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Universal Inner Functions on the Ball

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Abstract. It is shown that given any sequence of automorphisms $(\phi_k)_k$ of the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$ such that $\|\phi_k(0)\|$ tends to 1, there exists an inner function I such that the family of "non-Euclidean translates" $(I \circ \phi_k)_k$ is locally uniformly dense in the unit ball of $H^\infty(\mathbb{B}_N)$.

1 Introduction

The existence of universal objects in spaces of holomorphic functions is a subject which was thoroughly studied during the last decade by many mathematicians. Its origin goes back to the work of Birkhoff in 1929, who proved that there exists an entire function whose integer translates are dense in the space of entire functions endowed with the compact-open topology. Since this work, many other universal phenomena related to holomorphic functions have been exhibited, such as universal Taylor series [7] or universal radial limits of holomorphic functions [2].

Universality is often considered in Fréchet spaces of holomorphic functions, where the properties of the norm of the functions is irrelevant. In this paper, however, bounding our functions is one of our primary concerns. The universal phenomenon which is discussed here was first exhibited by Heins [6], who established the existence of a Blaschke product B and of a sequence $(\phi_k)_k$ of automorphisms of the unit disk $\mathbb{D}$ such that the functions $(B \circ \phi_k)_k$ are locally uniformly dense in the unit ball of $H^\infty(\mathbb{D})$. In fact, Gorkin and Mortini [5] have shown that for every sequence of automorphisms of $\mathbb{D}$ with $(\phi_k(0))_k$ tending to the boundary, there exists a Blaschke product that is universal for the unit ball of $H^\infty(\mathbb{D})$ relative to the sequence of automorphisms $(\phi_k)_k$.

For the ball $\mathbb{B}_N$ of $\mathbb{C}^N$, Chee [3] proved in 1979 that there is a function $f \in H^\infty(\mathbb{B}_N)$, bounded by 1, and a sequence of automorphisms $(\phi_k)_k$ of $\mathbb{B}_N$ such that $(f \circ \phi_k)_k$ is dense in the unit ball $\mathcal{B}$ of $H^\infty(\mathbb{B}_N)$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{B}_N$. In 1979, nobody knew whether an inner function on $\mathbb{B}_N$ exists. It was just recently that Gauthier and Xiao [4], applying Alexandrov's density theorem, have shown that a universal function in the sense of Chee can be chosen to be inner.

Our aim in this paper is to prove a result similar to that of Gorkin and Mortini for the unit ball.

Theorem 1.1 *Let $(\phi_k)_k$ be a sequence of automorphisms of the ball such that $(\phi_k(0))_k$ converges to the boundary. There exists an inner function I such that the sequence $(I \circ \phi_k)_k$ is dense in $\mathcal{B}$.*

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Our method differs from that of [5], since in that paper the authors work in the corona of $H^\infty(\mathbb{D})$, and the corona of $H^\infty(\mathbb{B}_N)$ is not so well known. It differs also from the work of [4], because we must take into account that $(\phi_k(0))_k$ may approach the boundary tangentially. The paper is organized as follows. In Section 2, we collect several results which will be useful for our purpose. Section 3 contains our main construction, which is done by induction. Finally, in Section 4 we prove Theorem 1.1.

We end this introduction with a word on notations. As was mentionned previously, $\mathbb{B}_N = \{z \in \mathbb{C}^N ; \|z\| < 1\}$ is the unit ball of $\mathbb{C}^N$, whereas $\mathcal{B} = \{f \in H^\infty(\mathbb{B}_N) ; \|f\|_\infty \leq 1\}$ is the (closed) unit ball of $H^\infty(\mathbb{B}_N)$. Throughout this paper, $\mathcal{B}$ will be equipped with the topology of uniform convergence on compacta of $\mathbb{B}_N$ and $A(\mathbb{B}_N)$ is the ball algebra, namely

$$A(\mathbb{B}_N) = \{f \in H^\infty(\mathbb{B}_N),\ f \text{ is continuous on } \overline{\mathbb{B}_N}\}.$$

A class of holomorphic functions between $A(\mathbb{B}_N)$ and $H^\infty(\mathbb{B}_N)$ will play an important role in the construction of our inner function, and $\tilde{A}(\mathbb{B}_N)$ is the class of all $f \in H^\infty(\mathbb{B}_N)$ that have a continuous extension to $\mathbb{B}_N \cup \Gamma_f$ for some set $\Gamma_f \subset \partial\mathbb{B}_N$ with $\sigma(\Gamma_f) = 1$, where σ is the normalized Lebesgue measure on $\partial\mathbb{B}_N$. Lastly, if K is a compact set and $f \in C(K)$, $\|f\|_{C(K)}$ will mean $\sup_{z\in K} |f(z)|$.

2 Useful Lemmas

We will need several lemmas for our purpose. The first one can be found in [3, 4].

Lemma 2.1 *If a sequence of holomorphic functions on a domain is bounded by* 1*, and converges at some point to a value of modulus* 1*, then the sequence converges to this value uniformly on compact subsets of the domain.*

Our next result is a lemma of approximation of functions in $\mathcal{B}$ by functions in $A(\mathbb{B}_N)$. It is due to Chee [3].

Lemma 2.2 *Let h be in $\mathcal{B}$, $0 < R < 1$, $\varepsilon > 0$ and $u \in \partial\mathbb{B}_N$. Then there exists $g \in A(\mathbb{B}_N)$ such that*

- $\|g\|_\infty \leq 1$,
- $\|h - g\|_{C(R\overline{\mathbb{B}_N})} < \varepsilon$,
- $g(u) = 1$.

The existence of inner functions on $\mathbb{B}_N$ was a long-standing open problem, until it was proved by Aleksandrov that they are dense in $\mathcal{B}$. We need the following version of Alexandrov's interpolation theorem [1]

Lemma 2.3 *Let g be in $A(\mathbb{B}_N)$ and $G \in \tilde{A}(\mathbb{B}_N)$ such that there exists a non-zero function $U \in A(\mathbb{B}_N)$ with $|g| + |U| \leq 1$ on $\overline{\mathbb{B}_N}$. Then there exists an inner function I such that $I - g \in GH(\mathbb{B}_N)$.*

In particular, I interpolates the values of g on the zero-variety of G.

3 Main Construction

We suppose in this section that $(\phi_n)_{n\geq 1}$ is a sequence of automorphisms of $\mathbb{B}_N$ such that $\phi_n(0) \to v \in \partial\mathbb{B}_N$, $\phi_n^{-1}(0) \to w \in \partial\mathbb{B}_N$. We fix (h_j) a dense sequence in $\mathcal{B}$, and let $K_j = \{z \in \mathbb{B}_N\ ;\ \|z\| \leq 1 - 1/2^j\}$. By induction, we build

- a sequence of inner functions $I_1, \dots, I_k$,
- a sequence of integers $m_1, \dots, m_k$,
- for each k, a sequence of integers $(p_l^k)_{l\geq 1}$, which will be a subsequence of $(p_l^{k-1})_{l\geq 1}$

satisfying the following five properties:

(P1): $\forall j \leq k,\ I_j \circ \phi_{m_j}^{-1}(\phi_{p_l^k}(0)) \longrightarrow_{l\to+\infty} 1$;
(P2): $\forall z \in K_k$, $|I_k \circ \phi_{m_k}^{-1}(z) - 1| \leq 1/2^k$;
(P3): $\forall j \leq k-1$, $|I_k \circ \phi_{m_k}^{-1}(\phi_{m_j}(0)) - 1| \leq 1/2^k$;
(P4): $\forall j \leq k-1$, $|I_j \circ \phi_{m_j}^{-1}(\phi_{m_k}(0)) - 1| \leq 1/2^k$;
(P5): $\|I_k - h_k\|_{C(K_k)} \leq 1/2^k$.

We first explain how to obtain I_1. First, by Lemma 2.2 one may approximate h_1 by a function e_1 in $A(\mathbb{B}_N)$ such that $\|e_1\|_\infty = 1,\ e_1(w) = 1$ and

$$\|e_1 - h_1\|_{C(K_1)} < 1/8. \tag{3.1}$$

By continuity of e_1 at w, we may choose m_1 large enough such that

$$\forall z \in K_1,\ |e_1(\phi_{m_1}^{-1}(z)) - 1| < 1/8 \tag{3.2}$$

(this can be seen, for example, as an application of Lemma 2.1). This m_1 being kept fixed, the sequence $(\phi_{m_1}^{-1}(\phi_n(0)))_n$ converges to $u = \phi_{m_1}^{-1}(v)$ which belongs to $\partial\mathbb{B}_N$. We then "replace" e_1 by f_1 in $A(\mathbb{B}_N)$ such that

$$\|f_1\|_\infty = 1,\ f_1(u) = 1, \quad \|f_1 - e_1\|_{C(K_1\cup\phi_{m_1}^{-1}(K_1))} < 1/16.$$

We set $P_u(z) = \langle z, u\rangle$. $(P_u(\phi_{m_1}^{-1}(\phi_n(0))))_n$ is a sequence of $\mathbb{D}$ that converges to 1. Hence, we may consider an increasing sequence of integers $(p_l^1)_l$ such that the sequence $w_l = P_u(\phi_{m_1}^{-1}(\phi_{p_l^1}(0)))$ is a Blaschke sequence in $\mathbb{D}$. Let $M \geq 1$ to be fixed later, and consider B the Blaschke product whose zeros are each w_l (of order 1), and 0 (of order M). The function $G(z) = B(P_u(z))$ belongs to $\tilde{A}(\mathbb{B}_N)$ (discontinuities on $\partial\mathbb{B}_N$ can only occur when $|P_u(z)| = 1$). We are almost ready to apply Alexandrov's interpolation theorem. The last step is to consider

$$g_1(z) = (1-\delta)f_1(z) + \delta\left(\frac{1+P_u(z)}{2}\right)^2$$

and $U(z) = \delta\left(\frac{1-P_u(z)}{2}\right)^2$ where $\delta > 0$ is so small that

$$\|g_1 - e_1\|_{C(K_1\cup\phi_{m_1}^{-1}(K_1))} < 1/8. \tag{3.3}$$

Then U does not vanish on $\mathbb{B}_N$, and a straighforward computation shows that $|g_1| + |U| \leq 1$. By Lemma 2.3, there exists an inner function I^M such that $I^M - g_1$

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belongs to $GH(\mathbb{B}_N)$. In particular, the partial derivatives $D^\alpha I^M$ satisfy $(D^\alpha I^M)(0) = (D^\alpha g_1)(0)$ for all multi-indices α with $|\alpha| \leq M$. A simple normal family argument leads to the existence of M such that

$$(3.4) \qquad \|I^M - g_1\|_{C(K_1 \cup \phi_{m_1}^{-1}(K_1))} < 1/4.$$

We set I_1 to this I^M and prove that it verifies all the properties (P1)–(P5). For (P3) and (P4), there is nothing to prove. (P5) is an easy consequence of (3.1), (3.3) and (3.4), whereas (P2) follows from (3.2), (3.3) and (3.4). To prove (P1), it suffices to observe that

$$I_1 \circ \phi_{m_1}^{-1}(\phi_{p_l^1}(0)) = g_1 \circ \phi_{m_1}^{-1}(\phi_{p_l^1}(0)).$$

Indeed, each $\phi_{m_1}^{-1}(\phi_{p_l^1}(0))$ is a zero of G. Since g_1 belongs to $A(\mathbb{B}_N)$, $g_1(u) = 1$ and $(\phi_{m_1^{-1}}(\phi_{p_l^1}(0)))_l$ converges to u, (P1) is established.

We now suppose that the construction has been done up to step k, and explain how to perform step $k+1$. The proof is almost the same, except that we have to take into account properties (P3) and (P4). We first extract from $(p_l^k)_{l\geq 1}$ a subsequence $(p_l')_{l\geq 1}$ such that

$$(3.5) \qquad \forall l \geq 1, \forall j \leq k, \ \left| I_j \circ \phi_{m_j}^{-1}(\phi_{p_l'}(0)) - 1 \right| \leq 1/2^{k+1}.$$

By Lemma 2.2, one may find a function $e_{k+1} \in A(\mathbb{B}_N)$ such that

$$\|e_{k+1}\|_\infty = 1, \ e_{k+1}(w) = 1 \quad \text{and} \quad \|e_{k+1} - h_{k+1}\|_{C(K_{k+1})} < 1/2^{k+3}.$$

We then choose m_{k+1} from the sequence p_l' such that

$$(3.6) \qquad \forall j \leq k, \ |e_{k+1}(\phi_{m_{k+1}}^{-1}(\phi_{m_j}(0))) - 1| < 1/2^{k+3},$$

and for all $z \in K_{k+1}$, $|e_{k+1}(\phi_{m_{k+1}}^{-1}(z)) - 1| < 1/2^{k+3}$. Let $u_{k+1} \in \partial\mathbb{B}_N$ be the limit of $\left(\phi_{m_{k+1}}^{-1}(\phi_n(0))\right)_n$. We find f_{k+1} in $A(\mathbb{B}_N)$ such that $\|f_{k+1}\| = 1$, $f_{k+1}(u_{k+1}) = 1$, and

$$(3.7) \qquad \|f_{k+1} - e_{k+1}\|_{C\left(K_{k+1} \cup \phi_{m_{k+1}}^{-1}(K_{k+1}) \cup \left\{\phi_{m_{k+1}^{-1}}(\phi_{m_j}(0)); j\leq k\right\}\right)} < 1/2^{k+3}.$$

As was done for the first step, using a Blaschke product, a third function g_{k+1}, and Alexandrov's density theorem it is possible to extract from $(p_l^k)_l$ a subsequence $(p_l^{k+1})_l$ and to build an inner function I_{k+1} such that

$$(3.8) \qquad |I_{k+1}(z) - f_{k+1}(z)| < 1/2^{k+3}$$

for all z belonging to $K_{k+1} \cup \phi_{m_{k+1}}^{-1}(K_{k+1}) \cup \{\phi_{m_{k+1}}^{-1}(\phi_{m_j}(0)); j \leq k\}$ and

$$I_{k+1}(\phi_{m_{k+1}}^{-1}(\phi_{p_l^{k+1}}(0))) \to 1 \text{ as } l \text{ goes to } \infty.$$

We claim that I_{k+1} satisfies all the properties (P1)–(P5).

- (P1): the property is straightforward for $j \leq k$ because $(p_l^{k+1})_l$ is a subsequence of $(p_l^k)_l$. For $j = k+1$, it follows immediately from the contruction.
- (P2) and (P5) are easy.
- (P3): this follows from (3.6), (3.7), and (3.8).
- (P4) is satisfied by (3.5), since m_{k+1} is one of the p_l'.

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4 Proof of Theorem 1.1

We consider a sequence of automorphisms $(\phi_n)_n$ of the unit ball of $\mathbb{C}^N$ such that $(\|\phi_n(0)\|)_n$ tends to 1. Using the representation of automorphisms of $\mathbb{B}_N$ (see [8, Theorem 2.2.5]), it is easy to check that $(\|\phi_n^{-1}(0)\|)_n$ converges to 1, too. Picking a subsequence if necessary, one may assume that $(\phi_n(0))_n$ converges to $v \in \partial\mathbb{B}_N$, and that $(\phi_n^{-1}(0))_n$ converges to $w \in \partial\mathbb{B}_N$. We then consider the inner functions (I_k) and the integers (m_k) exhibited in the previous section. By property (P2), the product $I = \prod_{j\geq 1} I_j \circ \phi_{m_j}^{-1}$ converges uniformly on compact subsets of $\mathbb{B}_N$. On the other hand, it is well known that the product of inner functions is again an inner function (see [4, Lemma 5]). By property (P5), the sequence (I_k) is dense in $\mathcal{B}$, and it suffices to prove that $I \circ \phi_{m_k} - I_k$ tends to 0 uniformly on compact subsets of $\mathbb{B}_N$. Writing

$$I \circ \phi_{m_k} - I_k = \Big(\prod_{j\neq k} I_j \circ \phi_{m_j}^{-1} \circ \phi_{m_k} - 1\Big) I_k,$$

and using Lemma 2.1, it is enough to prove that $\prod_{j\neq k} I_j \circ \phi_{m_j}^{-1} \circ \phi_{m_k}(0)$ converges to 1 as k tends to $+\infty$. Let us write $I_j \circ \phi_{m_j}^{-1} \circ \phi_{m_k}(0) = 1 + u_{j,k}$. By property (P4), for $j \leq k-1$, one has $|u_{j,k}| \leq 2^{-k}$. Now, by property (P3), for $j \geq k+1$, $|u_{j,k}| \leq 2^{-j}$. Hence we get

$$\begin{aligned}\Big|\prod_{j\neq k} I_j \circ \phi_{m_j}^{-1} \circ \phi_{m_k}(0) - 1\Big| &\leq \prod_{j\neq k}(1 + |u_{j,k}|) - 1\\ &\leq (1 + 2^{-k})^{k-1} \prod_{j=k+1}^{+\infty} (1 + 2^{-j}) - 1,\end{aligned}$$

and this last term goes to 0 as k goes to infinity.

Remark 4.1 It is trivial that if (ϕ_k) is a family of automorphisms of the ball such that $\|\phi_k(0)\| \leq r < 1$ for any k, then for any inner function I, the sequence $(I \circ \phi_k)$ cannot be dense in $\mathcal{B}$. So our result reads :

> Let(ϕ_k) be a sequence of automorphisms of $\mathbb{B}_N$. There exists an inner function I such that the set $(I \circ \phi_k)$ is dense in $\mathcal{B}$ if and only if there is a subsequence (ϕ_{n_k}) such that $\|\phi_{n_k}(0)\| \to 1$.

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