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L^p $(1 \leq p \leq \infty)$ ESTIMATES FOR $\overline{\partial}$ ON A CERTAIN PSEUDOCONVEX DOMAIN IN $\mathbb{C}^n$

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Abstract. Let $\Psi \in C^2[0,1]$ be a positive real valued function on $(0,1]$. Under certain assumptions on Ψ, the set $D = \{z \in \mathbb{C}^n; \sum_{j=1}^{n-1} |z_j|^2 + \Psi(|z_n|^2) < 1\}$ is a pseudoconvex domain with C^2-boundary which may be infinite type. If Ψ has flatness at 0 so that $\int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds < \infty$, then we can obtain $L^p(1 \leq p \leq \infty)$ estimates for $\overline{\partial} u = f$ on D.

§1. Introduction and statement of the result

In this paper we study $L^p(1 \leq p \leq \infty)$ estimates for $\overline{\partial}$ on a certain pseudoconvex domain in $\mathbb{C}^n$ with some boundary condition. The estimation is deduced from the explicit formula of the solution for $\overline{\partial}$ represented by integral kernels.

$L^p(1 \leq p \leq \infty)$ estimates for $\overline{\partial}$ on strongly pseudoconvex domains were obtained by Kerzman [5] in case f is a (0,1)-form, and in the general case by Øvrelid [6]. With the finite type condition, L^p estimates were obtained by Chang-Nagel-Stein [2] for pseudoconvex domains of finite type in $\mathbb{C}^2$. In dimension $n \geq 3$, Bruna-Castillo [1] got L^p estimates for $\overline{\partial}$ on some convex domains of ellipsoid type. However, there is a counter-example which show that $L^p(2 < p \leq \infty)$ estimates for $\overline{\partial}$ are not true on general pseudoconvex domains [3]. In the geometric convex case, Polking [7] obtained $L^p(1 < p < \infty)$ estimates for $\overline{\partial}$ on convex domains in $\mathbb{C}^2$ without the finite type condition. Polking's results and ours show that it will be possible to get L^p estimates for $\overline{\partial}$ with some weaker geometric conditions than the finite type condition.

Let $\Psi \in C^2[0,1]$ be a real valued function satisfying

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(A) $\Psi(0) = 0$ and $\Psi(1) = 1$;

(B) $\Psi'(t) > 0, \quad 0 < t \leq 1$;

(C) $\Psi'(t) + t\Psi''(t) > 0, \quad 0 < t \leq 1$;

(D) there exists $\tau \in (0,1)$ such that $\Psi''(t) > 0, \quad 0 < t < \tau$.

Define

$$D = \{z \in \mathbb{C}^n; |z_j| < 1, j = 1, \cdots, n, \sum_{j=1}^{n-1} |z_j|^2 + \Psi(|z_n|^2) < 1\},$$

so that D is a bounded Reinhardt domain. Without loss of generality, we assume that Ψ is defined and satisfies (B) and (C) in an interval of the type $[0, 1+\epsilon), \epsilon > 0$. Therefore, conditions (B) and (C) imply that D is a weakly pseudoconvex domain with a C^2-boundary defining function $\rho(z) = \sum_{j=1}^{n-1} |z_j|^2 + \Psi(|z_n|^2) - 1$. Define $\rho_n(z_n) = \Psi(|z_n|^2)$. Since the complex Hessian of ρ_n at z_n is $\Psi'(|z_n|^2) + |z_n|^2\Psi''(|z_n|^2)$ this shows that ρ_n is strictly subharmonic if $z_n \neq 0$. To exclude the strongly pseudoconvex case we assume of course $\Psi'(0) = 0$. We showed that $L^p (1 \leq p \leq \infty)$ estimates for $\bar\partial$ depend on flatness of Ψ at 0. Now we state our main result.

MAIN THEOREM. *Let $D = \{z \in \mathbb{C}^n; \sum_{j=1}^{n-1} |z_j|^2 + \Psi(|z_n|^2) < 1\}$, and let $f \in L^p_{0,q}(D), 1 \leq p \leq \infty$, be $\bar\partial$-closed. If $\int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds < \infty$, then there is a solution u of $\bar\partial u = f$ on D such that for each p with $1 \leq p \leq \infty$,*

$$\|u\|_{L^p(D)} \leq c(p) \|f\|_{L^p(D)}$$

where the constant $c(p)$ is independent of f.

EXAMPLES. (1) For m a positive integer, put $\Psi_m(t) = t^m, 0 \leq t \leq 1$. Then Ψ_m satisfies conditions (A)–(D) and the condition in Main theorem. In this case the domain defined by Ψ_m will be a complex ellipsoid.

(2) For $\alpha > 0$, write $\Psi_\alpha(t) = e \exp(-1/t^\alpha), 0 \leq t \leq 1$. Then Ψ_α satisfies all conditions (A)–(D) and it satisfies the condition in Main theorem if and only if $\alpha < \frac{1}{2}$.

Remark. In the case of $\Omega = \{z \in \mathbb{C}^2; |z_1|^2 + \Psi(|z_2|^2) < 1\}$, Verdera [9] obtained the L^∞ estimate for $\bar\partial$ on Ω. By the same method, we can obtain the L^∞ estimate for $\bar\partial$ on D. However, for L^p estimates we must extend integral kernels to the interior of D.

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§2. Construction of the solution operator for $\bar{\partial}$

From the condition (D) we have the following inequality.

LEMMA 2.1. ([9]) *There exists a constant $\eta = \eta(\Psi) > 0$ such that for $N = \frac{1}{16}$, the following inequality holds:*

$$\Psi(|\zeta + v|^2) - \Psi(|\zeta|^2) - 2\mathrm{Re}\left[\frac{\partial \Psi}{\partial \zeta}(|\zeta|^2)v\right]$$
$$\geq \Psi(N|v|^2), \zeta, v \in \mathbb{C}, |\zeta| < \eta, |v| < \eta.$$

Let $\widetilde{D}$ be an open neighborhood of $\overline{D}$ and suppose ρ to be defined in $\widetilde{D}$. Put

$$F_0(\zeta, z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j), \quad (\zeta, z) \in \widetilde{D} \times \mathbb{C}^n.$$

LEMMA 2.2. *There exists a positive constant η, depending only on Ψ, such that for $M = \frac{1}{32}$,*

$$\mathrm{Re}\ F_0(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{j=1}^{n-1} |\zeta_j - z_j|^2 + \Psi(M|\zeta - z|^2),$$

where $(\zeta, z) \in \widetilde{D} \times \mathbb{C}^n, |\zeta_n| < \eta, |\zeta - z| < \eta$.

Proof. From Lemma 2.1 it follows that

$$\begin{aligned}
\rho(z) &= \rho(\zeta) - 2\mathrm{Re}\ F_0(\zeta, z) + (\rho(z) - \rho(\zeta) + 2\mathrm{Re}\ F_0(\zeta, z)) \\
&= \rho(\zeta) - 2\mathrm{Re}\ F_0(\zeta, z) + \sum_{j=1}^{n-1} |z_j|^2 - \sum_{j=1}^{n-1} |\zeta_j|^2 + \Psi(|z_n|^2) - \Psi(|\zeta_n|^2) \\
&\quad + 2\mathrm{Re} \sum_{j=1}^{n-1} \bar{\zeta}_j(\zeta_j - z_j) + 2\mathrm{Re}\left[\frac{\partial \Psi}{\partial \zeta_n}(|\zeta_n|^2)(\zeta_n - z_n)\right] \\
&\geq \rho(\zeta) - 2\mathrm{Re}\ F_0(\zeta, z) + \sum_{j=1}^{n-1} |\zeta_j - z_j|^2 + \Psi(N|\zeta_n - z_n|^2),
\end{aligned}$$

where $(\zeta, z) \in \widetilde{D} \times \mathbb{C}^n, |\zeta_n| < \eta, |\zeta - z| < \eta$. From conditions (A) and (D), we have for small $|x|, |y|$,

$$\Psi\left(\frac{x^2+y^2}{2}\right) \leq \frac{1}{2}(\Psi(x^2) + \Psi(y^2)) \lesssim x^2 + \Psi(y^2).$$

Thus we get

$$\sum_{j=1}^{n-1} |\zeta_j - z_j|^2 + \Psi(N|\zeta_n - z_n|^2) \gtrsim \Psi\left(\frac{N}{2}|\zeta - z|^2\right).$$

Thus we have the required inequality. $\square$

For $\epsilon, \delta > 0$, we define

$$D_\delta = \{z \in \widetilde{D}; \rho(z) < \delta\}, \quad V_\delta = \{z \in \widetilde{D}; |\rho(z)| < \delta\},$$

$$U_{\epsilon,\delta} = \{(\zeta, z) \in V_\delta \times D_\delta; |\zeta - z| < \epsilon\}, \quad Z = \{z; z_n = 0\}.$$

If we use Lemma 2.2, by the same method as in [9], we can obtain the following results.

LEMMA 2.3. *There exist $\epsilon, \delta, c, M > 0$, depending on Ψ, and C^1-functions $\Phi : V_\delta \times D_\delta \to \mathbb{C}, F, G : U_{\epsilon,\delta} \to \mathbb{C}$, holomorphic in $z \in D_\delta$ for each fixed $\zeta \in V_\delta$, such that*

(a) $\Phi = FG$ *in* $U_{\epsilon,\delta}$, $F(\zeta, \zeta) = 0$, $|G| > c$ *in* $U_{\epsilon,\delta}$, $|\Phi| > c$ *in* $(V_\delta \times D_\delta) \backslash U_{\epsilon,\delta}$;

(b) $F = F_0$ *in a neighborhood of* $bD \cap Z$, *and the following inequality holds*

$$\operatorname{Re} F(\zeta, z) \gtrsim \rho(\zeta) - \rho(z) + \sum_{j=1}^{n-1} |\zeta_j - z_j|^2 + \Psi(M|\zeta - z|) \text{ for } (\zeta, z) \in U_{\epsilon,\delta};$$

(c) $d_\zeta F(\zeta, z)|_{\zeta = z} = \partial\rho(z)$;

(d) $\Phi(\zeta, z) = \sum_{j=1}^n (\zeta_j - z_j) P_j(\zeta, z)$, *where* P_j *is continuously differentiable in* $V_\delta \times D_\delta$, *holomorphic in* $z \in D_\delta$ *for each fixed* $\zeta \in V_\delta, j = 1, \ldots, n$.

DEFINITION 2.4. A generating form $W(\zeta, z) = \sum_{j=1}^n w_j(\zeta, z) d\zeta_j$ with coefficients in $C^{1,\infty}(bD \times D)$ is a (1,0) form in ζ such that

(i) $< W(\zeta,z), \zeta - z >= \sum_{j=1}^n w_j(\zeta,z)(\zeta_j - z_j) = 1$ for $(\zeta,z) \in bD \times D$

(ii) $w_j(\zeta,z) \in C^{1,\infty}(bD \times D)$, $1 \le j \le n$.

DEFINITION 2.5. Let $W(\zeta,z) = \sum_{j=1}^n w_j(\zeta,z)d\zeta_j$ be a generating form with coefficients in $C^{1,\infty}(bD \times D)$ and let $\widehat{W}(\zeta,z,\lambda) = \lambda W(\zeta,z) + (1-\lambda)B(\zeta,z)$ where $\lambda \in I = [0,1]$ and $B = \frac{\partial_\zeta \beta}{\beta}, \beta = |\zeta - z|^2$. For $-1 \le q \le n$, the Cauchy-Fantappiè kernel $\Omega_q(\widehat{W})$ of order q generated by $\widehat{W}$ is defined by

$$\Omega_q(\widehat{W}) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \begin{pmatrix} n-1 \\ q \end{pmatrix} \widehat{W} \wedge (\bar\partial_{\zeta,\lambda}\widehat{W})^{n-q-1} \wedge (\bar\partial_z \widehat{W})^q$$

for $0 \le q \le n-1$, and 0 otherwise. $\Omega_q(W)$ is defined in the same way, with W instead of $\widehat{W}$.

By Lemma 2.3, $L_D = \frac{P}{\Phi}$ with $P = \sum_{j=1}^n P_j d\zeta_j$ is a generating form which is holomorphic in $z \in D$. Thus it follows that (see [8, Theorem 3.6 in IV])

$$f = \bar\partial_z(T_q f) \quad (q \ge 1),$$

where

$$T_q f = \int_{bD \times I} f \wedge \Omega_{q-1}(\widehat{L}_D) - \int_D f \wedge \Omega_{q-1}(B).$$

§3. Extension of integral kernels to the interior of D

For convenience, we briefly recall the extension of integral kernels to the interior of D. We will use the same notations as in [8]. We only need to consider the case $0 \le q-1 \le n-2$. In the following, we replace $q-1$ by q for convenience. Since $P(\zeta,z)$ is holomorphic in z, for $0 \le q \le n-2$ and any $f \in C_{0,q+1}(bD)$ one has

$$\int_{bD \times I} f \wedge \Omega_q(\widehat{L}_D) = \int_{bD} f \wedge A_q(L_D, B),$$

where the double form $A_q(L_D,B)$ is given by

$$A_q(L_D,B) = \sum_{j=0}^{n-q-2} a_q^j A_q^j(L_D,B), \tag{3.1}$$

with numerical constants a_q^j and

$$(3.2)\quad A_q^j(L_D, B) = \frac{P \wedge \partial_\zeta \beta \wedge (\overline{\partial}_\zeta P)^j \wedge (\overline{\partial}_\zeta \partial_\zeta \beta)^{n-q-2-j} \wedge (\overline{\partial}_z \partial_\zeta \beta)^q}{\Phi^{j+1} \beta^{n-(j+1)}}$$
$$= \frac{A_q^j(P, \partial_\zeta \beta)}{\Phi^{j+1} \beta^{n-(j+1)}}.$$

Before applying Stokes' theorem we must extend the generating forms L_D and B from $bD \times D$ to $\overline{D} \times D$ without singularities. Let $\widehat{F}(\zeta, z) = F(\zeta, z) - 2\rho(\zeta)$. Then for $\zeta \in V_\delta$,

$$(3.3)\quad \operatorname{Re} \widehat{F}(\zeta, z) \gtrsim \begin{cases} -\rho(\zeta) - \rho(z) + \sum_{j=1}^{n-1} |\zeta_j - z_j|^2 + \Psi(M|\zeta - z|^2), \\ \qquad\qquad\qquad\qquad (\zeta, z) \in U_{\epsilon,\delta} \\ -\rho(\zeta) + c, \quad (\zeta, z) \in (V_\delta \times D_\delta) \backslash U_{\epsilon,\delta}, \end{cases}$$

and so $\widehat{F}(\zeta, z)$ never vanishes for $(\zeta, z) \in V_\delta \times D_\delta$. Set $\widehat{\beta} = \beta + \rho(\zeta)\rho(z)$. Then

$$\widehat{B} = \frac{\partial \beta}{\widehat{\beta}} \in C^\infty(\overline{D} \times \overline{D} \backslash \Delta_{bD}), \quad \Delta_{bD} = \{(\zeta, \zeta); \zeta \in bD\},$$

and $\widehat{B} = B$ for $\zeta \in bD$. We fix $\epsilon_0 > 0$ such that $\{\zeta \in \widetilde{D}; |\rho(\zeta)| < 2\epsilon_0\} \Subset V_\delta$, and choose $\chi \in C^\infty(\overline{D})$ with $\chi(\zeta) \equiv 1$ for $\zeta \in V_\delta$ with $\rho(\zeta) \geq -\epsilon_0$ and $\chi(\zeta) \equiv 0$ for $\zeta \in D$ with $\rho(\zeta) \leq -2\epsilon_0$. Set $\widehat{\Phi}(\zeta, z) = \widehat{F}(\zeta, z) G(\zeta, z)$ and

$$\widehat{A}_q^j(L_D, B) = \chi(\zeta) \frac{A_q^j(P, \partial \beta)}{\widehat{\Phi}^{j+1} \widehat{\beta}^{n-(j+1)}}$$

and

$$\widehat{A}_q(L_D, B) = \sum_{j=0}^{n-q-2} a_q^j \widehat{A}_q^j(L_D, B).$$

It follows directly from the definition that

$$\widehat{A}_q(L_D, B) = A_q(L_D, B) \quad \text{on} \quad bD \times D.$$

Thus, by Stokes' theorem, we have the integral solution operator $\widehat{T}_q : L^1_{0,q}(D) \to L^1_{0,q-1}(D)$, $1 \leq q \leq n$, defined by

$$\widehat{T}_q f = (-1)^q \int_D f \wedge \overline{\partial}_\zeta \widehat{A}_{q-1}(L_D, B) - \int_D f \wedge \Omega_{q-1}(B).$$

§4. Proof of Main theorem

Let $M(\zeta, z)$ be any of the coefficients of the double form $\bar{\partial}_\zeta \widehat{A}_{q-1}(L_D, B)$ which make up the kernels of $\widehat{T}_q$. From (3.1) and (3.2) we see that $\widehat{A}_{q-1}(L_D, B)$ is a linear combination of terms $\frac{N(\zeta,z)}{\widehat{\Phi}^l \widehat{\beta}^{n-l}}$, $1 \le l \le n-q-1$, where $N \in C^{1,\infty}(\overline{D} \times \overline{D})$ and $N(\zeta, z) = \mathcal{O}(|\zeta - z|)$. Hence M is a linear combination of terms

$$\frac{\bar{\partial}_\zeta N}{\widehat{\Phi}^l \widehat{\beta}^{n-l}} - \frac{l(\bar{\partial}_\zeta \Phi)N}{\widehat{\Phi}^{l+1}\widehat{\beta}^{n-l}} - (n-l)\frac{[\bar{\partial}_\zeta \beta + (\bar{\partial}_\zeta \rho)\rho(z)]N}{\widehat{\Phi}^l \widehat{\beta}^{n-l+1}}, \quad \text{with} \quad 1 \le l \le n-q-1.$$

Notice that

$$\left|\frac{\bar{\partial}_\zeta N}{\widehat{\Phi}^l \widehat{\beta}^{n-l}}\right| \lesssim \frac{1}{|\widehat{\Phi}|^l |\zeta - z|^{2(n-l)}}; \tag{4.1}$$

$$\left|\frac{(\bar{\partial}_\zeta \Phi)N}{\widehat{\Phi}^{l+1}\widehat{\beta}^{n-l}}\right| \lesssim \frac{1}{|\widehat{\Phi}|^{l+1} |\zeta - z|^{2(n-l)-1}}; \tag{4.2}$$

$$\left|\frac{(\bar{\partial}_\zeta \beta)N}{\widehat{\Phi}^l \widehat{\beta}^{n-l+1}}\right| \lesssim \frac{1}{|\widehat{\Phi}|^l |\zeta - z|^{2(n-l)}}; \tag{4.3}$$

$$\left|\frac{[(\bar{\partial}_\zeta \rho)\rho(z)]N}{\widehat{\Phi}^l \widehat{\beta}^{n-l+1}}\right| \lesssim \frac{|\rho(z)|}{|\widehat{\Phi}|^l |\zeta - z|^{2(n-l)+1}} \lesssim \frac{1}{|\widehat{\Phi}|^{l-1} |\zeta - z|^{2(n-l)+1}}. \tag{4.4}$$

Thus from (4.1)–(4.4) it follows that

$$|M(\zeta, z)| \lesssim \sum_{l=1}^{n-q-1} \left\{ \frac{1}{|\widehat{\Phi}|^{l-1}|\zeta - z|^{2(n-l)+1}} + \frac{1}{|\widehat{\Phi}|^{l}|\zeta - z|^{2(n-l)}} + \frac{1}{|\widehat{\Phi}|^{l+1}|\zeta - z|^{2(n-l)-1}} \right\} \quad \text{for all } \zeta, z \in D. \tag{4.5}$$

Lemma 4.6. ([4]) *There is a positive constant γ, depending on Ψ, such that for each z sufficiently close to bD, one can find in $B(z, \gamma)$ a smooth (of class C^1) change of coordinates $t^{(z)}(\zeta) = (t_1, \ldots, t_{2n})$ satisfying*

(a) $t_1 + it_2 = \rho(\zeta) + i\,\mathrm{Im}\, F(\zeta, z)$;

(b) $t_3(z) = \ldots = t_{2n}(z) = 0$;

(c) $|t^{(z)}(\zeta) - t^{(z)}(\zeta')| \approx |\zeta - \zeta'|, \quad \zeta, \zeta' \in B(z, \gamma)$.

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It is enough to consider the first term in the representation of $\widehat{T}_q f$. Thus the proof of Main theorem will be a consequence of standard results in analysis once we prove the following estimates.

PROPOSITION 4.7.

$$\int_D |M(\zeta,z)| dV(\zeta) \lesssim \int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds \quad \textit{for all} \quad z \in D$$

and

$$\int_D |M(\zeta,z)| dV(z) \lesssim \int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds \quad \textit{for all} \quad \zeta \in D.$$

Proof. We will prove the necessary estimates for the second term among righthand terms in (4.5). The corresponding estimates for the first and third terms can be handled by analogous methods. Let us first prove for $1 \le l \le n-q-1$,

$$\int_{D\cap V_\delta} \frac{1}{|\widehat{\Phi}|^l |\zeta - z|^{2(n-l)}} dV(\zeta) \lesssim \int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds \quad \text{for all} \quad z \in D.$$

Integrals are significant only for points near to bD. Therefore, consider $\delta, \gamma > 0$, and z close to bD, such that Lemmas 2.3 and 4.6 can be applied. We may assume that γ is the smaller of the constant ϵ in Lemma 2.3. On $V_\delta \setminus B(z,\gamma)$, the denominators of the righthand terms in (4.5) are bounded from below away from zero, uniformly in z, by (a) of Lemma 2.3. Hence we must prove that

$$\int_{D\cap B(z,\gamma)} \frac{1}{|\widehat{\Phi}|^l |\zeta - z|^{2(n-l)}} dV(\zeta) \lesssim \int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds \quad \text{for all} \quad z \in D. \tag{4.8}$$

We use the coordinates $t = (t_1, \dots, t_{2n})$ given by Lemma 4.6 on the neighborhood $B(z,\gamma)$, where $\gamma > 0$ is independent of z. Recall that $t_1 = \rho(\zeta)$ and $t_2 = \text{Im } F(\zeta,z) \approx \text{Im } \widehat{\Phi}(\zeta,z)$. Thus from (3.3) it follows that

$$|\widehat{\Phi}(\zeta,z)| \gtrsim |t_1| + |t_2| + |\rho(z)| + \sum_{j=1}^{2(n-1)} t_j^2 + \Psi(M|t|^2).$$

It is now clear that (4.8) will follow from

$$I_l = \int_{|t|\le c} \frac{dt_1 \cdots dt_{2n}}{|t|^{2(n-l)} [|t_1| + |t_2| + \sum_{j=3}^{2(n-1)} t_j^2 + \Psi(M|t|^2)]^l} < \infty$$

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for $l = 1, \ldots, n-q-1$.

If $l = 1$, by using the inequality $a^{1-\epsilon}b^\epsilon \le a + b$ for $a, b > 0, 0 < \epsilon < 1$, it follows that

$$\begin{aligned}
I_1 &= \int_{|t|\le c} \frac{dt_1 \cdots dt_{2n}}{|t|^{2(n-1)}[|t_1| + |t_2| + \sum_{j=3}^{2(n-1)} t_j^2 + \Psi(M|t|^2)]} \\
&\lesssim \int_{\substack{|t|<c \\ t=(t_1,t_2,t')}} \frac{dt_1 dt_2 dt'}{|t'|^{2(n-1)(1-\epsilon)}(|t_1| + |t_2|)^{1+2(n-1)\epsilon}} \\
&\lesssim 1.
\end{aligned}$$

Now, if $l > 1$, we obtain

$$\begin{aligned}
I_l &= \int_{|t|\le c} \frac{dt_1 \cdots dt_{2n}}{|t|^{2(n-l)}[|t_1| + |t_2| + \sum_{j=3}^{2(n-1)} t_j^2 + \Psi(M|t|^2)]^l} \\
&\lesssim \int_{\substack{|t'|<c \\ t'=(t_3,\ldots,t_{2n})}} \frac{dt'}{|t'|^{2(n-l)-1}} \int_0^c \frac{d\tau}{[\tau + \sum_{j=3}^{2(n-1)} t_j^2 + \Psi(M|t'|^2)]^l} \\
&\lesssim \int_{|t'|<c} \frac{dt'}{|t'|^{2(n-l)-1}[\sum_{j=3}^{2(n-1)} t_j^2 + \Psi(M|t'|^2)]^{l-1}} \\
&\lesssim \int_{\substack{|t''|<c \\ t''=(t_{2n-1},t_{2n})}} \frac{dt''}{|t''|} \int_0^c \frac{r^{2(n-3)} r\,dr}{r^{2(n-l-1)}[r^2 + \Psi(M|t''|^2)]^{l-1}} \\
&\lesssim \int_{|t''|<c} \frac{dt''}{|t''|} \int_0^c \frac{r\,dr}{r^2 + \Psi(M|t''|^2)} \\
&\lesssim \int_{|t''|<c} \frac{|\log \Psi(M|t''|^2)|}{|t''|} dt'' \\
&\lesssim \int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds.
\end{aligned}$$

The proof of the estimates

$$\int_D |M(\zeta, z)| dV(z) \lesssim \int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds \quad \text{for all} \quad \zeta \in D \cap V_\delta,$$

is similar. For $\zeta \in D$ fixed near bD there are coordinates $u = (u_1, \ldots, u_{2n})$ for $z \in B(\zeta, \tilde\gamma)$, where $\tilde\gamma$ is independent of ζ, such that $u_1(z) = \rho(z), u_2(z) = \operatorname{Im} F(\zeta, z)$, and $u(\zeta) = (\rho(\zeta), 0, \ldots, 0)$, which has properties analogous to those of the coordinates $t = (t_1, \ldots, t_{2n})$ on $B(z, \gamma)$. From the estimate in

(b) of Lemma 2.3 we then obtain

$$|\widehat{\Phi}(\zeta, z)| \gtrsim |u_1| + |u_2| + \sum_{j=3}^{2(n-1)} u_j^2 + \Psi(M|u|^2),$$

for $u = u(z)$ and $z \in D \cap B(\zeta, \widetilde{\gamma})$, so that one can proceed as in the proof of the estimate of I. □

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