with observations. In case b) the latitude dependence of the conductivity is similarly described by a single parameter.

c) A third mechanism for generating differential rotation is a variation on the theme of mechanism b). In the presence of rotation, the critical temperature gradient at which convection sets in is not equal to the adiabatic gradient. Instead the superadiabaticity at onset is:

$$\Delta\nabla T_{rc} \equiv G_c = 4\,\frac{(\vec{\Omega}\cdot\vec{k})^2}{k_\theta^2 + k_\phi^2}\,\frac{T}{g} \tag{1}$$

where $\vec{k}$ is the wavenumber of the particular convective mode studied, T the temperature, g the acceleration of gravity, $\vec{\Omega}$ the rotation vector (Cowling, 1951; Tayler, 1973). For $\vec{k}\cdot\vec{\Omega}=0$ the Schwarzschild criterium is recovered. In a fully turbulent convection zone however, the dominant modes will not have $\vec{k}\cdot\vec{\Omega}=0$. Therefore it seems reasonable to modify the usual mixing length expression for the heat flux $\vec{F}$ as follows:

$$F_r = \rho\, c_p\, \kappa\, (\Delta\nabla T_r - \langle G_c \rangle) \tag{2}$$

where $\langle G_c \rangle$ is an average of (1) over the modes assumed dominant, and $\kappa$ the turbulent thermal diffusivity, the latitude dependence of which is used for mechanism b).

We report here on a formalism designed to deal with mechanisms a), b), c) simultaneously. Instead of introducing the viscosity, conductivity and $\langle G_c \rangle$ as separate parameters, they are derived on a common basis from a simple model of turbulence in the presence of rotation. In the following we describe the steps taken to derive these quantities. A full description is given in Durney and Spruit (1979). Ideas similar to ours have been given earlier by Gough (1978) (see also Section 5).

## II. MODEL OF TURBULENCE

We assume that the turbulence can be represented as consisting of cells with a given size, shape, and lifetime $\tau$ (though these quantities can vary with position in the star). During the interval $\tau$ the flow is assumed to be steady. At the end of the lifetime the cells are destroyed and replaced by a new cell pattern shifted arbitrarily in position with respect to the old pattern. This way, the continous production of new cells and their decay into smaller scales of motion is approximated by a series of discrete events, like in the conventional mixing length formalism. The velocity field in the cell is

$$\vec{v} = v_0\, \vec{f}(\vec{r}) \quad . \tag{3}$$

For the velocity field $\vec{f}$ we use linear theory. The velocity field of a convective mode with wavenumber $\vec{k}$ in the presence of rotation was first given by Cowling (1951). If $L_r$, $L_\theta$, $L_\phi$ are the dimensions of

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the cell in r, $\theta$ and $\phi$ directions, a cellular velocity field is constructed from such modes by superposition of the eight modes with $|k_r| = \pi/L_r$, $|k_\theta| = \pi/L_\theta$, $|k_\phi| = \pi/L_\phi$ . The expresions for this velocity field are given in Durney and Spruit (1979).

The velocity amplitude $v_0$ is fixed by requiring that the kinetic energy in the cell be equal to the work done by the buoyancy force during the life time $\tau$ (as is done also in the mixing length formalism).

The life time of the cell is assumed to be comparable to its turnover time:

$$\tau = L_r / V_0 \ . \tag{4}$$

The dimension of the cell in radial direction is assumed to be proportional to the scale height:

$$L_r = \alpha H \ . \tag{5}$$

These assumptions together determine the flow in the cell. The free parameters left are the dimensions of the cells in $\theta$ and $\phi$ direction. Both can be functions of r and $\theta$ :

$$\begin{aligned} L_\theta / L_r &= f_\theta (r, \theta) \ , \\ L_\phi / L_r &= f_\phi (r, \theta) \ . \end{aligned} \tag{6}$$

The parameters $f_\theta$ and $f_\phi$ depend on the rotation rate. This dependence has to be guessed at, though there are some constraints on the possibilities. For example we must have $f_\theta / f_\phi \rightarrow 1$ as $\Omega \rightarrow 0$, since in the nonrotating case the two horizontal coordinates are equivalent. Also, at $\theta = 0$ and $\theta = \frac{\pi}{2}$ we have $f_\theta = f_\phi$ since at the poles the directions of $\vec{g}$ and $\vec{\Omega}$ coincide. Observations or numerical experiments may help to define the proper cell shape to be used. At high rotation rates, the influence of rotation on the critical temperature gradient is strong (Equation 1). The influence is minimized if $(\hat{k} \cdot \vec{\Omega})$ is small, i.e. if the cells are elongated along the axis. The amplitude of the flow in such cells would be small in the direction along the axis and high perpendicular to it. Hence the efficiency of such elongated cells in carrying heat would be poor near the poles since the flow would be mostly horizontal there, and good near the equator. Hence we may expect that the elongation of the cells would be most pronounced at the equator.

A simple form for $f_\theta$ and $f_\phi$ incorporating these ideas could be:

$$\begin{aligned} f_\phi &= 1 \ , \\ f_\theta &= 1 + (\tau \Omega)^2 \sin^2\theta \ . \end{aligned} \tag{7}$$

Such estimates, however, are too poor to rely on, so in the following we will treat $f_\phi$ and $f_\theta$ as free parameters of the formalism.

III. CALCULATION OF VISCOSITY AND CONDUCTIVITY

The critical superadiabaticity $\langle G_c \rangle$ in Equations 1, and 2 turns out to be, for the velocity field defined above:

$$\langle G_c \rangle = \frac{4\Omega^2 T}{g} \, \frac{f_\theta^2 \cos^2\theta + \sin^2\theta}{1 + f_\theta^2/f_\phi^2} \, , \tag{8}$$

which shows the dependence on the free parameters $f_\theta$ and $f_\phi$. With the approximation that the life time $\tau$ of the cell is 'small enough' (see also Section 6) the thermal diffusivity $\kappa$ in (3) can be calculated easily from the velocity field in the cell. It is:

$$\kappa^{ij} = \tau \langle v^i v^j \rangle \, , \tag{9}$$

where the bracket denotes the average over the cell volume. It is seen that $\kappa$ is a tensor, i.e. the heat flow in general makes an angle with respect to the direction of the temperature gradient. We write

$$\langle v^i v^j \rangle = \langle v_r^2 \rangle A^{ij} \, , \tag{10}$$

where $v_r$ is the radial velocity component. The tensor A is dimensionless, its components turn out to be (the tensor is of course symmetric):

$$\begin{aligned} A^{rr} &= 1 \\ A^{\theta\theta} &= [\omega^2 q^2 (f_\theta^2 \cos^2\theta + \sin^2\theta) + f_\theta^2]/(1+q^2)^2 \\ A^{\phi\phi} &= [\omega^2 (f_\theta^2 \cos^2\theta + \sin^2\theta) + q^2 f_\theta^2]/(1+q^2)^2 \\ A^{r\theta} &= 0 \\ A^{r\phi} &= -\omega \sin\theta/(1+q^2) \\ A^{\theta\phi} &= \omega \cos\theta \, (q^2-1)/(1+q^2)^2 \, , \end{aligned} \tag{11}$$

where $\omega = 2\tau\Omega$; $q = f_\theta/f_\phi$. It is seen that in the limit $\tau \to 0$ (solar surface) or $\Omega \to 0$ the tensor assumes a diagonal form with $A^{rr} = 1$; $A^{\theta\theta} = A^{\phi\phi} = 1/4$ (using the variation of $f_\theta$ and $f_\phi$ as in (7). For the viscosity, we adopt the formalism given by Elsässer (1966). This formalism also assumes that $\tau$ is 'small'. The kinematic viscosity is then:

$$\nu^{ij} = \tau \langle v^i v^j \rangle_f \tag{12}$$

where $\langle\rangle_f$ stands for the average over the velocity distribution of the turbulent flow. It is easy to show, for the turbulence model adopted, that this average is the same as the average over the cell volume (ergodicity). Comparison with (9) then shows that the kinematic viscosity is equal to the thermal diffusivity, i.e. the Prandtl number is unity. We note that this conclusion depends only on the validity of Equations (12) and (9) and not on the form of the turbulent velocity field assumed.

## IV. TURBULENT STRESS TENSOR AND VISCOUS FORCE

Elsässer (1966) gives the following expression for the viscous stress tensor in terms of the viscosity and the mean flow:

$$T^{ik} = \rho t^{ik} - \tau\rho\left(t^{ij}u^{k}_{;j} + t^{kj}u^{i}_{;j} + u^{j}t^{ik}_{;j}\right) \tag{13}$$

Here $t^{ik} = \langle v^i v^k \rangle = \nu^{ik}/\tau$ and $;$ denotes the covariant derivative in $r, \theta$ and $\phi$ coordinates, and $\vec{U}$ is the mean flow with respect to the coordinate frame adopted. The viscous force is the divergence of T:

$$R^{i} = T^{ij}_{;j} \tag{14}$$

The first term in (13) is independent of the mean flow. It is called the turbulent pressure. It is important to note however (see also Gough, 1978) that because of its tensorial nature, it can in general not be balanced by a hydrostatic readjustment of the gas pressure. In this sense, it is not a 'real' pressure. An exception is the non-rotating case (see Section 5). There the turbulent pressure is still not isotropic, but the force it creates can indeed be balanced by a pressure gradient (i.e. is conservative).

## V. ROTATING AND INERTIAL FRAMES, COMPARISON WITH OTHER WORK

In the derivation of the viscosity above we have implicitly assumed a rotating frame of coordinates, rotating with the average solar angular velocity. For example, in (12) the turbulent velocities are measured with respect to this frame. We wish to point out here that the appearance of Equation 13 for the turbulent stress depends on the coordinate frame adopted. In particular, the 'turbulent pressure' part is different in different frames. To see this, write (13) as

$$T^{ik} = P^{ik} - V^{ik}(\vec{u}), \tag{15}$$

where $P^{ik}$ is the turbulent pressure (defined as the part that is independent of $\vec{U}$), and $V^{ik}$ the 'viscous' part. Suppose P and V are known in an inertial frame. Then the result can be transformed to a rotating frame by writing

$$\vec{u} = \vec{u}_d + \vec{u}_\Omega , \tag{16}$$

where $\vec{u}_\Omega = \vec{\Omega}\, r \sin\theta$ ,

and $\vec{U}_d$ is the mean flow with respect to the rotating frame. Then, since V is linear in $\vec{U}$;

$$T^{ik} = P^{ik} - V^{ik}(\vec{u}_\Omega) - V^{ik}(\vec{u}_d) . \tag{17}$$

Thus the turbulent pressure in rotating coordinates is $P - V(\vec{U}_\Omega)$ instead of P, and it can have a quite different behavior. The form of the turbulent pressure depends on the coordinate frame used. As an example, consider the formalism used by Kippenhahn (1963) and Köhler (1970). In this formalism one works entirely in an inertial frame. The rotation is treated as part of the mean flow $\vec{U}$, so that the zero order state (U=0) is nonrotating. In this formalism, the tensor t has the form

$$t^{rr} = t_o(r) \,, \qquad t^{\theta\theta} = t^{\phi\phi} = s \cdot t_o(r) \,, \tag{18}$$

(the other components are zero). Here s could in principle be a function of r as well, though this is not usually done. The turbulent pressure in inertial coordinates, $P^{ij} = t^{ij}$ is neglected because its divergence is a conservative force that can be compensated by a small hydrostatic adjustment of the gas pressure. If we transform to rotating coordinates according to (17), the components of the turbulent pressure $P_\Omega$ is this frame are the same as those of P, except for the $r\phi$ component:

$$P_\Omega^{r\phi} = -\omega \sin\theta \, (1-s) \rho \, t_o \quad . \tag{19}$$

As a result of this additional off-diagonal component, the force P is not conservative and hence uniform rotation is not possible. Thus, the turbulent pressure measured in the rotating frame drives a differential rotation. This is a rewording of the familiar conclusion by Biermann (1951). If we take into account the influence of rotation on convection according to the formalism described in this paper (working in the rotating frame), the form of $P_\Omega$ turns out to be qualitatively the same as in the Biermann-Kippenhahn-Köhler formalism. Hence, the same driving mechanism for differential rotation is present. In addition however, there are other effects like the latitude dependent heat transport, and a different (much more complicated) form of the viscous part $V^{ij}$ (see 16) of the stress tensor.

Gough (1978) has analyzed the influence of rotation on the turbulent Reynolds' stress and conductivity, working in a rotating frame, along the same line. He limits himself to the limit of slow rotation (only terms of order $\omega$ are retained). He gives an expression for the turbulent pressure which is equivalent to ours (11) in this limit. He does not give the viscous part of the stress tensor. For heat transport, he introduces the additional assumption that the conductivity is still a scalar in the presence of rotation. As is seen from Equations 10 and 11 however, the conductivity has off-diagonal components of order $\omega$ , which cannot be neglected even in the slow rotation limit.

## VI. LIMITS OF VALIDITY

In the derivations the assumption has been made that the life time $\tau$ of the cell is small compared with its turnover time, though in practice one takes the two comparable. This assumption is also commonly made in the derivation of the $\alpha$-effect (e.g. Moffat, 1978).

Secondly, expressions (12) for the viscosity and (13) for the stress are valid only if the variation of the mean flow across the cell is small enough:

$$\tau \, |\vec{v} \cdot \nabla \vec{u}| << |\vec{v}| \,, \tag{20}$$

where v is the turbulent flow and U the mean flow. In our case, U is the differential rotation flow. If $\Delta\Omega$ is a typical differential rotation rate, (20) means roughly:

$$\tau \Delta\Omega << 1 . \qquad (21)$$

In the Sun, $\tau\Delta\Omega \simeq 0.6$ in the middle of the convection zone and $\tau\Delta\Omega \simeq 2$ near the bottom. Thus (21) is not violated too badly. This is an improvement over formalisms in which the mean flow includes the entire rotation (as in the Kippenhahn-Köhler formalism). For these the corresponding inequality is $\tau\Omega << 1$ which is about a factor of 5 more stringent.

Finally we note that as long as (21) is satisfied, our formalism is in principle valid for arbitrary rotation rates, since the velocity field used is general.

REFERENCES

Biermann, L., 1951, Z.f. Ap., 28, 304.
Belvedere, G., and Paterno, L., 1977, Solar Phys., 47, 525.
Cowling, T.G., 1951, Astrophys. J., 114, 272.
Durney, B.R., and Roxburgh, I.W., 1971, Solar Phys. 16, 3.
Durney, B.R., and Spruit, H.C., 1979, Astrophys. J., Dec. 1979.
Elsässer, K., 1966, Z.f. Ap., 63, 65.
Gough, D., 1978, in G. Belvedere and L. Paterno, eds., Proc. of the Catania Workshop on Solar Rotation, University of Catania.
Köhler, H., 1970, Solar Phys., 13, 3.
Moffat, H.K., 1978, Magnetic Field Generation in Electrically Conducting Fluids, University Press, Cambridge.
Tayler, R., 1973, Mon. Not. R. Astron. Soc., 165, 39.
Weiss, N.O., 1965, Observatory, 85, 37.