THE BOUNDARY OF THE NUMERICAL RANGE

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1. Introduction. In [**1**] it was shown that for a compact normal operator on a Hilbert space the numerical range was the convex hull of the point spectrum. Here it is shown that the same holds for a semi-normal operator whose point spectrum satisfies a density condition (Theorem 1). In Theorem 2 a similar condition is shown to imply that the numerical range of a semi-normal operator is closed. Some examples are given to indicate that the condition in Theorem 1 cannot be relaxed too much.

2. The bounded operator T on a Hilbert space H is hyponormal if $T^*T \geqq TT^*$ and semi-normal if either T or T^* is hyponormal. A useful reference is [**7**]. Some properties are now given.

LEMMA 1. *Let T be semi-normal on H. Then $\overline{W(T)} = \text{co Sp } T$, where $W(T)$ denotes the numerical range of T,* Sp T *denotes the spectrum of T and* co *denotes convex hull.*

This is due to Putnam [**6**] and Stampfli [**8**] for hyponormal operators; the extension is trivial.

LEMMA 2. *Let T be semi-normal on H. Then the extreme points of $W(T)$ lie in the point spectrum of T.*

Proof. For normal operators this is due to MacCluer [**4**] and for hyponormal operators to Stampfli [**9**]. So suppose T^* is hyponormal. Write ex A for the extreme points of A and pSp T for the point spectrum of T. Then $z \in \text{ex } W(T)$ implies $z^* \in \text{ex } W(T^*)$; but then $z^* \in \text{pSp } T^*$ and so $z \in \text{pSp } T$ ([**8**, Lemma 2]).

LEMMA 3. *Let T be a hyponormal operator, L a support line for $W(T)$ and*

$$N = \bigcup_{z \in L} \{x : (Tx, x) = z\,\|x\|^2\}.$$

Then N is a reducing subspace for T and $T \mid N$ is normal.

Proof. It is shown in [**2,** Lemma 2] that N is a subspace and that

$$N = \{x : e^{-i\theta}(T - zI)x = e^{i\theta}(T^* - z^*I)x\}$$

for all $z \in L$, where θ is the acute angle between L and the x-axis. So if z is any fixed point on L and $S = T - zI$,

$$N = \{x : Sx = e^{2i\theta}S^*x\}.$$

But as S is hyponormal, Lemma 3 of [**9**] gives that N reduces S and $S \mid N$ is normal. So the same holds for T. That, for any operator, such a subspace N is reducing whenever it is invariant is observed in [**3**] (page 406).

3. The main results

THEOREM 1. *Let T be a semi-normal operator and assume* pSp $T \cap C$ *dense in*

$$C = \text{Sp } T \cap \partial(\text{co Sp } T).$$

Then $W(T) = \text{co(pSp } T)$.

Proof. We suppose first that T is hyponormal. For $z \in W(T)$ we wish to show $z \in \text{co(pSp } T)$ and we may suppose $z = 0$.

(i) Suppose $0 \in \partial W(T)$, the boundary of the numerical range, and that im $W(T) \leqq 0$. Let L be the horizontal support line at 0. Let N be as in Lemma 3 so that $T_0 = T \mid N$ is Hermitian. Write $L_0 = L \cap W(T)$. If $L_0 = \{0\}$ or 0 is an end-point of L_0, then $0 \in \text{ex } W(T)$ by Lemma 2. Suppose L_0 is the interval (α, β) where $\alpha < 0 < \beta$; if either end-point is in L_0 it is in pSp T and the argument is shortened. Then Sp $T_0 \cap (\alpha, 0)$ and Sp $T_0 \cap (0, \beta)$ meet pSp T_0. For α and β are not isolated points of pSp T_0, so Sp $T_0 \cap (\alpha, 0)$ and Sp $T_0 \cap (0, \beta)$ are non-empty. Also we can find $z, z', y \in \text{Sp } T_0$ with $\alpha < z' < z < 0 < y < \beta$. Let M be a circular neighbourhood of z not meeting z' or 0. If M meets $\partial(\text{co Sp } T)$ in the lower half-plane, at z'' say, choose a neighbourhood M_0 of z within the triangle z', z'', y, with $M_0 \subset M$ so that $M_0 \cap C \subset L_0$. Then by the density hypothesis pSp T meets $M_0 \cap C$. Similarly it meets $(0, \beta)$, so $0 \in \text{co(pSp } T)$.

(ii) Now suppose $0 \in \text{int } W(T)$ and that a square centered at 0 with a corner at $(4d, 4id)$ lies in $W(T)$. If $A = (2d, 2id)$, $A \in \overline{\text{co}}(\text{ex(co Sp } T))$. So choose A' in the same quadrant with $|A - A'| < d$ and $A' \in \text{co(ex(co Sp } T))$, say

$$A' = \sum_{i=1}^{n} \lambda_i z_i$$

where $z_i \in \text{Sp } T \cap \partial(\text{co Sp } T)$, λ_i are non-negative and $\sum_{i=1}^{n} \lambda_i = 1$. Let $z_i' \in \text{pSp } T$ with $|z_i - z_i'| < d/n$. So $A'' = \sum \lambda_i z_i'$ lies in co(pSp T) and A'' is in the first quadrant. Similarly for the other quadrants. So $0 \in \text{co(pSp } T)$.

Now suppose that T^* is hyponormal. Suppose L^* is a support line for $W(T^*)$ and N^* is the reducing subspace for T^* provided by Lemma 3. Then L supports $W(T)$. Also N^* reduces T; indeed N^* is just the subspace N since $(Tx, x) = z \|x\|^2$ whenever $(T^*x, x)^* = z \|x\|^2$. Since $T \mid N$ is normal, the proof in (i) proceeds as before. For (ii) we only need the convexoid property provided by Lemma 1 for T^* and hence for T.

COROLLARY. *If T is hyponormal and satisfies the conditions of the theorem then* $W(T^*) = \text{co(pSp } T^*)$.

Proof. Since $(\text{pSp } T)^* \subseteq \text{pSp } T^*$, we have $W(T^*) \subset \text{co(pSp } T^*)$ and the result follows.

The following theorem is of the same kind. The spectral case when T is normal was proved by Meng [**5**] by a direct method.

THEOREM 2. *Let T be semi-normal. Then a necessary and sufficient condition that $W(T)$ be closed is that* ex(co Sp T) *lie in* pSp T.

Proof. The condition implies $\text{co}\,\text{Sp}\,T \subseteq \text{co}(\text{pSp}\,T)$ so $\overline{W(T)} \subseteq W(T)$, using Lemma 1, so the condition is sufficient. Also the condition is necessary for, if $W(T)$ is closed, $\text{ex}\,W(T) = \text{ex}(\text{co}\,\text{Sp}\,T)$. But $\text{ex}\,W(T) \subseteq \text{pSp}\,T$ (Lemma 2).

Some examples are now given to illustrate the conditions.

(i) Let T be the compact diagonal operator with diagonal elements

$$1, i, \frac{1+i}{2}, \frac{1+i}{3}, \frac{1+i}{4}, \ldots.$$

Here the set C of Theorem 1 consists of $1, i, 0$ so the condition of Theorem 1 is not satisfied but the conclusion holds. This case is covered by the theorem of [**1**] referred to in §1. Here $W(T)$ is not closed. Note that for compact operators normality and semi-normality are equivalent.

(ii) Let T_1 be the diagonal operator with diagonal elements

$$1, 1/2, 1/3, \ldots,$$

let $T_2 = T_1 + iI$ and let $T = T_1 \oplus T_2$. Here the set C is $\text{Sp}\,T$. Also $W(T)$ is the square having corners $0, 1, 1+i, i$, without the side $[0, i]$. Theorem 1 applies, Theorem 2 does not.

(iii) The following example shows that $\text{pSp}\,T$ dense in $\partial\,\text{Sp}\,T$ is not sufficient for Theorem 1. Let T_1 be a diagonal operator with diagonal entries on the lines

$$L_n = \left[\frac{1}{n}, \frac{1}{n} + i\right], \qquad n = 1, 2, \ldots$$

and including the end-points. Let T_2 be Hermitian with $W(T_2) = (0, 1)$ and let $T = T_1 \oplus iT_2$. Then $W(T)$ is the square having corners $0, 1, 1+i, i$ but with the corners $0, i$ removed. So $W(T) \neq \text{co}(\text{pSp}\,T)$ but $\text{pSp}\,T$ is dense in $\partial\,\text{Sp}\,T = \text{Sp}\,T$.

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