ALGEBRAIC APPROXIMATION OF CURVES

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Introduction. In his paper on the algebraic approximation of differentiable manifolds Nash **(1)** introduced the concept of a sheet of a real algebraic variety (see the definition in §16 below) and raised certain questions of a general nature. In attempting to answer these questions it has been necessary to evolve some sort of technique for manipulating curves on algebraic varieties, and, in particular, to set up a criterion for the possibility of approximating a sequence of analytic arcs (definition in §1) joined end to end by a single analytic arc. The greater part of this paper is devoted to this latter topic, the results being applied in the last section to the problems suggested in Nash's paper.

The work falls naturally into three parts. The first deals with the approximation of plane curves by algebraic curves, the second with the corresponding problem in higher dimensional spaces and on varieties in general, while the third is concerned with the sheets of real algebraic varieties. The separate preliminary treatment of the case of plane curves is natural in the sense that plane algebraic curves present a specially simple situation, being represented each by a single equation.

The following is a brief sketch of the paper. Part I deals with plane curves consisting of analytic arcs placed end to end, the object being to approximate these by parts of algebraic curves, smoothing off the joins of the arcs in some way but preserving in some way the other singularities of the original curve. The corresponding question in Euclidean n-space is then taken up in Part II. Finally, for a curve on a real algebraic variety the technique is to project into a suitable linear space, approximate the projected curve and then lift back on to the variety again. It is here that the preservation of the singularities of the given curve is important. For a bit of experimentation soon shows that the approximation of the projected curve may not lift into an approximation of the original curve unless attention is given to this point. A disturbance of the structure of a singularity may result in the lifted curve going off, so to speak, in the wrong direction.

In Part III the sheets of a real algebraic variety are defined, namely, as sets maximal with respect to the property that any two points can be joined by an analytic arc in the set. This property of analytic connectivity is not transitive, and so the concept is a bit tricky to handle, but the use of the approximation theorems of Part II gives a partial transitivity property. Next, a local study of a real algebraic variety shows that it has locally the structure of a cell complex in the sense of Whitehead (with a little more trouble the

Received September 30, 1957.

corresponding global result could be obtained, but it is not needed here). Finally, the following three questions of Nash **(1)** are answered. Are sheets closed sets? Does a real algebraic variety have just a finite number of sheets? Does each sheet have a point on it with a neighbourhood containing no points of the variety not on that sheet? The answers are respectively yes, yes, no.

In the local cell decomposition of a variety mentioned above, each cell is contained in a sheet. It is natural to ask when cells with common frontier points belong to the same sheet. The answer is not hard to see when two r-cells meet along a variety of dimension $r - 1$, but the general case seems a bit more difficult, and so far no satisfactory answer has been worked out.

I should like here to draw attention to a recent paper by Whitney **(4)** in which some further connectivity properties of real algebraic varieties are obtained.

PART ONE: PLANE CURVES

1. Definitions. All the curves to be discussed in this paper will be contained in Euclidean spaces; thus, a curve is specified by setting the co-ordinates $x_1, x_2, \ldots, x_n$ in the relevant space equal to continuous functions of a real parameter t, which will, in general, be assumed to vary from 0 to 1. The equations $x_i = f_i(t)$ so obtained are the parametric equations of the curve.

An analytic arc in Euclidean n-space is defined to be an arc with parametric equations $x_i = f_i(t)$, where the f_i are real analytic functions of the variable t, $0 \leqslant t \leqslant 1$.

Let C be an analytic arc in n-space and let P be a point of C, with co-ordinates $(x_1', x_2', \ldots, x_n')$, say. Let t_0 be a value of t for which $f_i(t_0) = x_i'$, where $x_i = f_i(t)$ $(i = 1, 2, \ldots, n)$ are the parametric equations of C. Let $P(t_0)$ be the variable point with co-ordinates $(g_1(t), g_2(t), \ldots, g_n(t))$, where the g_i are the expansions of the f_i in powers of $t - t_0$; a similar definition is to be made for all t_0 such that $f_i(t_0) = x_i'$ $(i = 1, 2, \ldots, n)$. If A is the ideal of C in the ring of power series in the $x_i - x_i'$ with real coefficients, then it is known that A has just a finite number of prime components. The points $P(t_0)$, for all possible t_0, are generic zeros for these prime components, and so t_0 has only a finite number of possible values. That is to say, in a neighbourhood of each of its points, C consists of a finite number of irreducible algebroid branches.

Continuing with the notation of the last paragraph, suppose that, in a neighbourhood of P, C consists of exactly one algebroid branch, and let $F_1, F_2, \ldots, F_r$ be a basis for the ideal of power series in the $x_i - x_i'$ vanishing on C around P. Then P is said to be simple on C if the matrix $(\partial F_i/\partial x_j)$ is of rank $n - 1$ at P. If the rank of this matrix is less than $n - 1$, or if C consists near P of more than one branch, then P is said to be a singular point of C.

If at least one of the functions $f_i(t)$ appearing in the parametric equations of C has a non-zero derivative at P, it is not hard to see that P is a simple point of C. Thus, the singular points must be among those at which the derivatives of all the f_i vanish. Since the f_i are analytic, it is clear that there can be only a finite number of such points. Thus, an analytic arc has at most a finite number of singularities. Note, incidentally, that the derivatives of all the f_i may vanish at a simple point; consider, for example, the origin on the plane curve given by $x = t^2$, $y = t^6$.

It will be assumed for convenience in what follows that the parameter values $t = 0$ and $t = 1$ are always mapped on simple points of any analytic arc under consideration. These points will be called the end-points of the arc.

An algebraic arc will be defined to be an analytic arc which lies entirely on some real algebraic curve.

Let $P_1, P_2, \ldots, P_m$ be a finite collection of points of Euclidean n-space and for each pair P_i, P_{i+1} let C_i be an analytic arc with these points as end-points, not passing through any other of the P_j. Then the point-set union C of the C_i will be called a piecewise analytic curve. The C_i will be called the arcs belonging to C and the P_j will be called the joints of C.

A piecewise algebraic curve is a piecewise analytic curve all of whose arcs are algebraic arcs. A variation of this definition is obtained by taking $P_1 = P_n$, when the resulting piecewise analytic or algebraic curve will be called closed. To avoid repetitive descriptions later it will be convenient to agree that the term "singularities of a piecewise analytic or algebraic curve" means the set of singularities of the individual arcs along with all intersections of the arcs other than the joints; thus, the joints of the curve are not counted among the singularities.

Let C and C' be piecewise analytic curves. Then C' will be called an ϵ-approximation of C, for a given positive number ϵ, if there is a homeomorphism $f:C \rightarrow C'$ such that the distance of $f(p)$ from p is less than ϵ for all $p \in C$.

Let C and C' be piecewise analytic curves and let $p \in C$, $p' \in C'$. Then C and C' are said to be analytically equivalent at the pair p, p' if there are neighbourhoods U and U' of p and p', respectively, and an analytic homeomorphism f of U onto U' which carries $U \cap C$ onto $U' \cap C'$. An analytic homeomorphism will map the point $(x_1, x_2, \ldots, x_n)$ on the point $(X_1, X_2, \ldots, X_n)$ given by the formulae

$$X_i - a'_i = \sum_{j=1}^{n} a_{ij}(x_j - a_j) + F_i, \qquad i = 1, 2, \ldots, n$$

where $(a_1, a_2, \ldots, a_n)$ and $(a_1', a_2', \ldots, a_n')$ are the co-ordinates of p and p' respectively, the determinant $|a_{ij}|$ is not zero and the F_i are power series in the $x_j - a_j$ of order not less than two (the order of a power series being the degree of the lowest terms appearing). The analytic equivalence will be said to be of order r if f can be so chosen that all the series F_i are of order not less than r.

Let C' be an ϵ-approximation of the piecewise analytic curve C, and let f be the appropriate homeomorphism of C onto C'. Then, if $f(p)$ is simple on

C' whenever p is simple on C, and if $f(p) = p$ for each singular point p of C, and if C and C' are analytically equivalent at each pair $p, f(p) = p$, for each singularity p of C, then C' will be called a singularity preserving ϵ-approximation of C.

The main idea to be treated in what follows is that of the smoothing approximation. This notion will now be introduced in two forms. In the first place let C denote the figure in the (x, y)-plane consisting of two analytic arcs with a common point P simple on each of them, and not an end point of either of them, and assume that the tangents to the two arcs at P are distinct. The curve C' will be called an ϵ-approximation of C smoothed at P if:

(1) There is a continuous mapping $f\colon C' \to C$ such that $f^{-1}(P)$ consists of two distinct points P_1 and P_2 on C', and f is a homeomorphism on $C - P_1 - P_2$;

(2) The distance of p from $f(p)$ is less than ϵ for all $p \in C$;

(3) There is a neighbourhood U of P in the plane and an analytic homeomorphism F of U on a circle V of centre $(0, 0)$ in the (X, Y)-plane such that $F(U \cap C)$ consists of the parts of the X and Y axes in V and $F(U \cap C')$ consists of the part of the hyperbola $XY = 1$ contained in V.

The second form in which this idea will be wanted is as follows. Let C be a piecewise analytic curve in the plane consisting of the union of two analytic arcs C_1 and C_2 with the joint P, and suppose that the tangents to C_1 and C_2 at P are distinct. Then an ϵ-approximation of C smoothed at P is defined as above, with the modification that $F(U \cap C)$ consists of the positive parts of the X- and Y-axes in V, and $F(U \cap C')$ consists of the part of $XY = 1$ in the first quadrant of V.

Smoothing approximations will later be required not only in the plane but also in spaces of any dimension. Let C be a figure in n-space consisting either of two analytic arcs crossing at P or of a piecewise analytic curve with the joint P, the tangents at P being distinct in each case. Then C' will be called an approximation of C smoothed at P if there is an analytic homeomorphism F of a neighbourhood U of P onto a sphere V of centre P' such that $F(U \cap C)$ and $F(U \cap C')$ lie in a plane through P', and $F(U \cap C')$ is an approximation of $F(U \cap C)$, in the sense already defined, smoothed at P'.

2. Analytic equivalence and smoothing. The object of this section is to show how singularity preserving and smoothing approximations of plane curves can be explicitly constructed.

LEMMA 2.1. *Let F be a power series in x and y free of multiple factors, let G be a power series in x and y and let λ be a real number. Assume F and G to be of order $\geqslant 1$, so that $F = 0$ and $F + \lambda G = 0$ are the equations, in a neighbourhood of the origin, of curves C and C', each consisting of a finite number of analytic arcs through the origin. Then:*

(1) *If the integer r is pre-assigned, there is an integer s such that if G is of order* $\geqslant s$, *C and C′ are analytically equivalent at the origin* (*a self-corresponding point*), *the analytic equivalence being of order* $\geqslant r$.

(2) *If the analytic equivalence of* (1) *is induced by an analytic homeomorphism* $f\colon U \to U'$, *where U and U′ are neighbourhoods of the origin, then f depends analytically on* λ.

Proof. The proof of this lemma is due to Samuel **(2)**, with some minor changes. Write

$$F_1 = \frac{\partial F}{\partial x}, \quad F_2 = \frac{\partial F}{\partial y}.$$

In the ring of power series in x and y with real coefficients let $\mathfrak{a}$ be the ideal (F_1, F_2) and let $\mathfrak{p}$ be the ideal (x, y). The ideal $(F, \mathfrak{a})$ has an isolated zero at the origin and so there is an integer d such that $\mathfrak{p}^d \subset (F, \mathfrak{a})$. It follows at once that, for any integer k, $\mathfrak{p}^{2d+k} \subset F\mathfrak{p}^k + \mathfrak{a}^2\mathfrak{p}^k$. Then if G is of order $2d+k$, that is to say, if $G \in \mathfrak{p}^{2d+k}$, $F + \lambda G$ can be written as $F + \lambda H + \lambda FK$, where $H \in \mathfrak{a}^2\mathfrak{p}^k$ and $K \in \mathfrak{p}^k$. $1 + \lambda K$ has an inverse in the ring of power series in x and y, convergent in a neighbourhood of the origin, and so the equation $F + \lambda G = 0$ is equivalent to the equation $F + \lambda H(1 + \lambda K)^{-1} = 0$. The last equation can be written as $F + \Sigma A_{ij}F_iF_j = 0$, where the A_{ij} are in $\mathfrak{p}^k$ and have co-efficients analytic in λ, and vanishing for $\lambda = 0$. It must now be shown that there is an automorphism S of the power series ring in x and y given by equations of the form:

$$S(x) = x + u_{11}F_1 + u_{12}F_2,$$
$$S(y) = y + u_{21}F_1 + u_{22}F_2,$$

where the u_{ij} are power series in x and y vanishing at the origin, such that $S(F) = F + \Sigma A_{ij}F_iF_j$. The existence of S will establish the required analytic equivalence between C and C'.

Now, by Taylor's theorem, $S(F) = F(S(x), S(y)) = F + \Sigma u_{ij}F_iF_j +$ terms of degree $\geqslant 2$ in the u_{ij} and in the F_i. Thus, to prove the existence of S, the u_{ij} must be determined so that

$$\sum u_{ij}F_iF_j + \ldots = \sum A_{ij}F_iF_j,$$

where the dots denote the higher terms. A solution can be obtained by picking out from this equation the terms in F_iF_j for each pair i, j. The following four equations are thus obtained:

$$u_{ij} = A_{ij} + \text{terms of degree} \geqslant 2 \text{ in the } u\text{'s}.$$

These equations can be solved formally by iteration for the u_{ij} as power series in x and y; convergence is assured by the implicit function theorem, provided x and y are small enough.

To see that the analytic equivalence between C and C' obtained in this way satisfies (1) and (2) in the statement of the lemma, note that the orders of the A_{ij} are all $\geqslant k$, and so the orders of the u_{ij} also satisfy this inequality. Condition (1) of the lemma follows at once. To verify condition (2), note that the presence of λ in the terms A_{ij} (remembering that the A_{ij} have coefficients analytic in λ vanishing for $\lambda = 0$) implies that the coefficients of the u_{ij}, written as power series in x and y, will be analytic in λ, as required.

It is an immediate corollary of this lemma that, if the neighbourhoods U and U' are fixed so that the equations of the automorphism S, or what is essentially the same, of the homeomorphism f: $U \to U'$ are convergent, then, if λ is taken small enough, $C \cap U$ will be approximated arbitrarily closely by $C' \cap U'$.

LEMMA 2.2. *Let C be an algebraic curve in the plane with a singular point P at which exactly two simple branches meet with distinct tangents, and let U be a neighbourhood of P such that $U \cap C$ is homeomorphic to two crossed line segments. Let $F = 0$ be the irreducible equation of C and let G be any polynomial in x and y not vanishing at P. Then if ϵ is pre-assigned and λ is taken small enough $F + \lambda G = 0$ is a smoothed ϵ-approximation of C within a sufficiently small neighbourhood U_0 of P.*

Proof. Take P as origin, and let U_0 be a neighbourhood of P such that $G \neq 0$ in U_0. If U_0 is small enough, G^{-1} is a convergent power series in x, y in U_0. Also, if U_0 is small enough, F can be factorized into fgh, where f, g, h are convergent power series in U_0, $f = 0$ and $g = 0$ are the equations of the two branches meeting at P, and $h \neq 0$ at P. f and g are thus of order one. Then the equations $X = f$, $Y = ghG^{-1}$ define an analytic homeomorphism of U_0 onto a neighbourhood of the origin in the (X, Y)-plane. It is not hard to see that if λ is small enough, this homeomorphism defines the required smoothing approximation.

The above lemmas will be combined to give a proof of the following theorem:

THEOREM 1. *Let C be a plane algebraic curve with singularities at (x_i, y_i) $(i = 1, 2, \ldots, n)$. Let exactly two simple branches of C with distinct tangents meet at (x_1, y_1). Then, if K is a circular disc containing the (x_i, y_i) and ϵ is a pre-assigned positive number, there exist algebraic curves C_1 and C_2 such that $C_1 \cap K$ and $C_2 \cap K$ are ϵ-approximations of $C \cap K$, smoothed in the two complementary ways at (x_1, y_1) (corresponding to the two complementary hyperbolas $xy = \pm 1$) and otherwise singularity preserving, with analytic equivalence of pre-assigned order at each singularity.*

Proof. Let $F(x, y) = 0$ be the equation of C, free from multiple factors and define the polynomial $G(x, y)$ by the equation

$$G(x, y) = \prod_{i=2}^{n} [(x - x_i)^2 + (y - y_i)^2]^r.$$

G vanishes at each of the (x_i, y_i) for $i = 2, \ldots, n$ and is of order $2r$ at each of these points. And so, by Lemma 2.1, if r is large enough, the curve

$F+\lambda G = 0$ is a singularity preserving approximation of $F = 0$ in suitably chosen neighbourhoods of the (x_i, y_i) $(i \neq 1)$, and the approximation can be made arbitrarily good by taking λ small enough. Also $G \neq 0$ at (x_1, y_1) and so, by Lemma 2.2, $F + \lambda G = 0$ is, in a suitable neighbourhood of (x_1, y_1), a smoothed approximation of $F = 0$, and again the approximation can be made arbitrarily good by making λ sufficiently small.

On the other hand, if p is a simple point of C, there is a neighbourhood of p in which there is an admissible system of co-ordinates (in the sense of the real analytic structure of the (x, y)-plane) one of which is the arc length along C while the other is λ. Thus, if γ is a non-singular arc on C, γ has a neighbourhood in which the only part of $F + \lambda G = 0$, for λ sufficiently small, is a non-singular arc whose points are at arbitrarily small distance from γ.

Let C' be the curve $F + \lambda G = 0$. Then, to complete the proof of the theorem, a mapping f: $C' \to C$ must be constructed to satisfy the conditions of the definitions of smoothing and singularity preserving approximations. In order to construct this mapping take an open covering of $C \cap K$ as follows.

(i) About each singularity (x_i, y_i) take a neighbourhood U_i such that in U_i, for λ sufficiently small, $F + \lambda G = 0$ gives a singularity preserving or smoothing approximation of $F = 0$, as the case may be. Writing g for the inverse of the mapping f which is to be constructed (and remembering that g will be a mapping, and in fact a homeomorphism on C with (x_1, y_1) removed, the latter point being carried by g into two distinct points), this means that g is now constructed on the $U_i \cap C$.

(ii) $C \cap K$ with the (x_i, y_i) removed consists of a finite number of simple arcs joining singular points to each other, or joining singular points to frontier points of K, or joining frontier points of K to each other. Let γ be one of these arcs and let the end-point p_1 be one of the (x_i, y_i). Let q_1 be the point on γ furthest from p_1 for which the operation g is already defined and let q_1' be a point between p_1 and q_1 on γ. Proceed similarly at the other end p_2 of γ, marking points q_2 and q_2' on γ. If an end point of γ, say p_1, is non-singular on C but lies on the frontier of K, take $q_1 = p_1$ and take q_1' near q_1 but outside K. Let $U(\gamma)$ be the union of normal line segments to γ at all points from q_1' to q_2' of length $\delta(\gamma)$ on either side of γ. If $\delta(\gamma)$ is small enough, $U(\gamma)$ is fibred by these normal segments. Repeat this procedure for each arc of $C \cap K$ with the singularities removed.

The U_i along with the $U(\gamma)$ form the required covering of $C \cap K$. Let U be the union of these sets; then U is a neighbourhood of $C \cap K$. Since $F \neq 0$ in $K - U$, it is clear that, for λ sufficiently small, $C' \cap K$ lies entirely in U. Also, fixing attention on the arc γ, and using the notation introduced above, it follows at once from Lemma 2.1 or 2.2, whichever is relevant, that, for λ small enough, $g(q_1)$ and $g(q_2)$ are in $U(\gamma)$. A similar statement can be made for all the other arcs of $C \cap K$. Then, remembering that λ can be taken as one of the local co-ordinates in the plane in $U(\gamma)$, it follows at once that $C' \cap U(\gamma)$ consists of a simple arc with $g(q_1)$ and $g(q_2)$ near its end-points.

It is now easy to see that g can be extended to the whole of $C \cap K$ by mapping the arc q_1q_2 of C on the arc $g(q_1)g(q_2)$ of C' in $U(\gamma)$, proceeding similarly for each arc γ of $C \cap K$. For ϵ pre-assigned, $f = g^{-1}$ is an ϵ-approximation if λ is small enough, and the theorem is completely proved.

The curves C_1 and C_2 mentioned in the statement of the theorem refer to the two complementary ways of smoothing at (x_1, y_1), corresponding respectively to positive and negative values of λ.

It is clear that the above theorem could be modified to yield an approximation of C which is smoothed at several singularities, while preserving the rest.

Also, it is clear that the order of the analytic equivalence between C and C' at each singularity can be made arbitrarily high by taking the exponent r in G large enough.

3. Preliminary approximation theorems. The principal object of this part of the paper is the application of Theorem 1 to the approximation of piecewise analytic curves by circuits of algebraic curves. The way in which this is to be done will now be sketched, the details being completed later in this section and in the next.

Fix attention first on a closed piecewise algebraic curve C in the plane, and let the arcs of C be C_i $(i = 1, 2, \ldots, n)$. Each C_i is part of an algebraic curve $\bar{C}_i$, and C is part of the composite algebraic curve $\bar{C} = \bigcup \bar{C}_i$. If matters are suitably arranged, Theorem 1 can be applied to approximate $\bar{C}$ by an algebraic curve $\bar{C}'$, smoothing at the joins of the C_i. Thus, C is approximated by a circuit C' of the algebraic curve $\bar{C}'$. Parts of $\bar{C}' - C'$ may, however, meet C'. The next step is to show that the curve $\bar{C}'$ can be modified in such a way that the approximating circuit becomes isolated, that is to say, does not meet any other part of the curve. The lemmas which establish the procedure for isolating the approximating circuit will be dealt with first in this section. The sequence of operations can be summarized as follows:

(1) Lift $\bar{C}'$ into 3-space in such a way that C' is separated from the rest.

(2) Make a transformation of 3-space so that the unwanted circuits are removed.

(3) Project back onto the plane.

LEMMA 3.1. *Let C be a real plane algebraic curve and let (x_i, y_i), $(i = 1, 2, \ldots, r)$, be singularities at each of which exactly two simple branches meet with distinct tangents (C may, of course, have other singularities as well). Then there is a curve C' in 3-space which, under the projection P onto the (x, y)-plane, projects onto C and is such that $P^{-1}(x_i, y_i)$, for each i, consists of exactly two distinct points, while, apart from these points, P is a homeomorphism on C'. Also, if (x_0, y_0) is a singularity of C other than the (x_i, y_i) and if $P^{-1}(x_0, y_0) = (x_0, y_0, z_0)$, then C' and C are analytically equivalent at the pair of points (x_0, y_0, z_0) and $(x_0, y_0, 0)$, to an arbitrarily high order.*

Proof. Assume co-ordinates chosen so that the lines $x = x_1$, $x = x_2, \ldots,$ $x = x_r$ meet C in simple points, apart from the points (x_i, y_i) $(i = 1, 2, 3, \ldots, r)$ and suppose that at all these simple points the tangents are not parallel to the y-axis. Also, at each of the points (x_i, y_i) $(i = 1, 2, \ldots, r)$ there are two tangents, and it is to be assumed that none of these tangents is parallel to the y-axis. Let $Y_1, Y_2, \ldots, Y_s$ be the y-co-ordinates of the singularities of C, other than the (x_i, y_i). Let the line $x = x_i$ meet C at the simple points (x_i, y_{ij}), $j = 1, 2, \ldots, m$, and let f be the product of all the distinct expressions picked from the $y - Y_i$, the $y - y_i$ and the $y - y_{ij}$, for all i, j; that is to say, if a number of these expressions should happen to be equal, the corresponding factor of f is nevertheless to appear just once. Similarly, let g be the product of distinct factors picked from the set $x - x_i$. Let $F(x, y)$ be the rational function f/g.

Examining the behaviour of F on the curve C, it is clear that the only possible points of indeterminacy are the zeros of g, namely, the points (x_i, y_i) and (x_i, y_{ij}) for all i, j. Representing y as a power series in $x - x_i$ for points of C around (x_i, y_{ij}) it turns out that F is continuous on C at that point. On the other hand, at the point (x_i, y_i) there are two distinct branches of C with distinct tangents. Making use of the two corresponding expansions of y in powers of $x - x_i$, it follows this time that F is continuous on each of the branches of C at (x_i, y_i) taken separately. The fact that the tangents to these two branches are distinct ensures that the two limits of F as (x, y) approaches (x_i, y_i) along the two branches of C are different.

Now let C' be the curve in 3-space whose points are of the form

$$(x, y, F^m(x, y)),$$

where (x, y) is on C. Then C' is the curve required by the statement of the present lemma. For if P is the projection of C' on C it is clear that $P^{-1}(x_i, y_i)$ consists of two distinct points, corresponding to the two limits of F along the branches at (x_i, y_i) and that P is one-one on C' except at these points which project doubly. To complete the proof of the lemma, the behaviour of P at points projecting on singularities of C other than the (x_i, y_i) must be examined. Around such a singularity (x_0, y_0), F is a real analytic function of x and y, and so P extends to an analytic homeomorphism of a neighbourhood of

$$(x_0, y_0, z_0) = P^{-1}(x_0, y_0)$$

on a neighbourhood of $(x_0, y_0, 0)$ in 3-space. For example, the mapping of (x, y, z) on $(x, y, z - F^m(x, y))$ gives such an extension which is an analytic equivalence of order m. This completes the proof.

LEMMA 3.2. *Let C be an algebraic arc in n-space E_n with a singularity Q projecting, under rectangular projection, on an arc C' in r-space E_r with a singularity at Q'. Let C and C' be analytically equivalent at Q, Q', the equivalence being induced by an analytic homeomorphism F of a neighbourhood U of Q on a neigh-*

bourhood U' of Q', such that, on $F^{-1}(U' \cap E_r)$, F coincides with the projection. Then a sufficiently good approximation C_1 of C projects on an arbitrarily good approximation C_1' of C'. Also, if C_1 is a singularity preserving approximation of C with analytic equivalence of sufficiently high order, and if the order of the analytic equivalence induced by F is of sufficiently high order, then C_1' will be a singularity preserving approximation of C', and the corresponding analytic equivalence at Q' will be of pre-assigned order.

Proof. The first part of the lemma, that a sufficiently good approximation of C projects into an arbitrarily good approximation of C', is practically trivial, and so attention will be fixed on the second part. Let P be the orthogonal projection of E_n on E_r. Let F_1 be an analytic homeomorphism of U (the same neighbourhood as in the above statement; no generality is lost as U can, if necessary, be shrunk to suit both situations) on a neighbourhood U_1 of Q, such that F_1 induces the analytic equivalence assumed between C and C_1 at Q. Let U_0 be a neighbourhood of Q' in E_r, say $U' \cap E_r$. Define F' as $PF_1 F^{-1}$ restricted to U_0. It is clear that F' defines an analytic equivalence of C_1' and C' at Q' as required. Also, it is not hard to see that, since P is a linear transformation, the order of F' can be made as high as one pleases by making those of F_1 and F large enough.

LEMMA 3.3. *Let $P_1, P_2, \ldots, P_r$ be a set of points in n-space E_n contained in a sphere A with centre the origin. Let P be a point outside A and let U be a pre-assigned neighbourhood of P. Then there exists a rational mapping F of E_n onto itself such that:*

(1) *F approximates the identity mapping arbitrarily closely on A;*

(2) *F carries all points outside a sufficiently large sphere B into U;*

(3) *$F(P_i) = P_i$ for each i, and if $(a_{i1}, a_{i2}, \ldots, a_{in})$ are the co-ordinates of P_i then the equations of F in a neighbourhood of P_i are of the form $X_j = x_j + F_{ij}$, where F_{ij} is a power series in the $x_j - a_{ij}$ of pre-assigned order.*

Proof. The idea involved here is similar to that of (**3**, Lemma 2, §3). The mapping constructed there is the composition of stereographic projection of E_n on a sphere in E_{n+1} and an oblique projection back onto E_n. Such a mapping would be rational and would satisfy (1) and (2) above, but not (3). The required mapping will be constructed by making a suitable modification of stereographic projection.

Let r be an integer such that $2r$ is greater than the pre-assigned orders referred to in (3) of the present theorem, let $d(P_i, x)$ be the distance of the point $(x_1, x_2, \ldots, x_n)$ from P_i, and define the polynomial

$$G(x) = \prod_i d^{2r}(P_i, x).$$

Then the equations of the mapping F are to be:

$$(1) \qquad \begin{aligned} X_1 &= (k^2 x_1 + kG(x)\tan\alpha)/(k^2 + G(x)) \\ X_i &= k^2 x_i/(k^2 + G(x)), \qquad i = 2, 3, \ldots, n, \end{aligned}$$

where $(X_1, X_2, \ldots, X_n)$ is the transform of $(x_1, x_2, \ldots, x_n)$ under F. It must be shown that the constants k and α can be chosen so that the conditions of the lemma are satisfied.

In order to do this, it is convenient to consider the geometrical meaning of the mapping F with the equations (1). Let the n-space E_n be the hyperplane $x_{n+1} = 0$ in $(n + 1)$-space, and let the co-ordinates be chosen so that P is on the x_1-axis. Let α be the angle between the x_{n+1}-axis and the line joining P to the point $(0, 0, \ldots, 0, k)$ in $(n + 1)$-space. Then F is the composition of the mappings f and g where $f(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_{n+1})$ and $g(y_1, y_2, \ldots, y_{n+1}) = (X_1, X_2, \ldots, X_n)$ these mappings being defined by the equations:

$$\text{(2)} \qquad y_i = k^2x_i/(k^2 + G(x)), \qquad i = 1, 2, \ldots, n,$$
$$y_{n+1} = kG(x)/(k^2 + G(x)).$$

$$\text{(3)} \qquad X_1 = y_1 + y_{n+1} \tan \alpha$$
$$X_i = y_i, \qquad i = 2, \ldots, n.$$

The mapping g is, of course, the projection of E_{n+1} onto E_n along lines parallel to the join of P and $(0, 0, \ldots, 0, k)$ while f, on the other hand, is a mapping similar to stereographic projection, and would coincide with it if G were replaced by $\sum x_i^2$. f can be described geometrically as the projection from $(0, 0, \ldots, 0, k)$ of E_n on the hypersurface H in E_{n+1} having the equations (2) as parametric equations. f is a one-one mapping of E_{n+1} on H, and in fact a birational correspondence, the inverse mapping being given by

$$x_i = ky_i/(k - y_{n+1}), \qquad i = 1, 2, \ldots, n.$$

Comparing H with the sphere of centre $(0, 0, \ldots, 0, \frac{1}{2}k)$ and radius k in E_{n+1}, given by the parametric equations

$$Y_i = k^2x_i/(k^2 + \sum x_j^2), \qquad i = 1, 2, \ldots, n,$$
$$Y_{n+1} = k \sum x_j^2/(k^2 + \sum x_j^2),$$

it is easy to see that, for points $(x_1, x_2, \ldots, x_n)$ in some bounded set in E_n the distance of $(y_1, y_2, \ldots, y_{n+1})$ from $(Y_1, Y_2, \ldots, Y_{n+1})$ can be made as small as one likes by taking k large enough. Also it is a simple computation to show that, still confining $(x_1, x_2, \ldots, x_n)$ within a bounded set the partial derivatives of y_{n+1} with respect to the x_i, calculated from the equations (2), are as small as one pleases if k is sufficiently large, while $\partial y_j/\partial x_i$ approximates δ_{ij}, the Kronecker δ, for large k. It follows that the normal to H at points corresponding to values of $(x_1, x_2, \ldots, x_n)$ in a bounded set will make an arbitrarily small angle with the x_{n+1}-axis in E_{n+1} if k is large enough. Taking the sphere A as the bounded set in these remarks, it follows at once that a line parallel to the join of P to $(0, 0, \ldots, 0, k)$ will, if k is large enough, meet H at not more than one point corresponding to values of $(x_1, x_2, \ldots, x_n)$ in A. That is to say, $g \circ f = F$ is one-one on A for sufficiently large values of k.

An easy calculation shows that for k sufficiently large, F also approximates the identity mapping arbitrarily closely on A. Thus (1) in the statement of the lemma is proved. To verify (2) note that outside a large sphere B, the polynomial $G(x)$ will be large, and so, if B is large enough, the distance of $(y_1, y_2, \ldots, y_{n+1})$, given by equations (2), from $(0, 0, \ldots, 0, k)$ will be less than a pre-assigned number. On the other hand, a sufficiently small neighbourhood of $(0, 0, \ldots 0, k)$ will project under g into U, and so (2) is proved. To prove part (3) of the lemma, note that $G(x)$, as a power series in the $x_j - a_{ij}$ around P_i, is of order $2r$, and so the series expansions of the X_j in equations (1) are of the required form.

LEMMA 3.4. *Let C be a real algebraic curve and let $C = C_1 \cup C_2$ where C_1 is a closed circuit and C_2 is the rest of the curve. Assume that the intersections of C_1 and C_2 are all double points at which exactly two simple branches meet. Then there exists an algebraic curve $C' = C_1' \cup C_2'$ of which the circuit C_1' is a singularity preserving ϵ-approximation of C_1 for pre-assigned ϵ, while C_2' is contained in an arbitrary neighbourhood of a pre-assigned point P, arbitrarily far from C_1. In addition, the order of the analytic equivalence of C_1 and C_1' at each singularity can be made greater than a pre-assigned integer.*

Proof. The first step is to apply Lemma 3.1 to C, taking the points of intersection of C_1 and C_2 as the (x_i, y_i). In this way a space curve $K = K_1 \cup K_2$ is obtained such that $K_1 \cap K_2 = \phi$. Also, K_1 projects in a one-one manner on C_1, and if (x_0, y_0, z_0) on K_1 projects on a singularity of C_1, then K_1 and C_1, regarded as space curves, are analytically equivalent to an arbitrarily high order at the pair (x_0, y_0, z_0) and $(x_0, y_0, 0)$. The next step is to move K_2 to a great distance from K_1. Take the 3-space with co-ordinates (x, y, z) as the hyperplane $t = 0$ in (x, y, z, t)-space. Let A and B be spheres in the latter space with the origin as centre, and such that $K_1 \subset A \subset B$. Let B be of radius R. Let $\phi(x, y, z)$ be a continuous function equal to zero on K_1 and equal to $2R$ on $K_2 \cap B$. Apply the Weierstrass approximation theorem to approximate ϕ on B by means of a polynomial f. Let g be a rational function of x, y, z which is equal to zero to a pre-assigned order at each singularity of K_1, satisfies everywhere the inequality $0 \leqslant g < 1$, and also satisfies $g > 1 - \delta$ outside pre-assigned neighbourhoods of the singularities of K_1, δ being a pre-assigned positive number. A method of construction for g will be given in Lemma 3.5 below. Now, for each point (x, y, z) of K, set $G(x, y, z)$ equal to the point in 4-space with co-ordinates (x, y, z, fg), where f and g are evaluated at (x, y, z). Then G is a birational transformation of K into a curve

$$K' = K_1' \cup K_2'.$$

K_1' projects in a one-one manner on K_1 and is contained in A, while K_2' lies outside B, if the functions f and g have been suitably chosen. Also, there is analytic equivalence between K_1 and K_1' at each singularity, the order being that of g, regarded as a power series at the point in question.

The mapping F of Lemma 3.3, carrying (x, y, z, t)-space on itself is now to be applied. F carries K' into a curve

$$K'' = K_1'' \cup K_2''.$$

The points P_i of Lemma 3.3 are here to be taken as the singularities of K_1'. Now, the notation used here is intended to indicate that K_1'' and K_2'' are the images of K_1' and K_2' under F; it must, however, be checked that K'' constitutes the whole of a real algebraic curve. That this is so follows at once from the fact that F is birational on K', being the composition of the birational mapping f of Lemma 3.3 and the projection g of Lemma 3.3 which, being one-one on A, is certainly birational on $f(K')$. By Lemma 3.3 K_1'' is a singularity preserving approximation of K_1', which will be arbitrarily close if the constant k of Lemma 3.3 is taken large enough. Also, the analytic equivalence of K_1' and K_1'' at each singularity can be made of arbitrarily high order. If the sphere B has been made large enough, Lemma 3.3 implies that K_2'' will be contained in a preassigned neighbourhood of P in 4-space.

The proof will now be completed by a projection back onto the plane, which is the subset $z = t = 0$ of 4-space, applying Lemma 3.2 to obtain the required result. To apply Lemma 3.2, note first that, under the orthogonal projection of (x, y, z, t)-space on the (x, y)-plane K_1' is carried onto C_1, the singularities of the former being mapped on those of the latter, with analytic equivalence in each case of pre-assigned order. Also, by the above argument, K_1'' is a singularity preserving approximation of K_1', with analytic equivalence at each singularity of arbitrary high order. It follows at once from Lemma 3.2, that, if K_1'' is a sufficiently close approximation of K_1', and if the orders of analytic equivalence just mentioned are high enough, then C_1', the projection of K_1'' is as stated in this lemma. Also, K_2'' is contained in a neighbourhood of P, and so the same holds for its projection C_2', and the proof of the lemma is complete.

LEMMA 3.5. *Let $P_1, P_2, \ldots, P_m$ be points in Euclidean space of any dimension, and let $U_1, U_2, \ldots, U_m$ be spheres with these points as centres and radii $r_1, r_2, \ldots, r_m$, respectively. Then there is a rational function f such that f vanishes to a pre-assigned order at each P_i and $f > 1 - \epsilon$ outside the U_i, for a pre-assigned positive number ϵ, and at all points $0 \leqslant f < 1$.*

Proof. Let η be a positive number and let s be a positive integer. Denote by $d(P_i, x)$ the distance of P from the point with co-ordinates $(x_1, x_2, \ldots, x_n)$. Define

$$f_i(x) = d^{2s}(P_i, x)/(\eta r^{2s} + d^{2s}(P_i, x)).$$

Clearly $0 \leqslant f_i < 1$, and also f_i vanishes at P_i, being of order s there (regarded as a power series around P_i). It is not hard to verify that $f_i > 1 - \eta$ at points outside U_i. Now set

$$f(x) = \prod_{i=1}^{m} f_i(x).$$

It is easy to see that if η is small enough, f satisfies the requirements of the lemma.

4. The main approximation theorems in the plane. The main results of Part I will now be obtained, namely, approximation theorems for piecewise analytic and algebraic curves in the plane. Attention will first be fixed on closed curves, and the required result approached in two stages, namely, Theorems 2 and 3.

THEOREM 2. *Let C be a closed piecewise algebraic curve with arcs C_i, joints P_j, satisfying the following conditions:*

(1) C_i is part of an algebraic curve $\bar{C}_i$ and at each P_i exactly two C_j meet, namely C_i, and C_{i+1}, and P_i is to be simple on both, the two tangents being distinct.

(2) If $\bar{C}_i - C_i$ meets C_j (this is to include the case $i = j$) then it does so at a point which is simple both on C_j and on $\bar{C}_i - C_i$ and the two tangents there are distinct.

Then there exists an algebraic curve $C' = C_1' \cup C_2'$, where the circuit C_1' is an ϵ-approximation of C (ϵ being pre-assigned) smoothed at the P_i and singularity preserving, with analytic equivalence of pre-assigned order at the singularities, while C_2' is contained in an arbitrarily pre-assigned set U.

Proof. Apply Theorem 1 to $\bar{C} = \bigcup \bar{C}_i$. Within a disc containing C this gives an arbitrarily good approximation of C, smoothing at the P_i, and otherwise singularity preserving with analytic equivalence of arbitrarily high order at the singularities. Let the resulting curve be $C^* = C^*_1 \cup C^*_2$ where C^*_1 approximates C. Then the above conditions ensure that the intersections of C^*_1 and C^*_2 are all points where two simple branches meet with distinct tangents. Applying Lemma 3.4 to C^*, the required result follows.

Theorem 3 will now generalize Theorem 2 and remove restrictions placed temporarily on the C in that Theorem.

THEOREM 3. *Let C be a closed piecewise analytic curve in the plane with arcs C_i, and joints P_j. Then there exists an algebraic curve $C_1' \cup C_2'$ with one circuit C_1' giving an ϵ-approximation of C, where ϵ is pre-assigned, smoothed at the P_i, and otherwise singularity preserving with analytic equivalence of arbitrarily high order, while C_2' lies in a pre-assigned set.*

Proof. Let Q_i be a singular point of C; it may be a singular point of just one C_j or a point at which several of these arcs meet, being either singular or simple on each one of these. Let U_i be a sufficiently small neighbourhood of Q_i and let $F_i = 0$ be the equation of $C \cap U_i$, where F_i is a power series in x and y. If all sufficiently high powers of x and y in F_i are dropped a polynomial equation $F_i' = 0$ is obtained which represents in U_i a curve analytically equivalent to $C \cap U_i$, the analytical equivalence being of arbitrarily high order. The same can be said of $F_i' + \lambda_i G_i = 0$ where G_i is a polynomial containing only sufficiently high powers of x and y and λ_i is a real number.

Repeat the above procedure at each singularity Q_i of C, thus obtaining a set of algebraic curves $F_i' + \lambda_i G_i = 0$. By suitable adjustment of the λ_i it may be ensured that the curve $F_i' + \lambda_i G_i = 0$ does not contain Q_j for $i \neq j$. Now approximate the remainder of C outside the neighbourhoods U_i by straight line segments. These segments along with the parts of the curves $F_i' + \lambda_i G_i = 0$ in U_i are to play the part of the C_i of Theorem 2. It is easy to see that these line segments can be chosen in such a way that the conditions of that theorem hold. An application of Theorem 2 then gives the required result.

COROLLARY. *Theorem* 3 *will also hold if* C *is replaced by any piecewise analytic curve, not necessarily closed.*

Proof. For if the C_i are the arcs of a non-closed piecewise analytic curve then a closed curve may be obtained by joining the first and last end points of C by any analytic arc not meeting the C_i at any other point. Theorem 3 may then be applied to the resulting closed curve after which the unwanted portion of the approximation, corresponding to the additional analytic arc, can be discarded.

PART TWO: CURVES ON REAL ALGEBRAIC VARIETIES

5. Approximation of a curve in 3-space. It has now been shown that a piecewise analytic curve in the plane, that is to say, a sequence of analytic arcs joined end to end, can be approximated by part of an algebraic curve. As indicated in the introduction, that approximation theorem was the first stage towards a similar approximation theorem on a real algebraic variety. The second stage is to generalize the result in the plane to Euclidean spaces of higher dimension. The general result will be proved by induction on the dimension of the surrounding space starting with dimension 3. The case of dimension 3 needs special attention because it is not, in general, possible to make a projection of a curve on to a plane in a one-one manner.

The procedure for obtaining the required approximation in 3-space is as follows.

(1) Let C be a closed piecewise analytic curve in 3-space. Let C_y and C_z be its projections on the planes $y = 0$ and $z = 0$ respectively. Approximate these curves by circuits of algebraic curves C_y', C_z' with equations $f(x, z) = 0$, $g(x, y) = 0$, respectively. These approximations are to be smoothed at the joints and otherwise singularity preserving. Then it will turn out that in 3-space the curve C' with equations $f = g = 0$ has a circuit approximating C arbitrarily closely, smoothing at the joints and otherwise singularity preserving (provided that the approximations in the two planes are suitably made).

(2) Now C' can be represented alternatively by the equations $g(x, y) = 0$, $z = h(x, y)$ where h is a rational function with indeterminacies at points on which more than one point of C' or some singularity of C' projects. In particular the circuit of C' approximating C is obtained by allowing the argument

(x, y) of h to vary on a suitable circuit of C_z'. Approximate this circuit of C_z' by an isolated circuit of an algebraic curve C'' with equation $g'(x, y) = 0$ and then it will be shown that the equations $g' = 0, z = h$ define a curve C'' in space with an isolated circuit approximating C, smoothing at the joints and otherwise singularity preserving.

The details of the procedure sketched in (1) above will be carried out by a sequence of lemmas in the next three sections. These treat in turn various special points of C. First, there are the singularities of C which are to be preserved; second, points at which C is to be smoothed; and, finally, points at which the tangent to C is parallel to the (x, z)-plane. An example of the last type is the point $(0, 0, 0)$ on one loop of the intersection of the cylinders

$$x^2 + (y-1)^2 = 1, \qquad z^2 + (y-1)^2 = 1.$$

A slight displacement of the cylinders, unless subject to suitable restrictions, would clearly change the configuration at the origin.

6. Analytic equivalence. The object of this section is to give a criterion for the analytic equivalence of curves defined locally by suitably related sets of equations. As the results are to be used later in the proofs of the general approximation theorems, it will be convenient to state a lemma applicable to space of any dimension.

LEMMA 6.1. *Let $f_i(x)$, $i = 1, 2, \ldots, m$, be power series in $x_1, x_2, \ldots, x_n$, and let $F(x, z)$ be a power series in $x_1, x_2, \ldots, x_n, z$; similar meanings are to be attached to $f_i'(x)$, $i = 1, 2, \ldots, m$, and $F'(x, z)$. Suppose that there is an automorphism S of the power series ring in the x_i of the type $S(x_i) = x_i + h_i(x)$, where the h_i are power series of order not less than 2, and where $S(f_i)$ is a linear combination of the f_j', and $S^{-1}(f_i')$ a linear combination of the f_j, for each i. Then, if the orders of the h_i and of the difference $F - F'$ are sufficiently high and if the series $f_1, f_2, \ldots, f_m$, F, and F_z have no common zero in a neighbourhood of the origin, there will be an automorphism T of the power series ring in x_1, x_2, $\ldots, x_n$ and z, of the form*

$$T(x_i) = x_i + h_i(x),\ T(z) = z + l(x, z),$$

where the order of $l(x, z)$ is greater than a pre-assigned integer, and where T and T^{-1} carry each of the sets $f_1, f_2, \ldots, f_m$, F and $f_1', f_2', \ldots, f_m'$, F' into linear combinations of the other.

Proof. The power series $l(x, z)$ mentioned in the statement of the lemma is to be found in such a way that the set of equations $f_i(x + h) = 0 (i = 1, 2, \ldots, m)$, $F(x + h, z + l) = 0$ is a set of linear combinations of the equations $f_i'(x) = 0 (i = 1, 2, \ldots, m)$, $F'(x, z) = 0$, where $x + h$ denotes the set $x_j + h_j$ $(j = 1, 2, \ldots, n)$. Since the automorphism S already relates the f_i and the f_i' in this way, it will be sufficient to find l in such a way that

$$F(x+h, z+l) = \sum A_i f_i'(x) + BF'(x, z),$$

where the A_i and B are power series in the x_i and z, and B is invertible in a neighbourhood of the origin. The last condition is equivalent to saying that B does not vanish at the origin. Applying Taylor's theorem to the equation just written, it turns out that the power series l, the A_i and B must be determined to satisfy the following equation:

$$(1) \qquad F(x+h, z) + lF_z(x+h, z) + \tfrac{1}{2}l^2 F_{zz}(x+h, z) + \ldots$$
$$= \sum A_i f_i'(x) + BF'(x, z)$$

where the dots denote terms involving higher powers of l.

There are two cases to consider. If $F_z \neq 0$ at the origin, then $F_z(x+h, z)$ is invertible around the origin, and so equation (1) can be divided by it. Equation (1) can then be solved by setting the A_i all equal to zero and $B = 1$, and then calculating l iteratively. The first approximation to l will be $F'(x, z) - F(x+h, z)$, and the order of this, which will also be the order of l, can be made arbitrarily high by making the orders of the h_i and $F - F'$ high enough.

The second and more complicated case is where F_z is zero at the origin. In this case, in order to enable an iterative solution for l to be carried out, it will first be shown that the A_i and B in (1) can be chosen in such a way that the terms free from l, namely, $-F(x+h, z) + \sum A_i f_i'(x) + BF'(x, z)$, will be equal to a multiple of $F_z^s(x+h, z)$ for some pre-assigned integer s. By the hypothesis of the operation of the automorphism S on the f_i, this is equivalent to finding power series A_i' and B such that $-F(x+h, z) + \sum A_i' f_i(x+h) + BF'(x, z)$ is a multiple of $F_z^s(x+h, z)$.

In solving this auxiliary problem, it is convenient to make a change of notation, writing X_i for $x_i + h_i$. Thus, the auxiliary problem is as follows: if the $p_i(X)$ $(i = 1, 2, \ldots, n)$ are power series of sufficiently high order, then it is required to find power series P_i, Q and R, Q being invertible, such that

$$(2) \qquad -F(X, z) + \sum P_i f_i(X) + QF'(X+p, z) = RF_z^s(X, z).$$

A solution will actually be found in which $Q = 1 + Q''$, where Q'' is of order $\geqslant 1$. Writing $P_i' = -P_i Q^{-1}$, $Q^{-1} = 1 + Q'$ and $R' = Q^{-1}R$, equation (2) becomes

$$(3) \quad F'(X+p, z) = F(X, z) + \sum P_i' f_i(X) + Q'F(X, z) + R'F_z^s(X, z)$$

where this equation is to be solved for the P_i', Q' and R'. Now, since by hypothesis the f_i, F and F_z have an isolated common zero at the origin, there is an integer d such that all monomials of degree d in the X_i and z are in the ideal generated in the ring of power series in the X_i and z by the $f_i(X)$, $F(X, z)$ and $F_z^s(X, z)$. On the other hand, if the p_i are of sufficiently high order, $F'(X+p, z) - F(X, z)$ will be of order $\geqslant d$. It follows at once that the P_i', Q' and R' can be chosen so that (3) is satisfied.

Returning to the main question, the solution of this auxiliary problem implies that equation (1) can be rewritten as

$$(4) \qquad lF_z(x+h,z) + \tfrac{1}{2}l^2F_{zz}(x+h,z) + \ldots = C(x,z)F_z^s(x+h,z)$$

where C is some power series. This equation will now be solved by setting

$$l = u(x,z)F_z^{s-2}(x+h,z),$$

where u is a power series to be determined. By means of this substitution (4) becomes

$$u + G = CF_z(x+h,z)$$

where G is a power series in the x, z and u, involving only powers greater than the first of u. The power series u can now be obtained from this equation by iteration, the convergence of the process being assured by the implicit function theorem. Note that the first approximation to u in the iterative process is $CF_z(x+h,z)$, and so the first approximation to l is $CF_z^{s-1}(x+h,z)$. Since F_z is zero at the origin, this ensures that l will be of pre-assigned order, merely by taking s big enough. By following through the above proof step by step it is not hard to see the

COROLLARY. *If the h_i and F' depend analytically on one or more parameters in such a way that the h_i vanish and F' reduces to F when these parameters are set equal to zero, then the series l will also depend analytically on these parameters and will vanish when they are set equal to zero.*

Restrict attention now to curve branches in 3-space.

LEMMA 6.2. *Let C be a curve branch (not necessarily irreducible) in a neighbourhood of the origin in 3-space. C is part of a curve with equations $F(y,z) = 0$, $G(x,y) = 0$ where F and G are power series free of double factors. Then, if C' has equations $F'(y,z) = 0$, $G'(x,y) = 0$ where F' and G' are power series such that $F - F'$, $G - G'$ are of sufficiently high order, it will follow that C and C' are analytically equivalent to an arbitrarily high order at the origin.*

Proof. By Lemma 2.1 there exists a transformation of the type $(x,y) \to (x+h, y+k)$ carrying G into G' where h and k are power series in x and y whose orders are greater than a pre-assigned integer if G and G' are of sufficiently high order. Also, F and F_z have a common isolated zero at the origin. The result then follows from Lemma 6.1.

COROLLARY. *If F' and G' depend analytically on one or more parameters in such a way that they reduce to F and G when these parameters vanish, then the equations of the analytic equivalence referred to in this lemma will also depend analytically on these parameters and will reduce to the identity transformation when the parameters vanish.*

This follows itself from the corollary of Lemma 6.1.

7. Smoothing in 3-space. Let C_1 and C_2 be two analytic arcs in 3-space with common end point P where P is simple on both arcs and projects into simple points of the projections of these arcs on the (x, y) = plane and on the (y, z) = plane. Also, the tangents at P to C_1 and C_2 are to be distinct and are to project into distinct tangents under the above projections. It will be assumed that these tangents are not parallel to the (x, z) = plane. There are two cases to consider.

(1) If P has co-ordinates (x_0, y_0, z_0) then C_1 and C_2 both lie on the same side of the plane $y = y_0$ for a neighbourhood P.

(2) C_1 and C_2 lie on opposite sides of $y = y_0$ around P.

Case (2) can be dealt with straightforwardly, but around points presenting case (1) a modification will be made so that in fact only points presenting case (2) arise.

Suppose case (1) holds. If y_1 is suitably chosen near y_0, $y = y_1$ cuts C_1 and C_2 at uniquely defined points P_1 and P_2 respectively near P. Let C_3 be the circular arc $P_1 P P_2$. It is not hard to see that the arcs C_1 and C_3 at P_1 satisfy all the conditions stated above and present case (2). A similar statement may be made about C_3 and C_2 at P_2. In the sequel all points at which case (1) holds will be dealt with in this way. Attention from now onwards can be confined to case (2).

Let $F_1 = 0$ and $F_2 = 0$ be the irreducible power series equations for the projections C_1', C_2' of C_1, C_2 on the (x, y) = plane and write $F = F_1 F_2$. Let $G(x, y)$ be a power series $\neq 0$ at the projection (x_0, y_0) of P. $C_1' \cup C_2'$ has a smooth approximation of the form $F + \lambda G = 0$, where λ is small, the sign being chosen so that C_1' and C_2' are joined up (cf. Lemma 2.2). Examining this smoothing process again it is required to show that the lines $y = y_1$ meet $F + \lambda G = 0$ in just two points, one of which lies on the smooth approximation of $C_1' \cup C_2'$ where λ is sufficiently small and y_1 is sufficiently near y_0.

To do this, suppose that the linear terms of F_1 and F_2, written in powers of $x - x_0$, $y - y_0$ are

$$a_{11}(x - x_0) + a_{12}(y - y_0)$$
$$a_{21}(x - x_0) + a_{22}(y - y_0).$$

The condition that the tangents to C_1 and C_2 at P should not be parallel to the (x, z)-plane implies that a_{11} and a_{21} are both $\neq 0$. It follows that for small λ, $F + \lambda G = F_1 F_2 + \lambda G$ contains a term in $(x - x_0)^2$. The Weierstrass preparation theorem implies that there is a power series P in $x - x_0$, $y - y_0$ and λ not vanishing at $x = x_0$, $y = y_0$, $\lambda = 0$ such that

$$P(F_1F_2 + \lambda G) \equiv (x - x_0)^2 + a(y)(x - x_0) + b(y) \equiv F'(x, y, \lambda),$$

where a and b are power series in y and λ. The above equation shows that

$$\frac{\partial F'}{\partial \lambda} \neq 0 \text{ at } x - x_0 = y - y_0 = \lambda = 0.$$

And so b contains a linear term in λ, which means that, at $y = y_0$, the discriminant of F', regarded as a quadratic in $x - x_0$, changes sign as λ changes sign. It follows at once that for a suitable choice of sign for λ, $F' = 0$ will have two real roots. It is clear that one of them will lie on the smooth approximation of $C_1' \cup C_2'$. Summing up:

LEMMA 7.1. *If case* (2) *described above holds and if* C_1', *and* C_2', *are the projections of* C_1, *and* C_2, *respectively, on the* (x, y)*-plane and* (x_0, y_0) *is the projection of* P, *then, in a neighbourhood of* (x_0, y_0) *there exists an arbitrarily good smooth approximation of* $C_1' \cup C_2'$ *cut in one point by each line* $y = y_1$.

Suppose that $H = 0$ is the equation of the projection $C_1'' \cup C_2''$ of $C_1 \cup C_2$ on the (y, z)-plane. Then the lines $y = y_1$, $z = z_1$, with (y_1, z_1) near (y_0, z_0) cut the sheet of the surface $F + \lambda G = 0$ over the above constructed approximation of $C_1' \cup C_2'$ in one point. A homeomorphism ϕ of a neighbourhood of $(0, y_0, z_0)$ on the (y, z)-plane and a neighbourhood of (x_0, y_0, z_0) on that surface is thus defined. It is thus clear that for μ small enough and of suitable sign and $K \neq 0$ at (y_0, z_0), K being a power series in $y - y_0$, $z - z_0$, the surface $H + \lambda K = 0$ cuts the (y, z)-plane in a smooth approximation of $C_1'' \cup C_2''$. And so, applying the homeomorphism ϕ, $F + \lambda G = 0$, $H + \mu K = 0$ is a curve an arc of which is a smooth approximation of $C_1 \cup C_2$ in a neighbourhood of P.

The result so obtained may be summed up in

LEMMA 7.2. *Let* C_1, C_2 *be analytic arcs with the common end point* P, *case* (2) *holding, and suppose that* $C_1 \cup C_2$ *is part of the curve* $F(x, y) = 0$, $H(y, z) = 0$. *Then, in a sufficiently small neighbourhood* U *of* P *and for* λ, μ *sufficiently small and with suitable signs, the curve* $F + \lambda G = 0$, $H + \mu K = 0$, *where* $G(x, y)$ *and* $H(y, z)$ *are analytic functions* $\neq 0$ *at* P, *has an arc which is a smooth* ϵ*-approximation of* $(C_1 \cup C_2) \cap U$, *where* ϵ *is pre-assigned.*

8. Tangents parallel to the (x, z)**-plane.** Let P be a simple point of an analytic arc C and suppose that the tangent at P is parallel to the (x, z)-plane. Let the projections of C around P on the (x, y)-plane and (y, z)-plane have respectively the equations $F(x, y) = 0$ and $H(y, z) = 0$. The surfaces $F = 0$ and $H = 0$ touch at P and so the procedure of §7 would not give an approximation of C. The procedure to be followed here is to construct a curve through P analytically equivalent to $F = H = 0$.

In more detail, let (x_0, y_0, z_0) be co-ordinates of P and let G and K be power series in $(x - x_0, y - y_0)$ and $(y - y_0, z - z_0)$, respectively, of sufficiently high order. The conditions of Lemma 6.2 are satisfied by $F = H = 0$; it follows at once from that lemma and its corollary that $F + \lambda G = 0$, $H + \mu K = 0$ is analytically equivalent to $F = H = 0$ in a neighbourhood U of P and is an ϵ-approximation of it, with pre-assigned ϵ, if λ, μ are small enough (U being fixed). In particular, $F + \lambda G = H + \mu K = 0$ contains an arc C' approximating $C \cap U$.

9. First stage of approximation in 3-space. The lemmas of the preceding sections are now to be combined, the idea being to choose co-ordinates in such a way that the situations described in these sections all arise separately. Attention will be restricted meanwhile to closed piecewise algebraic curves.

Let C be a closed piecewise algebraic curve in 3-space with arcs C_i and joints P_j and assume that the tangents to the two arcs meeting at each joint are distinct. Assume that C_i is part of an algebraic curve $\bar{C}_i$ such that at each P_j just two branches belonging to these algebraic curves meet. Choose co-ordinates as follows.

(1) The projection of $\bar{C} = \bigcup \bar{C}_i$ on the (x, y)-plane is to be one-one with the exception that some double points, projections of two simple points of $\bar{C}$, may be introduced. The two tangents at each such double point are to be distinct. In particular, the joints P_j are to project regularly.

(2) The (x, z)-plane is not to be parallel to the tangents of the C_i at the joints.

It will be assumed, in addition, that all joints have been adjusted so that case (2) as described in §7 applies at each; this adjustment itself yields an arbitrarily good approximation of the curve.

For convenience the following points on the (x, y) and (y, z)-planes will be called special:

(*a*) The projections of all singularities of $\bar{C} = \bigcup \bar{C}_i$ except the P_j;

(*b*) Double points of the projection which are projections each of two simple points;

(*c*) Projections of points where the tangent to $\bar{C}$ is parallel to the (x, z)-plane;

Then the following theorem gives a first approximation to C.

THEOREM 4. *Let $F(x, y) = 0$ and $H(y, z) = 0$ be the equations of the projections of $\bar{C}$ on the (x, y) and (y, z)-planes respectively. Let $G(x, y)$ and $K(y, z)$ be polynomials not vanishing at the projections of the P_i. Then, if G and K are arranged to have suitable signs at the projections of the P_i and if they vanish to sufficiently high order at all special points and if λ, μ are small enough, $F + \lambda G = 0$, $H + \mu K = 0$ is a curve $\bar{C}'$ with a circuit C' which is an arbitrarily good approximation of C smoothed at the P_i and otherwise singularity preserving with analytic equivalence of arbitrarily high order at the singularities.*

Proof. The plan of the proof is similar to that of Theorem 1. The P_i and all points projecting on special points are surrounded by sufficiently small neighbourhoods for the appropriate lemma to apply. Thus, P_i has a neighbourhood $U(P_i)$ such that $\bar{C}'$ is a smooth approximation of C', and arbitrarily close if λ, μ are small enough. If P projects on a special point it has a neighbourhood $U(P)$ in which $\bar{C}'$ is analytically equivalent to $F = H = 0$. That is to say, part of $\bar{C}' \cap U(P)$ is analytically equivalent to C and this implies the existence of an operator f mapping $C \cap U(P)$ into $\bar{C}'$ such that the distance of Q from $f(Q)$ is arbitrarily small if λ, μ are small enough (see Lemma 6.2). Thus, in the union of the $U(P)$, where P is either a joint of projects or a special point, the operator f is constructed and is a one-valued continuous mapping. $f(Q)$ is in all cases arbitrarily close to Q if λ, μ are small enough. It is required to extend f to all of C. C outside the $U(P)$ is made up of non-singular arcs. In sufficiently small neighbourhoods of these arcs it is not hard to see that C' is a union of non-singular arcs. The extension of f is then made as in Theorem 1.

It will be noticed that the success of the proof of the above theorem depends upon the possibility of choosing polynomials G and K vanishing to a sufficiently high order at the singularities of C and with the correct signs at the P_i (see §15).

10. Second stage of approximation in 3-space. In this section C is a circuit of a real algebraic curve $\bar{C}$ in 3-space. Co-ordinates can be chosen so that $\bar{C}$ projects on a curve $\bar{C}_0$ in the (x, y)-plane, the correspondence between these curves being one-one except that a finite number of points of $\bar{C}_0$ are each projections of a pair of simple points of $\bar{C}$. Such points of $\bar{C}_0$ are to be double points where two simple branches meet with distinct tangents. If $\bar{C}_0$ has the equation $F(x, y) = 0$ then $\bar{C}$ can be represented by equations $F = 0$, $z = f/g$, where f/g is a rational function of x and y defined except possibly at singular points of $\bar{C}_0$.

THEOREM 5. *C being as above there exists a real algebraic curve $\bar{C}'$ of which an isolated circuit C' is an arbitrarily good approximation of C, singularity preserving with analytic equivalence of arbitrarily high order at each singularity.*

Proof. Apply Theorem 3 to $\bar{C}_0$, thus obtaining a curve $\bar{C}_0'$ of which one circuit C_0' is an arbitrarily good singularity preserving approximation of C_0 with analytic equivalence of order greater than a pre-assigned integer at each singularity, while $\bar{C}_0' - C_0'$ is contained in some pre-assigned set. Let the equation of $\bar{C}_0'$ be $F'(x, y) = 0$ and consider the curve $\bar{C}'$ with equations $F' = 0$, $z = f/g$. In particular let C' be a circuit of $\bar{C}'$ projecting on C_0'.

Let P be a singularity of C and apply Lemma 6.1, F and F' playing the part of the f_i, f_i' of that lemma, and both F and F' of the Lemma being replaced by $gz - f$. If the analytic equivalence of C_0 and C_0' is of sufficiently high order around the projection of P, then the lemma quoted implies that C and C' are analytically equivalent at P, to an arbitrarily high order.

A similar argument at points P_1 and P_2 of C projecting on the same point of C_0, shows that C' is analytically equivalent to C around P_1 and P_2.

Finally, the continuity of f/g at simple points of C_0 ensures that the correspondence between C and C' is one-one and is in fact the required approximation. It is clear that C' is an isolated circuit, and in fact $\bar{C}' - C'$ can be made to lie in a pre-assigned set. The proof is thus complete.

11. The final approximation theorems in 3-space.

THEOREM 6. *Let C be a closed piecewise algebraic curve with arcs C_i. C_i is to be part of an algebraic curve $\bar{C}_i$, and at each joint of C two simple branches of the $\bar{C}_i$ are to meet with distinct tangents. Then there exists an algebraic curve $\bar{C}'$ with an isolated circuit C' which is an arbitrarily good singularity preserving approximation of C, smoothed at the joints, and with analytic equivalence of arbitrarily high order at the singularities.*

Proof. The idea of the proof has already been sketched in §5. By Theorem 4, approximate C with a circuit C^* of an algebraic curve. Then apply Theorem 5 to C^*.

THEOREM 7. *Let C be a closed piecewise analytic curve in 3-space. Then there exists an arbitrarily good singularity preserving approximation of C by an isolated circuit of a real algebraic curve.*

Proof. In a sufficiently small neighbourhood of each singularity, C can be replaced by an analytically equivalent algebraic arc **(2)**. The remainder of C can be approximated by straight line segments joined end to end. Thus, C has been approximated by a closed piecewise algebraic curve and it is not hard to see that the condition imposed in Theorem 6 can be assumed to be satisfied. The result follows at once from that theorem.

COROLLARY. *A similar result holds for an open piecewise analytic curve, for such a curve can always be closed by an auxiliary arc joining its end points.*

12. Approximation of a piecewise algebraic curve in n-space. An approximation theorem of this type has already been obtained for $n = 3$. The general result will be obtained by induction. Let C be a closed piecewise algebraic curve in n-space. C is part of a composite algebraic curve $\bar{C}$. Coordinates are to be chosen in such a way that $\bar{C}$ projects on the hyperplane $x_n = 0$ in a one-one manner and also in such a way that no tangent to $\bar{C}$ is parallel to the x_n-axis. Thus, under this projection no fresh singularities are introduced. If the arcs of C are denoted by C_i, C_i being part of a real algebraic curve $\bar{C}_i$, then it will be assumed that the joints P_j of C are the only points common to the $\bar{C}_i$. It will be remembered that a similar restriction was imposed on curves in 3-space but was eventually removed in the proofs of the approximation theorems.

Let K be the projection of C, $\bar{K}$ that of $\bar{C}$, K_i that of C_i and $\bar{K}_i$ that of $\bar{C}_i$ on $x_n = 0$. Then a point $(x_1, x_2, \ldots, x_n)$ belongs to $\bar{C}$ if and only if $(x_1, x_2, \ldots, x_{n-1})$ is on $\bar{K}$ and $x_n = f(x_1, x_2, \ldots, x_{n-1})$, where f if a continuous function which is rational on each $\bar{K}_i$ separately, projection being a birational mapping on each $\bar{C}_i$. The approximation of C is to be made by approximating K, by the induction hypothesis, in $x_n = 0$ and at the same time approximating f by a rational function F such that $|f - F|$ is small except at singularities of

$\bar{K}$. Near such a singularity the numerator and denominator of F are to differ by terms of arbitrarily high order from those of f.

Attention will now be fixed on approximation by rational functions of the type just indicated. Let A be a bounded closed set in Euclidean space. A real valued function f on A is called quasi-rational on A if there exists a finite set S of points of A, to be called the singularities of f, such that f is continuous on $A - S$ and such that there is a polynomial ψ vanishing at each point of S but at no other point of A and having the property that $f\psi$ is equal to a polynomial in some neighbourhood of each point of S.

The function F on A is called a rational approximation of the quasi-rational function f if;

(1) F is rational;

(2) Outside prescribed neighbourhoods of the points of $S, f - F$ is less than a pre-assigned number ϵ;

(3) If $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is in S, $f = \phi/\psi$ around $(\bar{x})$, ϕ being a polynomial and $F = \Phi/\Psi$, the fractions not necessarily being in lowest terms, then $\phi - \Phi, \psi - \Phi$ have orders greater than a pre-assigned integer r in the $x_i - \bar{x}_i$.

In the notation of (3) the approximation is said to be of order $\geqslant r$ at the singularity $(\bar{x})$. If, in addition, f is one-valued (as in the above situation of a function of a curve) then the inequality $|f - F| < \epsilon$ will be required to hold on all of A and the accuracy of the approximation can be specified by ϵ and r.

THEOREM 8. *Let f be a quasi-rational function on the closed bounded set A, and let $P_1, \ldots, P_n$ be the set of singular points. Let U_i be a neighbourhood of P_i and let ϵ be a pre-assigned number. Then there exists a rational approximation of f approximating to within ϵ outside the U_i, and approximating to a pre-assigned order at the P_i.*

Proof. At P_i, f can be written as a rational function with denominator ψ; say $f = \phi_i/\psi$ in a neighbourhood of P_i. f is continuous outside the U_i, and so in particular ψf is continuous outside U_i.

Construct a polynomial Φ vanishing at the P_i in a similar way to the ϕ_i. A convenient method is as follows. Let g_i be a polynomial vanishing to order r at $P_j (j \neq i)$ and such that $g_i - 1$ vanishes to order r at P_i (cf. §15). Then set

$$\Phi(x) = \sum g_i \phi_i.$$

Now $\Phi - \psi f$ is continuous on $A - \bigcup U_i$, and so has a polynomial η_1-approximation G there. Let H be a rational function vanishing to order r at the P_i, and such that $1 - \eta_2 < H < 1$ outside the U_i (Lemma 3.5). Let $\phi = \Phi - GH$. Then

$$\begin{aligned} |\phi - \psi f| &= |\Phi - GH - \psi f| \\ &= |\Phi - \psi f - G + G(1 - H)| \\ &\leqslant \eta_1 + |G|\eta_2, \end{aligned}$$

outside the U_i. Hence, outside the U_i, $|\phi/\psi - f| \leqslant (\eta_1 + |G|\eta_2)/|\psi|$.

On $A - \bigcup U_i\ \psi$ is bounded below; and so, if η_1, η_2 are small enough, this last quantity is $< \epsilon$. Now compare the fractions ϕ/ψ and ϕ_i/ψ around P_i. They have the same denominators and the difference of their numerators is

$$\phi - \phi_i = \sum g_j\phi_j - GH - \phi_i$$
$$= \sum_{j\neq i} g_j\phi_j + (g_i - 1)\phi_i - GH$$

and the terms of this expression are all of order not less than r at P, by the mode of definition of the g_i and of H.

To apply the above result to the approximation of the curve C some preliminary adjustments may be necessary. Suppose that $\bar{C}_i$ is given by

$$x_n = \frac{f_i(x_1, x_2, \ldots, x_{n-1})}{g_i(x_1, x_2, \ldots, x_{n-1})},$$

where $(x_1, x_2, \ldots, x_{n-1})$ is on $\bar{K}_i$. Suppose that g_1 vanishes at some of the joints of the K_i, say $P_j, P_k, \ldots$. These points will not include the ends of of K_1 since projection is one-one and regular at these end points. Let $h_1(x_1, x_2, \ldots, x_{n-1})$ be a polynomial vanishing on K_1 but not at $P_j, P_k, \ldots$ (by hypothesis these are not on $\bar{K}_1$). Then $f_1/(g_1 + ch_1) = f_1/g_1$ on K_1, c being a constant, and the denominator is not 0 at any P_i. If necessary, a similar adjustment is to be made for all the g_i.

Now set the fractions f_i/g_i with adjusted denominators over a common denominator g and rewrite as f_i/g. Then f defined as f_i/g on K_i is a quasi-rational function on K whose singular points, namely, the zeros of g, are all different from the P_i. The following approximation theorem can now be proved.

THEOREM 9. *Let C be a closed piecewise algebraic curve in n-space. Then there exists an arbitrarily good singularity preserving approximation of C by an isolated circuit of an algebraic curve, with smoothing at the joints and analytic equivalence of arbitrarily high order at the singularities.*

Proof. The result is true for $n = 3$ and will now be proved by induction. Assume that it is true for $n - 1$. Then, in the notation introduced at the beginning of the section, K has an approximation K' by an isolated circuit of an algebraic curve with the analytic equivalence at all singularities of the quasi-rational function f (all projections of singularities of C are to be included among these). Let F be a rational approximation of f.

Then, by Lemma 6.1, if the analytic equivalence of K and K' at singular points of f and at singularities of K is of sufficiently high order and if the approximating order of F to f at these points is sufficiently high also, then C is analytically equivalent to the curve $x_n = F(x_1, x_2, \ldots, x_{n-1})$ with $(x_1, x_2, \ldots, x_{n-1}) \in K'$, at the appropriate points of C. Apart from singular points it is clear that this curve is an approximation of C and it is certainly an isolated circuit of an algebraic curve.

The usual extension (similar to that made in §11) can be made here to closed and open piecewise analytic curves in n-space.

13. Approximation on a hypersurface. It is convenient to make a few remarks here on real algebraic varieties. A real algebraic variety V is the set of all real points on a complex algebraic variety V'. V' is understood to be contained in affine n-space over the complex numbers while V is a subset of Euclidean n-space. V' is to be chosen as the smallest complex algebraic variety containing V. Thus, V' has a simple point on V for otherwise V' could be replaced by its singular locus. Denote by $\bar{V}'$ the variety whose points are obtained from those of V' by taking the complex conjugates of all co-ordinates. Then it can be assumed that $\bar{V}' = V'$. For otherwise V' could be replaced by $\bar{V}' \cap V'$. The equations of V' can therefore be chosen to have real coefficients.

It is known that if V' is of dimension r then co-ordinates can be chosen in such a way that the equations of V' are of the form $f(x_1, x_2, \ldots, x_{r+1}) = 0$, $x_{r+1+i} = f_i(x_1, x_2, \ldots, x_{r+1})$ where f is a polynomial and the f_i $(i = 1, 2, \ldots, n - r - 1)$ are rational functions of their arguments. By the arguments made above it can also be arranged that the coefficients appearing in f and the f_i are all real numbers. It is then not hard to see that if P is a real simple point of V' then there are uniformising parameters whose real parts are real local co-ordinates, in the sense of real analytic manifolds, on V around P. That is to say, P has a neighbourhood analytically homeomorphic to a Euclidean r-cell. P is then called a simple point of V. Also the dimension of V is defined by the dimension of V', namely, r.

Now let C be a closed or open piecewise algebraic curve on a real algebraic hypersurface H in n-space. Assume that C is not contained in the singular locus of H and that the joints of C are all simple on H. Assume for the moment that $n > 3$. Choose co-ordinates so that the following conditions hold:

(1) H has a polynomial equation $F = 0$ and C is not contained in the locus with equations $F = \partial F/\partial x_n = 0$ and in particular the joints of C are not in this locus.

(2) C projects in a one-one manner on a curve K in the space $x_n = 0$.

THEOREM 10. *There exists an arbitrarily good singularity preserving approximation of C by a circuit or arc* (*the latter if C is open*) *of an algebraic curve on H, smoothed at the joints and with analytic equivalence of arbitrarily high order at the singularities.*

Proof. A smoothing approximation K' which is singularity preserving is to be constructed for K. Let $Q_1, Q_2, \ldots, Q_m$ be the singularities of K along with all the intersections of K with the projection of $F = \partial F/\partial x_n = 0$. Then K

and K' are to be made analytically equivalent at all the Q_i (Theorem 9). If this approximation is close enough and if the analytic equivalences just mentioned are of high enough order, then Lemma 6.1 implies analytic equivalence of C and part of the curve C' given by $F = 0$, with $(x_1, x_2, \ldots, x_{n-1})$ on K' around the Q_i. It is easy to see that these local approximations can be extended to the required approximation; the details of the argument are similar to those of the proof of Theorem 1. In particular, the smoothing of the approximation K' of K lifts into H since $F = 0$ can be solved for x_n around the points in question.

In the case $n = 3$ it cannot be assumed that the projection on $x_3 = 0$ is one-one on C. New singularities may be introduced. In the approximation of K by K', we can also make these curves analytically equivalent at any such new singularities. The lifting into H is carried out as before with the aid of Lemma 6.1. The theorem is thus proved for all values of n.

It is clear from the proof that C' and C can be made analytically equivalent at any further finite set of points in addition to those projecting on the Q_i.

Theorem 10 can be extended at once to the approximation of piecewise analytic curves on a hypersurface. For, let C be such a curve on the hypersurface H, satisfying a condition similar to that imposed in Theorem 10, namely, that no arc of C lies in the singular locus of H, and in particular, the joints of C are simple on H. Subdivide C into arcs such that on each of them the projection on the hyperplane $x_n = 0$ is one-one. Then apply the method of Theorem 10 to approximate each of these arcs by an algebraic arc on H, with analytic equivalence at all the singularities of C and also at all points of intersection of C with the locus having the equations $F = 0$, $\partial F/\partial x_n = 0$.

14. Approximation on a variety of any dimension.

THEOREM 11. *Let C be a piecewise analytic curve on a real algebraic variety V having at most a finite number of points in common with the singular locus of V. In particular the joints of C are to be simple on V. Then there exists an algebraic curve on V with a circuit or arc approximating C arbitrarily closely, smoothed at the joints and otherwise singularity preserving, with analytic equivalence of arbitrarily high order at the singularities.*

Proof. Suppose that V is of dimension n and is contained in $(n + r)$-space. The result is then known, by the last section, for $r = 1$, and the object is to prove it in general by induction on r.

Choose co-ordinates so that the equations of V are $f(x_1, x_2, \ldots, x_{n+1}) = 0$, where f is a polynomial, along with $x_{n+i} = f_i(x_1, x_2, \ldots, x_{n+1})$, $(i = 2, \ldots, r)$, where the f_i are rational functions of their arguments. That such a choice of co-ordinates can be made is a well-known theorem of algebraic geometry. There is thus a sequence of varieties $V_1, V_2, \ldots, V_r$, where V_s is contained in Euclidean $(n+s)$-space in which the co-ordinates are $x_1, x_2, \ldots, x_{n+s}$, V_1 has the equation $f = 0$, and V_{s+1} projects on V_s in such a way that the points of V_{s+1} are of the form $(x_1, x_2, \ldots, x_{n+s+1})$ with $(x_1, x_2, \ldots, x_{n+s}) \in V_s$ and $x_{n+s+1} = f_{s+1}\ (x_1, x_2, \ldots, x_{n+1})$. The curve C on $V = V_r$ projects on a curve C_s on V_s. Thus, the points of C_s are defined by $x_{n+s} = f_s(x_1, x_2, \ldots, x_{n+1})$ with $(x_1, x_2, \ldots, x_{n+s-1})$ on C_{s-1}. It can also be assumed that the co-ordinates are chosen so that the f_i are indeterminate at only finitely many points of C_s and that none of these is a joint of C_s.

By the induction hypothesis C_s can be approximated by a circuit or arc C_s' of an algebraic curve on V_s with smoothing at the joints, otherwise singularity preserving with analytic equivalence of arbitrarily high order at the singularities of C_s and also at all points of C_s at which any of the f_i is indeterminate. Define C_{s+1}' as the curve whose points are $(x_1, x_2, \ldots, x_{n+s+1})$ with $(x_1, x_2, \ldots, x_{n+s}) \in C_s'$ and $x_{n+s+1} = f_{s+1}(x_1, x_2, \ldots, x_{n+1})$. Then C_{s+1}' is an arc or circuit of an algebraic curve on V_{s+1}. It is required to prove that it is an approximation of C_{s+1} with smoothing at the joints and otherwise singularity preserving. Clearly it is an approximation outside neighbourhoods of the following points: (a) singularities of C_{s+1}, (b) points of C_{s+1} singular on V_{s+1}, (c) points of C_{s+1} projecting on points of C_s at which some f_i is indeterminate. That C_{s+1}' is analytically equivalent to C_{s+1} around all such points follows at once from Lemma 6.1. The inductive proof is thus complete.

15. Some special polynomials. In this section explicit constructions are given for polynomials satisfying certain special conditions, such as were required in some of the proofs earlier in this paper.

The first such polynomial is to be a polynomial $F(P;Q;x)$ in the co-ordinates $x_1, x_2, \ldots, x_n$ in n-space vanishing to the order r at P and such that $1 - F$ vanishes to the order r at Q. For convenience in defining this polynomial take P as the origin and let the co-ordinates of Q be $(x'_1, x'_2, \ldots, x'_n)$. Then the definition is to be

$$F(P;Q;x) = 1 - \frac{\sum (x'^r_i - x^r_i)}{\sum x'^{r^2}_i}.$$

The next definition is to be that of a polynomial $F(P_1, P_2, \ldots, P_n; Q; x)$ vanishing to the order r at $P_1, P_2, \ldots, P_m$ and such that $1 - F$ vanishes to the order r at Q. A suitable definition for a polynomial with this property is the product of the $F(P_i; Q; x)$ for $i = 1, 2, \ldots, m$, using the definition just given for the individual factors.

Finally, a polynomial is to be constructed which vanishes to the order r at the points $P_1, P_2, \ldots, P_m$ and has given values $k_1, k_2, \ldots, k_p$ at another set of points $Q_1, Q_2, \ldots, Q_p$. A suitable definition for such a polynomial is

$$\sum k_i F(P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, \hat{Q}_i, \ldots, Q_p; Q_i; x)$$

where the circumflex denotes the omission of the letter marked.

PART III: SHEETS OF A REAL ALGEBRAIC VARIETY

16. Definition and examples. A subset S of a real algebraic variety V will be called analytically connected if every pair of points of S can be joined by an analytic arc contained entirely in S. A subset S of V is called a sheet of V if S is analytically connected and is not contained in any larger analytically connected set on V.

This definition is slightly weaker than that given by Nash. The term "sheet" in **(1)** is equivalent to the term "proper sheet" according to the following definition.

The sheet S of V is called proper if there is a point of S with a neighbourhood U such that $U \cap V \subset S$. If this condition is not satisfied S will be called embedded.

Examples. (1) Consider the surface in 3-space with the equation $(y^2 + z^2)^2 = z^2x^3$. The cross-section of this surface parallel to the (y, z)-plane for $x > 0$ consists of two circles touching while for $x \leqslant 0$ the only real points are in the x-axis. The two circles referred to have equal radii proportional to $x^{3/2}$. It is not hard to see that this surface has two sheets. One is the x-axis and the other is the part of the surface with $x \geqslant 0$. The first statement is clear since the x-axis is analytically connected and any analytic arc on the surface through a point with $x < 0$ must lie entirely on the x-axis. To prove the second statement take any points P and Q on S with $x \geqslant 0$ and project them on the (y, z)-plane. Let the projections be P' and Q'. If these points are on the same side of $z = 0$, join them by a straight line with parametric equations $z = z(t)$, $y = y(t)$. Then

$$x(t) = \frac{(y^2(t) + z^2(t))^{2/3}}{z(t)^{2/3}}$$

is real analytic and gives the required arc on the surface joining P and Q. If P' and Q' are on opposite sides of $z = 0$ join them by an analytic arc $y = y(t)$, $z = z(t)$ such that the origin corresponds to $t = 0$ and such that this arc crosses $z = 0$ only at the origin. Assume that around $t = 0$ y and z have power series expansions $y(t) = at + \ldots$, $z(t) = bt + \ldots$ with a and b non-zero. Then $x(t)$ is an analytic function of s where $t = s^3$. This again gives an analytic arc on the surface joining P and Q, all points of the arc satisfying $x \geqslant 0$. It should be noted that when P and Q are on opposite sides of $z = 0$ then an analytic arc joining them must necessarily pass through the origin. In this example both the sheets occurring are proper.

(2) Consider now the surface S

$$(y^2 + z^2)^4 - z^4x^6 + (y^2 + z^2)^r = 0$$

where $r \geqslant 8$. Let $F = (y^2 + z^2)^4 - z^4x^6$. Then

$$-\frac{2}{3}x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z}$$

is a constant multiple of $(y^2 + z^2)^4$ and so for $r \geqslant 8$, $(y^2 + z^2)^r$ is in the square of the ideal generated by F_x, F_y, and F_z. The theorem of Samuel **(1)** shows that the surface S near the origin is shaped like two copies of the surface in example (1) placed point to point. On the other hand, the equation of S can be written as

$$x^3z^2 = \pm (y^2 + z^2)^2(1 + (y^2 + z^2)^{r-4})^{\frac{1}{2}}.$$

The procedure of the example of (1) shows that the sets $x \geqslant 0$ and $x \leqslant 0$ on this surface are analytically connected. Also, the second method of writing the equation of S shows that no analytic arc connects points with $x > 0$ to points with $x < 0$. The two sets on S given by $x \geqslant 0$ and by $x \leqslant 0$ are thus two separate sheets and are proper. The x axis is also a sheet for it is analytically connected and not contained in either of the above sheets. It is embedded.

17. Local dimension of a sheet or variety. Let V as before be a real algebraic variety contained in the complex algebraic variety $V' = \bar{V}'$ of complex dimension n. It will be convenient to speak of n as being the dimension of V. Let S be a sheet of V and let p be a point of S.

The local dimension of S at p, written as $\dim_p S$, will be said to be equal to n if every neighbourhood of p contains a simple point of V lying on S. Otherwise $\dim_p$ will be said to be less than n.

THEOREM 12. *Let S be a sheet of the real n-dimensional algebraic variety V and let p be a point of S. If* $\dim_p S = n$, *then* $\dim_q S = n$ *for all q on S.*

Proof. Since $\dim_p S = n$, a neighbourhood U of p will contain a simple point p' of V lying on S. If q is on S then there is an analytic arc A on S joining p' and q. This arc is not entirely contained in the singular locus of V since p' is simple, and so it meets this locus at a finite number of points. The last statement is equivalent to the fact that an analytic function of t for $0 \leqslant t \leqslant 1$ has only a finite number of zeros. Therefore, there is a simple point of V on S, namely, on A, in any neighbourhood of q. Therefore, $\dim_q S = n$.

COROLLARY. *If* $\dim_p S < n$ *for some p on S then S consists entirely of singular points of V.*

Proof. For if q is a simple point of V lying on S then $\dim_q S = n$ and so, by the above theorem, $\dim_p S < n$ is impossible.

If $\dim_p S = n$ for some p on S the above theorem justifies defining the dimension of S by n. In the contrary case the dimension of S will be said to be less than n.

Now, in the case where the dimension of S is less than n, S is contained in the singular locus of V. Thus, S is a sheet of a subvariety of V. Let V_0 be the smallest subvariety of V containing S and let the dimension of V_0 be r. The above theorem and its corollary show that $\dim_p S$, S being regarded as a sheet of V_0, is r at each point p of S. That is to say, the dimension on S is r. For otherwise S would be contained in the singular locus of V_0, namely, a smaller subvariety than V_0.

Thus, the dimension of S can be defined in all cases as the dimension of the smallest real algebraic variety containing S.

Some properties of n-dimensional sheets of n-dimensional varieties can be deduced from the following semi-transitivity property of analytic connectivity.

LEMMA 17.1. *Let p, q, r be points on V, q being simple. Then, if there are analytic arcs on V joining p to q and q to r, there is also an analytic arc on V joining p to r and meeting a pre-assigned neighbourhood of q.*

Proof. The union of the two arcs joining p to q and q to r is a piecewise analytic arc on V. By Theorem 11 there is an algebraic arc approximating it arbitrarily closely, smoothing at q and otherwise singularity preserving. This give the required joint of p and r. Note incidentally that p and r may be singular on the given arcs, and these singularities must be preserved along with any others.

THEOREM 13. *Let V be a real algebraic variety of dimension n and let p be a simple point. Let S be the set of all points joined by analytic arcs to p on V. Then S is an n-dimensional sheet of V and every sheet of dimension n can be obtained in this way.*

Proof. Let q_1 and q_2 be points of S. Then there are analytic arcs on V joining q_1 to p and p to q_2. Let U be a neighbourhood of p homeomorphic to an n-cell. Lemma 17.1 implies that there is an analytic arc on V joining q_1 to q_2 and meeting U in some point q, say q being a simple point of V. Take q' on the arc q_1q_2. Then there is an analytic arc joining q' to q on V, namely, part of the arc q_1q_2, and there is also an analytic arc in the cell U joining q to p. Applying again Lemma 17.1 it follows that there is an analytic arc on V joining q' to p. Therefore, q' belongs to S and so the whole arc q_1q_2 lies in S. That is to say, it has been shown that S is analytically connected.

It must be shown now that S is a maximal analytically connected set. Assume that $S \subset S'$ where S' is analytically connected. If q is a point of S' there exists an analytic arc joining p and q in S' and so in V. It follows that $S' \subset S$ and the maximality of S is established.

Obviously $\dim_p S = n$ and so S is n-dimensional.

Conversely, let S be an n-dimensional sheet of V. Then S contains by definition a simple point p of V. Every point of S can be joined to p by an analytic arc lying in S and so lying in V. The above result and the maximal property of S show that S is the set of all points which can be joined to p in V by analytic arcs.

COROLLARY 1. *Each simple point of V belongs to exactly one sheet.*

COROLLARY 2. *Each n-dimensional sheet of a real n-dimensional variety V is proper.*

Proof. If S is of dimension n there is a simple point p of V on S. It follows that there is a neighbourhood U of p such that $U \cap V$ is an n-cell. All points of $U \cap V$ can be joined to p by analytic arcs on V and so $U \cap V$ lies in the sheet determined as in the above theorem by p. This sheet must be S and so S is proper.

The notion of local dimension can also be introduced for a real algebraic variety V (and, in fact, more generally for any real algebroid variety). If p is a point of V then the local dimension of V at p, written $\dim_p V$, will be said to be n if every neighbourhood of p contains a simple point of V, that is to say, a real simple point of V' in the terminology of §13. Otherwise $\dim_p V$ will be said to be less than n.

If $\dim_p V < n$ then there is a subvariety V_0 of V consisting entirely of singular points and there is a neighbourhood U of p such that $U \cap V_0 = U \cap V$. Let V_0 be the smallest real subvariety of V with this property. Then every neighbourhood of p must contain a simple point of V_0; for otherwise V_0 could be replaced by its singular locus, a smaller subvariety. If V_0 is of dimension r then $\dim_p V_0 = r$. Define now $\dim_p V = \dim_p V_0$.

Note that a variety is not homogeneous with respect to the notion of local dimension, whereas a sheet of a variety is. For example, on the surface of example (1) in §16 points satisfying $x \geqslant 0$ have local dimension 2 whereas those satisfying $x < 0$ have local dimension 1.

18. Local study of a real algebraic variety. To get further information of the sheets of a variety some results on the local structure of a real algebraic variety are required. These will be obtained in the following three lemmas.

LEMMA 18.1. *Let p be a point of a real algebraic variety V in n-space. Then, in any pre-assigned neighbourhood of p there is a neighbourhood U which can be written as the union of the closures of a finite number of disjoint open n-cells U_i such that the union of the frontiers of the U_i is of the form $W \cap U$ where W is a real algebraic variety containing V. In addition, each U_i has p on its frontier.*

Proof. The proof will be carried out by induction on n. Assume first that $\dim_p V = n - 1$. Take p as origin and choose co-ordinates in such a way that V, which is a hypersurface, has an equation of the form

$$F = x_n^r + a_1 x_n^{r-1} + \ldots + a_r = 0,$$

where the a_i are analytic in $x_1, x_2, \ldots, x_{n-1}$ at p. This simply means that the x_n-axis does not lie in V. Let V_0 be the projection on $x_n = 0$ of the locus with equations $F = \partial F/\partial x_n = 0$. The induction hypothesis implies that there is a neighbourhood U_0 of p in $x_n = 0$ such that U_0 can be written as $\bigcup \bar{Z}_i$, where the Z_i are disjoint open $(n - 1)$-cells and $\bigcup \mathrm{Fr} Z_i = U_0 \cap W_0$, where W_0 is a variety containing V_0. For each Z_i there are two possible cases to consider.

(1) There are sets on V, say $Z_i^{(1)}, Z_i^{(2)}, \ldots, Z_i^{(s)}$, projecting homeomorphically on Z_i and having p in their closures.

(2) There are no such sets as in (1).

Let C be a cylindrical neighbourhood of p, specified as the set of all points $(x_1, x_2, \ldots, x_n)$ with $(x_1, x_2, \ldots, x_{n-1})$ in some neighbourhood of p and x_n satisfying an inequality of the type $|x_n| < k$. C can be chosen as follows. If Z_i' is a set on V projecting homeomorphically on Z_i presenting case (2) or if Z_i' projects on a set Z_i presenting case (1) but is different from $Z_i^{(1)}, Z_i^{(2)}, \ldots, Z_i^{(s)}$, then $C \cap Z_i' = \phi$. Also, C is to be taken so that the subsets $x_n = \pm k$ of C do not meet V. This choice is always possible since the x_n-axis does not lie in V.

Shrink U_0 if necessary so that $U_0 \subset C$; this can be done by the induction hypothesis. Then define U as the set of points $(x_1, x_2, \ldots, x_n)$ such that $(x_1, x_2, \ldots, x_{n-1}) \in U_0$, $|x_n| < k$ for some positive number k. The cell decomposition of U is now to be defined. The part of U over a set Z_i presenting case (1) is divided into open cells by the $Z_i^{(j)}$. On the other hand, the part of U over a set Z_i presenting case (2) is itself an n-cell. Define the U_i as the collection of all these cells. It is at once clear that the U_i are disjoint and that p is in $\bar{U}_j$ for each j.

The union of the frontiers of the U_j consists of $V \cap U$ along with the top and bottom of U and the subset of U projecting on $\bigcup \mathrm{Fr} Z_i$. The last set can be written as $U_0 \cap W_0$ where W_0 is a real algebraic variety, by the induction hypothesis. Therefore, $\bigcup \mathrm{Fr}\, U_j$ is of the form required by this lemma. Also U can be taken arbitrarily small and so the proof is complete if $\dim_p V = n - 1$.

If $\dim_p V < n - 1$, repeat the above proof with V_0 taken as the projection of V on $x_n = 0$. Here only the sets Z presenting case (2) will appear but the rest of the proof is as above.

LEMMA 18.2. *Let p be a point on a real algebraic variety V of dimension n and let W be a subvariety of V containing p. In any pre-assigned neighbourhood of p there is a neighbourhood U of p such that $V \cap U$ is the union of the closures of a set of disjoint open cells of dimensions $\leqslant n$ such that:*

(1) $\bigcup \mathrm{Fr}\, U_i = U \cap W'$ *where W' is a variety on V containing W.*

(2) *Each r-cell in the decomposition of $V \cap U$ is contained in exactly one proper sheet of V of dimension r.*

(3) $p \in \bar{U}_i$ *for each i.*

Proof. Note first that Lemma 18.1 is the special case of this lemma with V replaced by n-space. The general proof will be carried out by induction, the result being obvious for a curve. Assume that the theorem is true for any variety V such that $\dim_p V < n$ in any space. The result is then to be proved for a variety of local dimension n at p. The proof will first be carried out for a variety V in $(n + 1)$-space with $\dim_p V = n$. V must thus be a hypersurface and so co-ordinates can be chosen so that it has an equation of the form

$$F = x_{n+1}^r + a_1 x_{n+1}^{r-1} + \ldots + a_r = 0$$

where the a_i are analytic at p which is to be taken as origin. Project on $x_{n+1} = 0$ and let W_0 be the union of the projections of W and of the locus with equations $F = \partial F/\partial x_{n+1} = 0$. Apply Lemma 18.1 and use the notation used there. Then there is an arbitrarily small neighbourhood U_0 of p in $x_{n+1} = 0$ such that $U_0 = \bigcup \bar{Z}_i$, where the Z_i are disjoint open n-cells the union of whose frontiers is a variety W_1 containing W_0. In the terminology of Lemma 1, if Z_i presents case (1) there exists a finite number of sets $Z_i^{(j)}$ on V projecting homeomorphically on Z_i, p lying in the closure of each of them. Let U be chosen as in Lemma 18.1 and let $q \in V \cap U$. Then there are two cases to consider according as $\dim_q V = n$ or $\dim_q V < n$.

If $\dim_q V = n$, every neighbourhood N of q contains a simple point q' of V. Then, in a suitable neighbourhood N' of q' contained in N, there is a point q'' which is simple on V and does not project on the variety W_1 which contains the frontiers of Z_i. Then a neighbourhood of q'' projects homeomorphically into a subset of some Z_i. That is to say, q'' is in some set Z_i' projecting homeomorphically on Z_i. It follows at once, since N is any neighbourhood of q, that q is in the closure of Z_i'. By the choice of U, namely, as in Lemma 18.1, Z_i' must be one of the $Z_i^{(j)}$ having p in its closure. Hence all points q of $V \cap U$ with $\dim_q V = n$ are in the closure of some $Z_i^{(j)}$.

All points q in $V \cap U$ with $\dim_q V < n$ are contained in a subvariety V_0 of V. Apply the induction hypothesis to V_0, shrinking U if necessary. Thus, $V_0 \cap U$ is the union of the closures of a number of cells which, if taken along with the $Z_i^{(j)}$ provide the required cell decomposition of V.

The conditions (1) (2) (3) of the theorem must now be checked. Condition (1) follows from the induction hypothesis on V_0 and from the mode of construction of $Z_i^{(j)}$; (3) follows in the same way. Now (2) will be checked. $Z_i^{(j)}$ lies on exactly one proper n-dimensional sheet of V, namely, that determined by any simple point on it (Theorem 13). Let U_1 be one of the open cells of the decomposition of V_0 assumed in the induction hypothesis. Then, by this hypothesis, U_1 is contained in exactly one proper sheet S of V_0. If S is a proper sheet of V the result is proved. Suppose S is not proper. Then every neighbourhood of every point of S contains points of V not in S. Such points are also not in V_0, since S is proper in V_0. V therefore has local dimension n at such points and so the cell U_1 can be discarded, being contained in one of the $\bar{Z}_i^{(j)}$.

The proof is thus complete for a variety V with $\dim_p V = n$ contained in $(n + 1)$-space. To prove the result for a variety V of dimension n in $(n + r)$-space project V into $(n + 1)$-space. Let V_1 be the projection and let W_1 be the union of the projection of W and the variety of all points which are the projections of more than one point of V. Apply the result already obtained to V_1 with the subvariety W_1 and lift the cell decomposition so constructed to V.

LEMMA 18.3. *Let V be a real algebraic variety, W a subvariety, and p a point of W. Then there is a neighbourhood U of p such that all points of $U \cap (V - W)$ can be joined to p by analytic arcs on V meeting W only at p.*

Proof. The proof is to be carried out by induction on $\dim_p V$. Assume that the result is true for any variety whose local dimension is less than n; the theorem is obvious in the case of a curve. The proof will first be carried out taking V as n-space and W as any variety through p. There are two cases to consider.

Case (1), $\dim_p W < n - 1$. Project W on the hyperplane $x_n = 0$, the projection being W'. Let p' be the projection of p. Apply the induction hypothesis taking V as the $(n - 1)$-space $x_n = 0$ and replacing W by W'. Then there is a neighbourhood U' of p' such that all points of $U' - W'$ can be joined to p' by analytic arcs meeting W' only at p'. Also apply the induction hypothesis with V, W replaced respectively by W_1, W where W_1 is the set of all points projecting on W'. Then there is a neighbourhood U of p such that all points of $U \cap (W_1 - W)$ can be joined to p by analytic arcs in W_1 meeting W only at p. It can be assumed that U is so small that it projects inside U' and it can also be assumed to be cylindrical.

Let q be any point of $U - W$. If $q \in W_1$ there is an analytic arc in $U \cap (W_1 - W)$ joining p to q, meeting W only at p. On the other hand, if $q \notin W_1$, q projects on $q' \in U' - W'$ and so there is an analytic arc in U' joining p' and q' and meeting W' only at p'. This arc can clearly be lifted into an arc joining p and q and meeting W only at p. This completes the proof of the lemma with $V = n$-space in case (1).

Case (2), $\dim_p W = n - 1$. This time W is a hypersurface. Choose coordinates so that p is the origin and W has an equation of the form

$$F = x_n^r + a_1 x_n^{r-1} + \ldots + a_r = 0$$

where the a_i are analytic in $x_1, x_2, \ldots, x_{n-1}$ at p. Let W' be the projection on $x_n = 0$ of the locus with equations $F = \partial F/\partial x_n = 0$ and let W_1 be the set of points projecting on W'. Let $W_2 = W_1 \cap W$.

Apply the induction hypothesis with V, W replaced by W_1, W_2 respectively, thus obtaining a neighbourhood U of p such that all points of $U \cap (W_1 - W_2)$ can be joined to p by analytic arcs in W_1 meeting W only at p. Assume that U is cylindrical and is shrunk, if necessary, so that it has the properties of the neighbourhood U in Lemma 18.1. Let Z_i and $Z_i^{(j)}$ be as in that lemma. Apply the induction hypothesis with V, W replaced by the hyperplane $x_n = 0$ and W' respectively. Then there is a neighbourhood U' of p in $x_n = 0$ whose points can be joined to p by analytic arcs meeting W' only at p. Assume that U is shrunk, if necessary, so that it projects into U'. Let q be a point of $U - W$. If $q \in W_1$, it has been shown that there is an analytic arc joining p to q in $U \cap (W_1 - W)_2$ meeting W only at p. On the other hand, if $q \notin W_1$ and if q does not project on a set Z_i covered by the $Z_i^{(j)}$ then proceed as in case (1). If $q \notin W_1$ and q projects on Z_i covered by some of the $Z_i^{(j)}$ then there is an analytic arc in the interior of Z_i joining the projection q' of q to p and meeting the frontier of Z_i only at p. For the sake of definiteness assume that q lies between $Z_i^{(1)}$ and $Z_i^{(2)}$ and suppose that the above-mentioned arc from q' to p in Z_i has parametric equations

$$x_j = f_j(t), \qquad j = 1, 2, \ldots, n - 1.$$

Suppose that the points of $Z_i^{(1)}$ and $Z_i^{(2)}$ lying over this arc are given respectively by $x_n = f_n^{(1)}(t)$ and $x_n = f_n^{(2)}(t)$. Then the arc with equations

$$x_j = f_j(t), \qquad j = 1, 2, \ldots, n - 1,$$
$$x_n = hf_n^{(1)} + kf_n^{(2)},$$

for suitable h, k, is an analytic arc joining q to p in U meeting W only at p. This completes case (2).

The proof will now be carried out for any real algebraic variety V with $\dim_p V = n$. By Lemma 18.2 there exists a neighbourhood U_1 of p such that $U_1 \cap V$ is the union of the closures of disjoint open cells whose frontiers lie on a variety W_1 containing W. Also the proof of Lemma 18.2 shows that the n-cells in this decomposition project homeomorphically on n-cells in n-space, the frontiers of the latter being contained in the projection of W_1. All the cells in this decomposition whose dimensions are less than n lie on a variety W_2 and it will be assumed that W_2 contains W_1.

Apply the induction hypothesis to W_2 with the subvariety W. Then there is a neighbourhood U_2 on p such that all points of $U_2 \cap (W_2 - W)$ can be joined to p by analytic arcs in W_2 meeting W only at p. If $q \notin W_2$ then q is in the interior of an n-cell Z' projecting on an n-cell Z in n-space. Apply the result already proved for n-space with the subvariety W_3 which is the projection of W_1. Then there is a neighbourhood U_3 of the projection of p in n-space such that p can be joined by an analytic arc to any point of $U_3 - W_3$, and in particular to the projection of q. Such an arc meets the frontier of Z only at the projection of p and so can be lifted into Z'. With a suitable choice of parameter a lifted arc is still analytic at p. The neighbourhood U required by the statement of this lemma can be taken as the smallest of U_1, U_2, U_3.

19. Further properties of sheets.

THEOREM 14. *Each sheet S of a real algebraic variety V is a closed set.*

Proof. If V is of dimension n it is sufficient to prove the theorem for an n-dimensional sheet. For every other sheet is of maximal dimension in some subvariety which is itself a closed set of V.

If $p \in \bar{S}$ then every neighbourhood U of p meets S; let $q \in U \cap S$ and assume that U is open. U is a neighbourhood of q and S is n-dimensional and so, by definition, U contains a simple point q' of V, q' lying on S. By Lemma 18.3, there exists an analytic arc joining p to q' on V if U is small enough. It follows that p lies on the sheet of V determined by the simple point q' as in Theorem 13. By the corollary of that theorem this sheet is S. Since p is any point of $\bar{S}$ this shows that S is closed as required.

COROLLARY. *Every point p of a real algebraic variety belongs to some proper sheet.*

Proof. By Lemma 18.2 (2), there is a neighbourhood U of p which can be written as the union of the closures of open cells each of which is contained in some proper sheet. The point p is in the closure of each such cell and so is in the closure of some proper sheet. By the theorem just proved p lies on that sheet.

THEOREM 15. *The number of sheets of a real algebraic variety in Euclidean space is finite.*

Proof. It is sufficient to prove the theorem for sheets of dimension n of an n-dimensional variety V because all other sheets are contained in some subvariety. Assume that the variety V has infinitely many n-dimensional sheets. Take a point on each sheet. This set of points will have a limit point p which may be a point at infinity. In the latter case apply some transformation, for example, inversion in some hypersphere, to make the limit point finite. Then every neighbourhood of p meets infinitely many n-dimensional sheets of V. But, by Lemma 18.2 there exists a neighbourhood U of p such that $U \cap V$ is a finite union of closures of cells, each n-cell lying on exactly one n-dimensional sheet. Since the sheets are closed (Theorem 14), the closure of each of these n-cells lies on exactly one n-dimensional sheet. Therefore, U meets only a finite number of these sheets. The contradiction so obtained proves the theorem.

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