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WHEN IS THE INTEGRAL CLOSURE COMPARABLE TO ALL INTERMEDIATE RINGS

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Abstract

Let $R \subset S$ be an extension of integral domains, with R^* the integral closure of R in S. We study the set of intermediate rings between R and S. We establish several necessary and sufficient conditions for which every ring contained between R and S compares with R^* under inclusion. This answers a key question that figured in the work of Gilmer and Heinzer ['Intersections of quotient rings of an integral domain', *J. Math. Kyoto Univ.* **7** (1967), 133–150].

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1. Introduction

All rings considered in this paper are integral domains, commutative with identity, and all subrings are unital domains. The field of fractions of a ring R is denoted by $qf(R)$ and the integral closure by R'. We frequently use $\mathrm{Spec}(R)$ (respectively, $\mathrm{Max}(R)$) to denote the set of all prime (respectively, maximal) ideals of R. For convenience, we will use the symbol $\mathrm{Rad}(R)$ to denote the Jacobson radical of R.

For a ring extension $R \subset S$, we denote the set of intermediate rings (that is, the set of all rings T such that $R \subseteq T \subseteq S$) by $[R, S]$, the conductor of R in S (that is, $(R : S) = \{x \in R : xS \subseteq R\}$) by $(R : S)$ and the integral closure of R in S by R^*. If in addition, $P \in \mathrm{Spec}(R)$ and $T \in [R, S]$, then T_P is the localisation $T_{R\setminus P}$. Also, $\mathrm{Supp}(S/R) = \{P \in \mathrm{Spec}(R) : R_P \neq S_P\}$ and $\mathrm{MSupp}(S/R) = \mathrm{Supp}(S/R) \cap \mathrm{Max}(R)$.

Given a ring extension $R \subset S$, we say that (R, S) is a normal pair if each ring in $[R, S]$ is integrally closed in S. The concept of a normal pair was introduced by Griffin [13]. These pairs where later studied in case R is an integral domain by Davis [7]. He proved that if R is quasilocal, then (R, S) is a normal pair if and only if there exists a divided prime ideal P of R (that is, $PR_P = P$) such that $S = R_P$ and R/P is a valuation ring [7, Theorem 1]. Recently, normal pairs have received considerable attention (see [3, 5, 9, 16]).

In recent years, there has been increasing interest in ring extensions $R \subset S$ satisfying FIP (finite intermediate rings property) or FCP (finite chain property). Following [1], the extension $R \subset S$ is said to satisfy FIP if there are only a finite number of rings contained between R and S. A ring extension $R \subset S$ is said to satisfy FCP if each chain of distinct intermediate rings of this extension is finite. Any minimal ring extension is an example of an FIP or FCP extension. (Recall from [10], that a ring extension $R \subset S$ is called minimal if $[R, S] = \{R, S\}$; as usual, $\subset$ denotes proper inclusion.) Moreover, if a ring extension has FCP, then any maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ of intermediate rings between R and S consists of a finite number of 'steps' $R_i \subset R_{i+1}$ that are minimal extensions. Later, in [11], Gilmer studied FIP and FCP for the case of overring extensions of integral domains. Several authors investigated the realisation of these two conditions in the more general setting of ring extensions (see [4, 8, 9, 16]). Notice that, in [4], Ben Nasr has independently studied the set $[R, S]$, in particular, when $R \subset qf(R)$ has FCP. He established an explicit description of any intermediate ring in terms of localisations of R (see [4, Theorem 2.4]). The main tool that we use to prove our results is Theorem 2.1 in which we generalise the last cited result.

As the title of this paper suggests, our goal is to obtain a necessary and sufficient condition under which the integral closure is comparable with each intermediate ring. Precisely, Theorem 2.7 states that if $R \subset S$ satisfies FCP such that $R \subset R^* \subset S$, then $[R, S] = [R, R^*] \cup [R^*, S]$ if and only if, for each $R \subseteq J \subset R^*$ such that $J \subset R^*$ is a minimal extension, $(J : R^*) \subseteq M$ for each $M \in \mathrm{MSupp}(S/R^*)$. As a consequence, we show that if $R \subset S$ is an FCP extension such that $[R, S] = [R, R^*] \cup [R^*, S]$, then $\mathrm{Supp}(S/R^*)$ contains at most two maximal ideals (Corollary 2.10). We also show that if $S = qf(R)$ and $R \subset R' \subset qf(R)$, then $[R, qf(R)] = [R, R'] \cup [R', qf(R)]$ if and only if each intermediate ring of $[R, R'] \setminus \{R'\}$ is quasilocal (Corollary 2.14). Further examples and counterexamples are given.

Finally, any unexplained terminology is standard as in [17].

2. Main results

For T an intermediate ring of $[R, S]$ and I a proper ideal of R, write

$$T_{(I)} = \bigcap_{(M \in \mathrm{Max}(T), M \supseteq I)} T_M.$$

If $I = R$, we adopt the convention $T_{(R)} = qf(R)$.

Recall from [10, Théorème 2.2(i) and Lemme 1.3] that if $R \subset S$ is a minimal extension, then there exists a unique maximal ideal M of R (called the crucial maximal ideal of $R \subset S$) such that the canonical injective ring homomorphism $R_M \to S_M$ can be viewed as a minimal ring extension, while the canonical ring homomorphism $R_P \to S_P$ is an isomorphism for all prime ideals P of R, except M. If, in addition, $R \subset S$ is an integral extension, then M is precisely the conductor $(R : S)$ of R in S [10, Théorème 2.2(ii)].

We begin by proving the following fundamental theorem.

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Theorem 2.1. *Let $R \subset S$ be an FCP extension and let $T \in [R, S]$ such that $T \cap R^* = J$. Set $C = (J : R^*)$. Then there exists a collection $F(T)$ of prime ideals of J such that $T = J_{(C)} \cap T_1$ and $T_1 = R^*_{(C)} \cap (\bigcap_{p \in F(T)} R^*_p)$.*

Proof. Let J^* denote the integral closure of J in S. It is clear that $J^* = R^*$. Also, the extension $J \subset S$ inherits FCP from $R \subset S$. So it suffices to prove the theorem for $J = R$. The result is trivial if $R^* = R$ or $R^* = S$. Now we assume that $R \subset R^* \subset S$. Let $T \in [R, S]$ with $T \cap R^* = R$, so that (R, T) is a normal pair [2, Proposition 4]. According to [7, Theorem 1], for each $m \in \mathrm{Max}(R)$, there exists a prime ideal $p \subseteq m$ with $T_m = R_p$. First, assume that R is quasilocal with maximal ideal m. We prove that $p = m$. If $p \subset m$, then $p \not\supseteq C$ [9, Corollary 3.2]. Thus, by virtue of [6, Proposition 0], $R_p = R^*_p$, and so $T = T_m = R^*_q$. It follows that $R = T \cap R^* = R^*_q \cap R^* = R^*$, which is a contradiction. Consequently, $T = R$.

We turn now to the general case, that is, R need not be quasilocal. Let $m \in \mathrm{Max}(R)$. If $m \not\supseteq C$, then *a fortiori* $p \not\supseteq C$, and hence $R_p = R^*_p = R^*_q$ for a prime ideal q of R^* lying over p [6, Proposition 0]. Thus $T_m = R^*_p$. Now, if $m \supseteq C$, we wish to show that $T_m = R_m$. Since $R \subset R^*$ satisfies FCP, there is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = R^*$ of rings from R to R^*. For each $i = 0, \ldots, n-1$, let C_i be the crucial ideal of $R_i \subset R_{i+1}$ and let $m_i = C_i \cap R$. Then, by [9, Corollary 3.2], $m = m_i$ for some i. We have $R_{m_i} \subset R^*_{m_i}$. Indeed, if $R_{m_i} = R^*_{m_i}$, then, necessarily, $(R_i)_{m_i} = (R_{i+1})_{m_i}$. Hence $(R_i)_{C_i} = (R_{i+1})_{C_i}$, although C_i is the crucial maximal ideal of the extension $R_i \subset R_{i+1}$, which is the desired contradiction. As, in addition, $(T \cap R^*)_m = T_m \cap R^*_m = R_m$ and R^*_m is the integral closure of R_m in S_m, we conclude from the 'quasilocal' case that $T_m = R_m$.

Let $F(T) = \{p \in \mathrm{Spec}(R) : R^*_p = T_m, \text{ for some } m \in \mathrm{Max}(R), m \not\supseteq C\}$. Then

$$T = \bigcap_{m \in \mathrm{Max}(R)} T_m = \Big(\bigcap_{i=0}^{n-1} T_{m_i}\Big) \cap \Big(\bigcap_{m \in \mathrm{Max}(R), m \not\supseteq C} T_m\Big) = \Big(\bigcap_{i=0}^{n-1} R_{m_i}\Big) \cap \Big(\bigcap_{p \in F(T)} R^*_p\Big).$$

Finally, take $T_1 = [R^*_{(C)} \cap (\bigcap_{p \in F(T)} R^*_p)]$. We have $T_1 \in [R^*, S]$ and $T = R_{(C)} \cap T_1$. This completes the proof. □

Proposition 2.2. *Let $R \subset S$ be an FCP extension. If each $R \subseteq J \subset R^*$ is quasilocal, then $[R, S] = [R, R^*] \cup [R^*, S]$.*

Proof. Let $T \in [R, S]$ such that $T \cap R^* = J$, $J \neq R^*$. Using Theorem 2.1, $T = J_{(C)} \cap T_1$, where $C = (J : R^*)$ and $T_1 \in [R^*, S]$. Since, by assumption, J is quasilocal, then its maximal ideal M necessarily contains the conductor C. Thus $J_{(C)} = J_M = J$ and $T = J$. Therefore $[R, S] = [R, R^*] \cup [R^*, S]$. □

Corollary 2.3. *Suppose that R^* is quasilocal and $R \subset S$ satisfies FCP. Then $[R, S] = [R, R^*] \cup [R^*, S]$.*

Proof. Since R^* is quasilocal, then by integrality, each intermediate ring of $[R, R^*]$ is quasilocal. Hence the conclusion follows readily from Proposition 2.2. □

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Theorem 2.4 of Gilmer and Heinzer [12] leads to an investigation of domains with Prüfer integral closure and which have a unique minimal intermediate ring. As a consequence of Proposition 2.2, we generalise the above cited result for FCP extensions.

COROLLARY 2.4. *Let $R \subset S$ be an FCP extension such that R is quasilocal and $R \subset R^*$ is a minimal extension. Then R^* is the least element in the set $[R, S] \setminus \{R\}$.*

For an FCP extension $R \subset S$, the converse of Corollary 2.4 is not true, in general. Indeed, in [2, Example 29], Ayache showed that there exists an FCP extension $R \subset S$ such that $[R, S] = \{R\} \cup [R^*, S]$ but R cannot be quasilocal. However, if $S = qf(R)$ and $[R', qf(R)]$ is finite, then $[R, qf(R)] = \{R\} \cup [R', qf(R)]$ if and only if R is quasilocal and $R \subset R'$ is a minimal extension [2, Corollary 28].

To facilitate the proof of Theorem 2.7, we isolate the following Proposition, which is of some independent interest.

PROPOSITION 2.5. *Let $R \subset S$ be an FCP extension such that $R \neq R^*$ and $R^* \subset S$ is a minimal extension with crucial maximal ideal N. If $[R, S] = [R, R^*] \cup [R^*, S]$, then $(R : R^*) \subseteq N \cap R$.*

PROOF. Suppose, by way of contradiction, that $(R : R^*) \nsubseteq N \cap R$. In the light of [9, Theorem 4.2(a)], $R/(R : R^*)$ is an Artinian ring and $\{m \in \mathrm{Max}(R) : m \supseteq (R : R^*)\}$ is finite. Let $\{m_0, \dots, m_n\}$ be the set of maximal ideals of R containing $(R : R^*)$. Since $R \subset R^*$ satisfies FCP, we can always find $J \in [R, R^*]$ such that $J \subset R^*$ is a minimal extension. Hence $(J : R^*)$ is the crucial maximal ideal of $J \subset R^*$, and so $(J : R^*) \cap R = m_i$ for some $i \in \{0, \dots, n\}$ [9, Corollary 3.2]. Put $N' = N \cap J$. Then $N' \neq (J : R^*)$ (indeed, if $N' = (J : R^*)$, then $N \cap R = (J : R^*) \cap R = m_i$ and $(R : R^*) \subseteq N \cap R$). It follows that $N \cap J \nsubseteq (J : R^*)$. Then the crosswise exchange lemma [9, Lemma 2.7] provides a ring $T \in [J, S]$ such that $J \subset T$ is an integrally closed minimal ring extension. Thus $T \cap R^* = J$. Since $[R, S] = [R, R^*] \cup [R^*, S]$, it follows that $T = J$, which is a contradiction. Therefore $(R : R^*) \subseteq N \cap R$. □

The following example shows that the converse of Proposition 2.5 is false.

EXAMPLE 2.6. Let x be an indeterminate over $K = \mathbb{Q}(\sqrt{2})$. Let $S = K[x]_{(x)} = K + xK[x]_{(x)}$ and $V = K[x]_{(x+1)} = K + (x + 1)K[x]_{(x+1)}$. Set $R^* = S \cap V$. Then R^* is a semilocal principal ideal domain with maximal ideals $M = (x)R^*$ and $N = (x + 1)R^*$ such that $R^*_M = S$, $R^*_N = V$ and $R^*/M \cong R^*/N \cong K$. Put $I = M \cap N$ and let $R = \mathbb{Q} + I$. Then R is a quasilocal domain with maximal ideal I and $(R : R^*) = I$. Let $D = \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}$ and set $T = \varphi^{-1}(D)$, the inverse image of D by the canonical epimorphism $\varphi : R^* \longrightarrow R^*/I = K \times K$. Then T is a ring of $[R, R^*]$ and T is a semilocal domain with two maximal ideals. Denote $\mathrm{Max}(T) = \{M', N'\}$ such that $M' = M \cap T$ and $N' = N \cap T$. Since $\mathbb{Q}(\sqrt{2}) \times \mathbb{Q} \subset K \times K$ is a minimal extension [8, Proposition III 4(b)], so is $T \subset R^*$ and hence $(T : R^*) \in \mathrm{Max}(T)$. Without loss of generality, we assume that $(T : R^*) = M'$. Take $T' = T_{M'}$. Then $T' \subset R^*_{M'}$ is a minimal extension. As $R^*_{M'} \subseteq R^*_M = S$, it follows that $T' \in [R, S]$. On the other hand, (R^*, S) is a normal

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pair and $R^* \subset S$ is a minimal extension. Necessarily, N is the crucial maximal ideal of $R^* \subset S$. Furthermore, it is easy to verify that R^* is the integral closure of R in S. Since $T' \cap R^* = T$ and $T \neq T'$, it follows that T' is incomparable with R^*. Therefore $[R, S] \neq [R, R^*] \cup [R^*, S]$, although $(R : R^*) \subseteq$ (in fact, $=$) $N \cap R$.

We now present the titular result.

Theorem 2.7. *If $R \subset S$ satisfies FCP and $R \subset R^* \subset S$, then the following conditions are equivalent:*

(i) $[R, S] = [R, R^*] \cup [R^*, S]$*; and*

(ii) *for each $R \subseteq J \subset R^*$ such that $J \subset R^*$ is a minimal extension, $(J : R^*) \subseteq M$ for each $M \in \mathrm{MSupp}(S/R^*)$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $J \in [R, R^*]$ such that $J \subset R^*$ is a minimal extension and set $C = (J : R^*)$. Suppose, by contradiction, that there is a maximal ideal $M \in \mathrm{Supp}(S/R^*)$ such that $M \not\supseteq C$. Set $S_1 = \bigcap\{R^*_q : q \in \mathrm{Spec}(R^*), q \neq M\}$. Then [3, Corollary 2.6] ensures that $S_1 \subseteq S$ and $R^* \subset S_1$ is a minimal extension with crucial maximal ideal M. As, in addition, R^* is the integral closure of J in S_1 and $M \not\supseteq C$, then, according to Proposition 2.5, $[J, S_1] \neq [J, R^*] \cup [R^*, S_1]$. Thus there exists $T \in [R, S]$ such that neither $T \nsubseteq R^*$ nor $R^* \nsubseteq T$, which is the desired contradiction.

(ii) $\Rightarrow$ (i) Let $T \in [R, S]$ such that $T \cap R^* = J$, $J \neq R^*$. By virtue of Theorem 2.1, $T = J_{(C)} \cap T' = (J_{(C)} \cap S) \cap T'$, where $C = (J : R^*)$, $T' \in [R^*, S]$. We prove that $J_{(C)} \cap S = J$. Suppose that $J \neq J_{(C)} \cap S$ and let $\{M_1, \ldots, M_n\}$ be the set of maximal ideals of J containing C. First, notice that, since $R \subset R^*$ satisfies FCP, there is $J \subseteq J' \subset R^*$ such that $J' \subset R^*$ is a minimal extension. By assumption, all maximal ideals of $\mathrm{Supp}(S/R^*)$ contain $(J' : R^*)$ and, according to [9, Corollary 3.2], for some $i \in \{1, \ldots, n\}$, $(J' : R^*) \cap J = M_i$. Hence $(J : R^*)$ is contained in all maximal ideals of $\mathrm{Supp}(S/R^*)$. Now, since $J \subset (J_{(C)} \cap S)$ has FCP, it follows from [9, Corollary 3.2] that $\mathrm{Supp}((J_{(C)} \cap S)/J)$ is nonempty. In addition, J is integrally closed in $((J_{(C)} \cap S)$, so, by [9, Remark 6.14(b)], $\mathrm{MSupp}((J_{(C)} \cap S)/J)$ is nonempty. Take $M \in \mathrm{MSupp}((J_{(C)} \cap S)/J)$. Since $(J_{(C)} \cap S)_{M_i} = (J_{(C)})_{M_i} \cap S_{M_i} = J_{M_i}$ for each $i \in \{1, \ldots, n\}$, by the definition of support, it follows that $M_i \notin \mathrm{Supp}((J_{(C)} \cap S)/J)$ for each i, and so $M \notin \{M_1, \ldots, M_n\}$. Now let $M' \in \mathrm{Max}(R^*)$ such that $M = M' \cap J$. Clearly, $\mathrm{Supp}((J_{(C)} \cap S)/J) \subseteq \mathrm{Supp}(S/J)$, and thus $MS = S$ and, since $MS \subseteq M'S$, $M'S = S$. It follows that $M' \in \mathrm{Supp}(S/R^*)$ and it is clear that $M' \not\supseteq C$, which is a contradiction. Hence $J_{(C)} \cap S = J$ and $T = J$. Therefore $[R, S] = [R, R^*] \cup [R^*, S]$. □

Example 2.8. Let x be an indeterminate over the field of rational numbers $\mathbb{Q}$. Let $V_i = \mathbb{Q}[x]_{(x+i)} = \mathbb{Q} + M_i$ with $M_i = (x + i)V_i$ for each $i = 0, 1, 2$. Set $T = V_0 \cap V_1 \cap V_2$. Then T is a Prüfer domain with quotient field $\mathbb{Q}(x)$ and the maximal ideals of T are $\{N_i = M_i \cap T = (x + i)T : 0 \leq i \leq 2\}$ such that $T_{N_i} = V_i$ and $T/N_i \cong \mathbb{Q}$. Set $R = \mathbb{Q} + (N_0 \cap N_1)$, $S = V_2$. It is easy to verify that $R^* = T$ and $C = (R : R^*) = N_0 \cap N_1$. On the other hand, it is clear that $R \subset R^*$ is a minimal extension and (R^*, S) is a normal pair with $\mathrm{Supp}(S/R^*) = \{N_0, N_1\}$. Since C is contained in all maximal ideals of $\mathrm{Supp}(S/R^*)$, from Theorem 2.7, we see that $[R, S] = \{R\} \cup [R^*, S]$.

In [12], Gilmer and Heinzer asked which domains admit a unique minimal overring. In light of Theorem 2.7, we obtain an answer to this key question.

COROLLARY 2.9. *If $R \subset S$ satisfies FCP and $R \subset R^* \subset S$, then the following conditions are equivalent:*

(i) *R^* is the unique minimal ring of the set $[R, S] \setminus \{R\}$; and*
(ii) *$R \subset R^*$ is a minimal extension and $(R : R^*) \subseteq M$ for each $M \in \mathrm{MSupp}(S/R^*)$.*

PROOF. (i) ⇒ (ii) From the condition (i), $R \subset R^*$ is a minimal extension and each ring of $[R, S]$ is comparable with R^*. Hence $[R, S] = \{R\} \cup [R, S]$. According to Theorem 2.7, $(R : R^*) \subseteq M$ for each $M \in \mathrm{MSupp}(S/R^*)$.

(ii) ⇒ (i) By virtue of Theorem 2.7, $[R, S] = [R, R^*] \cup [R^*, S]$. Moreover, since $R \subset R^*$ is a minimal extension, $[R, S] = \{R\} \cup [R^*, S]$. This implies that R^* is the unique minimal ring of the set $[R, S] \setminus \{R\}$. □

COROLLARY 2.10. *Let $R \subset S$ be an FCP extension such that $R \subset R^* \subset S$. If $[R, S] = [R, R^*] \cup [R^*, S]$, then $|\mathrm{MSupp}(S/R^*)| \leq 2$.*

PROOF. Suppose $R \subseteq J \subset R^*$ such that $J \subset R^*$ is a minimal extension. By Theorem 2.7, $(J : R^*) \subseteq M$ for each $M \in \mathrm{MSupp}(S/R^*)$. On the other hand, since $J \subset R^*$ is a minimal integral extension, $(J : R^*) \in \mathrm{Max}(J)$. According to [14, Theorem 3.3], there exist at most two maximal ideals M_1 and M_2 of R^* such that $(J : R^*) = M_1 \cap M_2$. It follows that $|\mathrm{MSupp}(S/R^*)| \leq 2$. □

REMARK 2.11. Suppose that $[R, S] = \{R\} \cup [R^*, S]$. In view of [14, Theorem 3.3] and Corollary 2.10, if $R \subset R^*$ is an inert (integral) minimal extension or ramified (integral) minimal extension, then $|\mathrm{MSupp}(S/R^*)| = 1$.

COROLLARY 2.12. *Suppose that $R \subset R^*$ and $R^* \subset S$ are two minimal extensions with crucial maximal ideals M and N, respectively. Then $[R, S] = \{R, R^*, S\}$ if and only if $M = N \cap R$.*

PROOF. Since $R \subset R^*$ is integral minimal extension, $(R : R^*) = M$. On the other hand, $R^* \subset S$ is a minimal integrally closed extension, so [15, Lemma 3.2] gives $|\mathrm{Supp}(S/R^*)| = 1$. In addition, $N \in \mathrm{Supp}(S/R^*)$, and hence $\mathrm{Supp}(S/R^*) = \{N\}$. Therefore, by Theorem 2.7 and the maximality of M, we conclude that $M = N \cap R$. □

EXAMPLE 2.13. Let x be an indeterminate over $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Let $S = K[x]_{(x)} = K + xK[x]_{(x)}$ and $V = K[x]_{(x+1)} = K + (x + 1)K[x]_{(x+1)}$. Then $R^* = S \cap V$ is a Prüfer domain with maximal ideals M and N such that $R^*_M = S$, $R^*_N = V$ and $R^*/M \cong R^*/N \cong K$. Set $R = \mathbb{Q}(\sqrt{2}) + M$. Then R^* is the integral closure of R in S and $R \subset R^*$ is a minimal extension with conductor (crucial ideal) M. Moreover, by [3, Corollary 2.6], $R^* \subset S$ is a minimal extension with crucial maximal ideal M. By Corollary 2.12, $[R, S] = \{R, R^*, S\}$.

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If $S = qf(R)$ and R is integrally closed in S, then, for each $P \in \mathrm{Spec}(R) \setminus \{0\}$, $PS = S$. Therefore $\mathrm{Supp}(S/R) = \mathrm{Spec}(R) \setminus \{0\}$. By the application of Theorem 2.7, we next recover the equivalence (i) $\Leftrightarrow$ (iii) of [2, Theorem 27].

Corollary 2.14. *If $[R, qf(R)]$ satisfies FCP and $R \subset R' \subset qf(R)$, then the following conditions are equivalent:*

(i) $[R, qf(R)] = [R, R'] \cup [R', qf(R)]$; *and*

(ii) *each intermediate ring J ($J \neq R'$) between R and R' is quasilocal.*

Proof. (i) $\Rightarrow$ (ii) Let $J \in [R, R']$, $J \neq R'$. Since $R \subset R'$ satisfies FCP, then there exists $J_1 \in [R, R']$ such that $J \subseteq J_1$ and $J_1 \subset R'$ is a minimal extension. Set $C = (J_1 : R')$, a maximal ideal of J_1. By virtue of Theorem 2.7, the conductor C is contained in all maximal ideals of $\mathrm{Supp}(qf(R)/R')$. Since $\mathrm{Supp}(qf(R)/R') = \mathrm{Spec}(R') \setminus \{0\}$, $C \subseteq \mathrm{Rad}(R')$. It follows from integrality that $C \subseteq \mathrm{Rad}(J_1)$. By maximality of C, we find that J_1 is quasilocal with maximal ideal C. Hence J is quasilocal with maximal ideal $C \cap J$.

(ii) $\Rightarrow$ (i) Apply Proposition 2.2. □

We close this section with the following result which treats the case where S is the product of a finite number of fields.

Proposition 2.15. *Let $R \subset S$ be an FCP extension such that $R = \prod_{i=1}^{n} R_i$, $S = \prod_{i=1}^{n} K_i$ and $R \subset R^* \subset S$. Then:*

(a) $R^* = \prod_{i=1}^{n} R_i^*$ *where R_i^* is a Prüfer domain; and*

(b) $[R, S] = [R, R^*] \cup [R^*, S]$ *if and only if there exists $i \in \{1, \ldots, n\}$ such that $R_j = R_j^* = K_j$ for each $j \neq i$, $R_i \subset R_i^* \subset K_i$ and $[R_i, K_i] = [R_i, R_i^*] \cup [R_i^*, K_i]$.*

Proof. (a) By [8, Lemma III.3(d)], $R^* = \prod_{i=1}^{n} R_i^*$, where $R_i \subseteq R_i^* \subseteq K_i$ for each i. If $R_i^* = K_i$, then R_i^* is a Prüfer domain. Suppose that $R_i^* \subset K_i$. Since $R^* \subset S$ has FCP, so does $R_i^* \subset K$ [9, Proposition 3.7 (d)]. Moreover, R_i^* is integrally closed in K_i, and hence (R_i^*, K_i) is a normal pair and so R_i^* is a Prüfer domain.

(b) First, notice that, by [8, Lemma III.3(d)], any intermediate ring T of $[R, S]$ is of the form $T = T_1 \times \cdots \times T_n$, where $R_i \subseteq T_i \subseteq K_i$ for each i. Suppose there exist $i < j$ such that $R_i \subset R_i^* \subset K_i$ and $R_j \subset R_j^* \subset K_j$. Let $T = R_1 \times \cdots \times K_i \cdots \times K_j \cdots \times R_n$. Then $T \in [R, S]$ and it is clear that $T \not\subseteq R^*$ and $R^* \not\subseteq T$, which contradicts the fact that $[R, S] = [R, R^*] \cup [R^*, S]$. The second assertion is clear.

Conversely, as mentioned above, each ring T in $[R, S]$ can be uniquely expressed as a product of rings $T_1 \times \cdots \times T_i \times \cdots \times K_n$, where $T_i \in [R_i, K_i]$ for each i and so $T = K_1 \times \cdots \times T_i \times \cdots \times K_n$, since $R_j = K_j$ for each $j \neq i$. By assumption, $T_i \subseteq R_i^*$ or $R_i^* \subseteq T_i$, and hence $T \subseteq R^*$ or $R^* \subseteq T$. □

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