INEQUALITIES ASSOCIATED WITH THE TRIANGLE

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(received December 14, 1964)

1. Introduction. Let R, r, s represent respectively the circumradius, the inradius and the semiperimeter of a triangle with sides a, b, c. Let f(R, r) and F(R, r) be homogeneous real functions. Let q(R, r) and Q(R, r) be real quadratic forms. The latter functions are thus a special case of the former. Our main result is to derive the strongest possible inequalities of the form

$$(1) \qquad q(R, r) \le f(R, r) \le s^2 \le F(R, r) \le Q(R, r) ,$$

with equality only for the equilateral triangle. Various applications are considered including a graphical representation of relations involving R, r, s. We prove the following theorems.

THEOREM 1. *Let f(R, r) and F(R, r) be homogeneous real functions. Then the strongest possible inequalities of the form*

$$f(R, r) \le s^2 \le F(R, r) ,$$

with equality only for the equilateral triangle, occur when

$$(2) \qquad f(R, r) = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{(R^2 - 2Rr)} ,$$

and

$$(3) \qquad F(R, r) = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{(R^2 - 2Rr)} .$$

Canad. Math. Bull. vol. 8, no. 5, October 1965.

THEOREM 2. Let $q(R, r)$ and $Q(R, r)$ be quadratic forms with real coefficients. Then the strongest possible inequalities of the form

$$q(R, r) \leq s^2 \leq Q(R, r) ,$$

with equality only for the equilateral triangle, occur when

$$q(R, r) = 16Rr - 5r^2 , \tag{4}$$

and

$$Q(R, r) = 4R^2 + 4Rr + 3r^2 . \tag{5}$$

2. A graphical representation. It is well known that a, b, c are the roots of the cubic equation

$$x^3 - 2sx^2 + (s^2 + 4Rr + r^2)x - 4Rrs = 0 . \tag{6}$$

A brief treatment of this equation is given in Blundon [1]. Let $D = (a - b)^2 (b - c)^2 (c - a)^2$ be the discriminant of the equation (6). The fact that a, b, c are real gives $D \geq 0$, with equality only for isosceles triangles (see Marsh [6]). It is known that

$$D = 4r^2[4R(R - 2r)^3 - (s^2 + r^2 - 10Rr - 2R^2)^2] ,$$

whence

$$(s^2 + r^2 - 10Rr - 2R^2)^2 \leq 4R(R - 2r)^3 , \tag{7}$$

with equality only for isosceles triangles. We may replace (7) by the simultaneous inequalities

$$s^2 \geq 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{(R^2 - 2Rr)} , \tag{8}$$

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{(R^2 - 2Rr)} . \tag{9}$$

We establish in the following lemma a representation in graphical form of relations involving R, r, s. This is convenient but not essential for the proof of Theorem 1. Further, it is hoped that such a representation may be a useful tool in the solution of related problems.

LEMMA. To within similarity, there is a one-one correspondence between all triangles and all points $(R/r, s/r)$ of a set S of the Cartesian plane.

Proof. The reference to similarity follows at once from considerations of homogeneity. Consider the set S of points (x, y) in the Cartesian plane satisfying (10). Let $R/r = x$ and $s/r = y$. From (7) it follows that every class of similar triangles is represented by a unique point in S such that y is positive and

$$(10) \qquad (y^2 + 1 - 10x - 2x^2)^2 \leq 4x(x-2)^3 .$$

This inequality is equivalent to the simultaneous inequalities

$$(11) \qquad y^2 \geq 2x^2 + 10x - 1 - 2(x-2)\sqrt{(x^2 - 2x)} ,$$

$$(12) \qquad y^2 \leq 2x^2 + 10x - 1 + 2(x-2)\sqrt{(x^2 - 2x)} ,$$

and equality holds in (10), (11), (12) only for isosceles triangles.

To complete the proof of the Lemma, we must show that to every point of the region S there corresponds a class of similar triangles. Choose an arbitrary point (x, y) in the region S; this is equivalent to choosing arbitrary positive real numbers R, r, s satisfying (7). Form the equation (6) and call its roots a, b, c. It is sufficient to prove that a, b, c are real and positive and further that they satisfy the triangular inequality. Since (7) holds for the numbers chosen, the discriminant of (6) is non-negative so that a, b, c are all real. Since the coefficients of (6) are alternately positive and negative, it follows that a, b, c are all positive. Finally, a, b, c satisfy the triangular inequality if and only if $s - a$, $s - b$, $s - c$ are all positive, which follows easily since these numbers are the roots of the equation

$$x^3 - sx^2 + (4Rr + r^2)x - r^2 s = 0 ,$$

and the signs of the coefficients in this equation are alternately positive and negative. This completes the proof of the Lemma.

3. Proof of Theorem 1. The inequalities of the theorem have already been established in (8) and (9). That they are best possible follows at once from the Lemma, since they together

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define the region S with a connected boundary which is the set of points for which equality holds in (10).

4. Proof of Theorem 2. We have

$$4R(R-2r)^3 = 4(R-2r)^2(R-r)^2 - 4r^2(R-2r)^2$$
$$\leq 4(R-2r)^2(R-r)^2,$$

with equality only if $R = 2r$, that is, when the triangle is equilateral. Combine this inequality with (7). Then

$$(s^2 + r^2 - 10Rr - 2R^2)^2 \leq 4(R-2r)^2(R-r)^2.$$

Taking the square root and arranging terms, we have at once the inequalities of the theorem, namely,

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2, \tag{13}$$

with equality only for the equilateral triangle. Various proofs of these inequalities have been given (see, for example, Gerretsen [2]).

It remains to be shown, using Theorem 1, that the inequalities (13) cannot be improved.

First, we have to show that α, β, γ are all zero in the inequalities

$$2x^2 + 10x - 1 + 2(x-2)\surd(x^2 - 2x)$$
$$\leq (4-\alpha)x^2 + (4-\beta)x + (3-\gamma) \leq 4x^2 + 4x + 3.$$

The case $x = 2$ gives $4\alpha + 2\beta + \gamma = 0$. Henceforth take $x > 2$. Then the right hand inequality gives $\alpha x^2 + \beta x + \gamma \geq 0$, that is,

$$\alpha x^2 + \beta x - 4\alpha - 2\beta \geq 0.$$

Then, on division by $x - 2$, which is positive, we have

$$\alpha x + 2\alpha + \beta \geq 0. \tag{14}$$

The left hand inequality gives

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$$2(x-2)\sqrt{(x^2-2x)} \leq (2-\alpha)x^2 - (6+\beta)x + (4\alpha + 2\beta + 4) .$$

Dividing by $x - 2$, we have

$$(15) \qquad 2\sqrt{(x^2-2x)} \leq (2-\alpha)x - (2\alpha + \beta + 2) ,$$

which reduces, on squaring, to

$$(16) \qquad \alpha(\alpha-4)x^2 + (4\alpha^2 + 2\alpha\beta - 4\alpha - 4\beta)x - (2\alpha + \beta + 2)^2 \geq 0 .$$

Secondly, we seek to show that α, β, γ are all zero in the inequalities

$$16x - 5 \leq \alpha x^2 + (16+\beta)x + (\gamma - 5)$$
$$\leq 2x^2 + 10x - 1 - 2(x-2)\sqrt{(x^2-2x)} .$$

These inequalities also reduce to (14) and (15). It remains to be proved that (14), (15) and (16) together imply $\alpha = \beta = 0$.

Since (14) holds for $x \to \infty$, we have $\alpha \geq 0$. Now (14) also holds for $x \to 2$. Therefore $4\alpha + \beta \geq 0$. Also, (15) holds for $x \to \infty$, so that $2 - \alpha \geq 0$, that is, $\alpha \leq 2$. Since (16) holds for $x \to \infty$, we have $\alpha(\alpha - 4) \geq 0$. The last two inequalities taken together give $\alpha \leq 0$. But we have already proved that $\alpha \geq 0$. Hence $\alpha = 0$. The relation $4\alpha + \beta \geq 0$ then gives $\beta \geq 0$. Putting $\alpha = 0$ in (16), we obtain $-4\beta x + (\beta + 2)^2 \geq 0$, and this inequality holds for $x \to \infty$, so that $\beta \leq 0$. Thus $\beta = 0$. This completes the proof of Theorem 2.

5. The graph. We now describe in more detail the graphical representation considered earlier. It was proved in the Lemma that there is a one-one correspondence between all triangles and all points of a plane region S defined by (10) (see the figure).

The boundary consists of two branches, one resembling roughly a hyperbola, the other a parabola, namely,

$$AB: \; y^2 = 2x^2 + 10x - 1 + 2(x-2)\sqrt{(x^2-2x)} ,$$

$$AC: \; y^2 = 2x^2 + 10x - 1 - 2(x-2)\sqrt{(x^2-2x)} .$$

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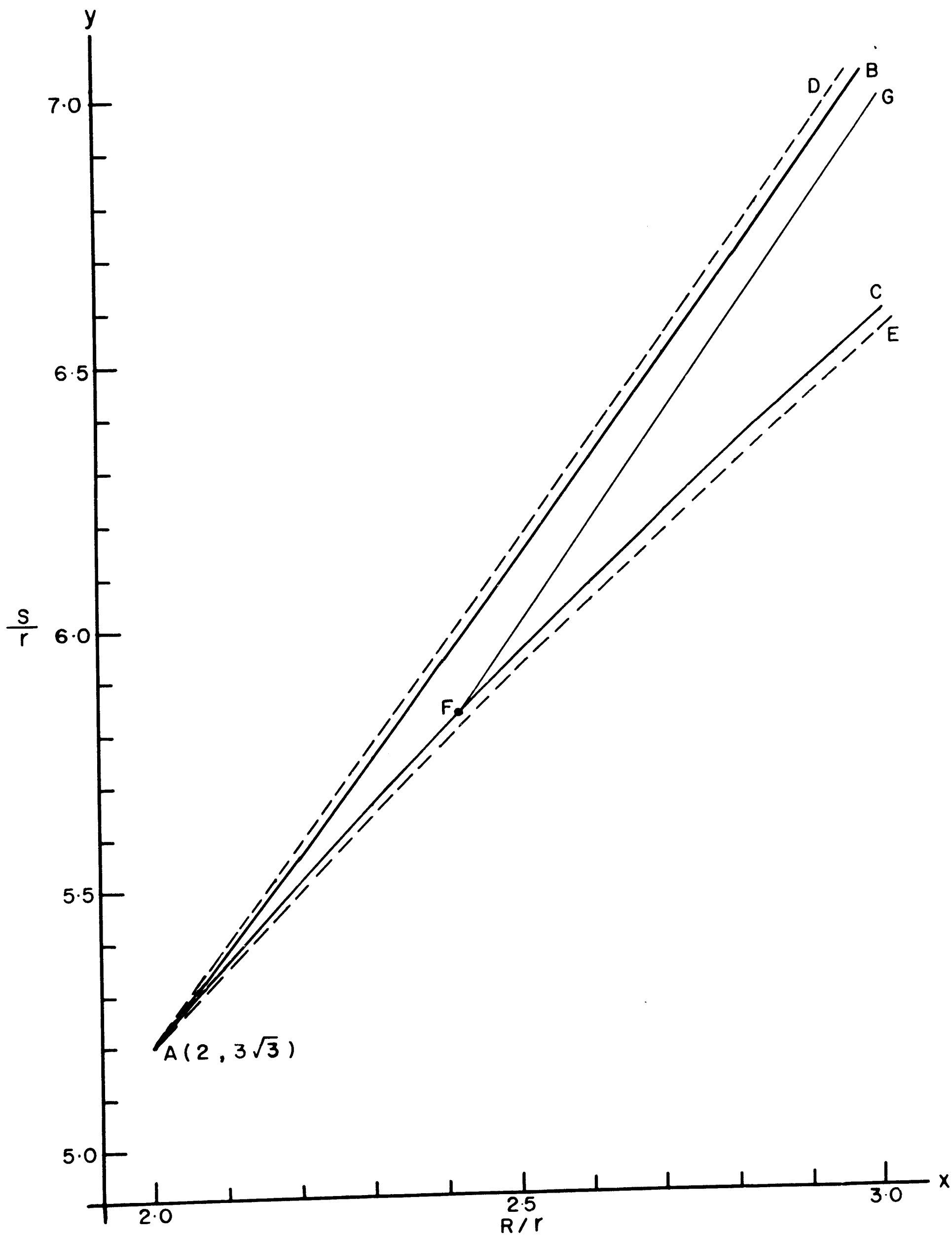

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By Theorem 2, the best approximations of the form $s^2 = \alpha x^2 + \beta x + \gamma$, with the curves passing through A, are AD : $y^2 = 4x^2 + 4x + 3$ and AE : $y^2 = 16x - 5$.

The curves AB, AC form a cusp at A, the common slope at A being $\sqrt{3}$. The slopes of AD and AE at A are $10\sqrt{3}/9$ and $8\sqrt{3}/9$ respectively. FG : $y = 2x + 1$ is an asymptote to both AB and AD. Clearly $A(2, 3\sqrt{3})$ represents the class of equilateral triangles. We have already proved that the boundary curves AB and AC represent isosceles triangles.

In applications to quadratic forms we replace in our consideration the region S by the region S' which is bounded by AD and AE.

To interpret another classification of triangles, we recall that $s - 2R - r$ is zero, positive or negative according as the triangle is right-, acute- or obtuse-angled. Hence all right triangles are represented by that part of the line $y = 2x + 1$ which lies in S. This is the ray FG. Its terminal point $F(1 + \sqrt{2}, 3 + 2\sqrt{2})$ represents isosceles right triangles. It follows easily that the region bounded by BA, AF, FG represents acute-angled triangles and the region bounded by FG, FC represents obtuse-angled triangles.

6. Applications. To prove a given inequality between R, r, s, we merely have to show that the region defined by the inequality contains S (or S', if only quadratic forms are being considered).

As an example, consider the problem of finding the strongest possible linear inequalities relating R, r, s. With reference to the graph, it is clear that the problem is equivalent to finding the intersection of all regions which are bounded by two rays with a common endpoint and which contains the region S or the region S'. The required region is bounded by the ray through A parallel to Ox and the ray through A parallel to FG. This region is defined by the inequalities

$$3\sqrt{3} \leq y \leq 2x + (3\sqrt{3} - 4) .$$

Hence, the strongest possible linear inequalities are

(17) $$3\sqrt{3}r \le s \le 2R + (3\sqrt{3} - 4)r ,$$

with equality only for the equilateral triangle. The first inequality of (17), usually given in the form $s^2 \ge 27r^2$, is well known. The second inequality I have not seen elsewhere. (17) represents the analogue of Theorem 2 when the degree is one.

The right hand inequality of (17) can now be applied to improve known inequalities of higher degree than the first when these are linear in s. For example, if Δ represents the area of a triangle, the inequality $4\Delta \le 3\sqrt{3}R^2$ is well known. Steinig [7] has proved the stronger inequality

$$\sqrt{3}\Delta \le 4Rr + r^2 .$$

By (17), we can state the still stronger inequality

$$\Delta = rs \le 2Rr + (3\sqrt{3} - 4)r^2 .$$

Similarly his inequality (in the same paper)

$$\sqrt{3}(abc)^{1/3} \le 2(R + r)$$

can be replaced by the stronger inequality

$$abc = 4Rrs \le 8R^2 r + (12\sqrt{3} - 16)Rr^2 .$$

To take another simple application of (17), we note that the relation $\sin A + \sin B + \sin C = s/R$ leads without difficulty to the well known inequality $\sin A + \sin B + \sin C \le \frac{3}{2}\sqrt{3}$. This can now be replaced by the stronger inequality

$$\sin A + \sin B + \sin C \le 2 + (3\sqrt{3} - 4)r/R .$$

Let us return to inequalities of degree two. The inequalities (13) have already been used by several authors to strengthen known inequalities. In addition to the paper of Steinig already mentioned, see papers by Leuenberger [4], Makowski [5], Gerretsen [2]. For example, the well known inequalities

$$18Rr \le ab + bc + ca \le 9R^2$$

have been replaced by the stronger inequalities

$$20Rr - 4r^2 \le ab + bc + ca \le 4R^2 + 8Rr + 4r^2 .$$

If we drop the restriction to quadratic forms, we can use Theorem 1 to derive the rather inelegant but best possible inequalities

(18) $$2R^2 + 14Rr - 2(R - 2r)\sqrt{(R^2 - 2Rr)}$$
$$\le ab + bc + ca$$
$$\le 2R^2 + 14Rr + 2(R - 2r)\sqrt{(R^2 - 2Rr)} .$$

Any number of such inequalities can now be written down. Thus

(19) $$4R^2 + 16Rr - 3r^2 - 4(R - 2r)\sqrt{(R^2 - 2Rr)}$$
$$\le a^2 + b^2 + c^2$$
$$\le 4R^2 + 16Rr - 3r^2 + 4(R - 2r)\sqrt{(R^2 - 2Rr)} .$$

The awkward radical in (18) and (19) as well as in (8) and (9) can be avoided by a parametric representation. Let $R/r = 1 + \frac{1}{2}(t + t^{-1})$, where $t \ge 1$. Then $R/r = (t + 1)^2/2t$. Since in this context all similar triangles are equivalent, we may put $r = 2t$ and $R = (t + 1)^2$, whence $R - 2r = (t - 1)^2$ and $\sqrt{(R^2 - 2Rr)} = t^2 - 1$. Then (8) and (9) become

(20) $$4(2t + 1)^3 \le s^2 \le 4t(t + 2)^3 .$$

The inequalities (18) and (19) now read

$$4(t + 1)(2t + 1)(5t + 1) \le ab + bc + ca$$
$$\le 4t(t + 1)(t + 2)(t + 5)$$

and

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$$8(2t + 1)(3t^2 + 6t + 1) \leq a^2 + b^2 + c^2$$

$$\leq 8t(t + 2)(t^2 + 2t + 3) .$$

We might now use the graph relating s and t to interpret various relations between R, r, s but we shall not pursue this here.

We can give t a simple geometrical interpretation. Hobson [3] gives a proof of Euler's result that the distance between the circumcenter and the incenter of a triangle is $\sqrt{(R^2 - 2Rr)}$. It is easily deduced that t is the ratio of the larger segment to the smaller segment in which a diameter of the circumcircle is divided by the incenter.

Since $t \geq 1$, with equality only for the equilateral triangle, we may use the relative distance from the circumcenter to the incenter as a measure of the 'skewness' or the 'eccentricity' of a triangle. Let t be such a measure. Then every $t \geq 1$ defines an equivalence class of triangles having the same 'skewness' measure. Thus each equivalence class of triangles is represented in our graph as the intersection of S and a line parallel to Oy. The general composition of classes is as follows:

$t = 1$	the equilateral triangle;
$1 < t < 1 + \sqrt{2}$	infinitely many acute-angled triangles, exactly two of which are isosceles;
$t = 1 + \sqrt{2}$	the right-angled isosceles triangle and infinitely many acute-angled triangles, exactly one of which is isosceles;
$t > 1 + \sqrt{2}$	infinitely many acute-angled triangles (one being isosceles), infinitely many obtuse-angled triangles (one being isosceles), and exactly one right triangle.

For the particular case of isosceles triangles, we can easily create a parametric representation which is rational in s as well as in R and r. For all triangles we have

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$r = 2t$, $R = (1+t)^2$. In addition, we have for isosceles triangles either $s^2 = 4t(t+2)^3$ or $s^2 = 4(2t+1)^3$. For the former put $t = (T-1)^2/2T$ and the latter put $t = (T^2-1)/2$. Then we have for isosceles triangles either $r = 4T(T-1)^2$, $R = (T^2+1)^2$, $s = 2(T-1)(T+1)^3$ or $r = 4(T^2-1)$, $R = (T^2+1)^2$, $s = 4T^3$.

Finally, the graph may be used to construct any number of relations. For example, the intersection of the region defined by $y \geq x^2$ and the region bounded by FG and FC is the point F. This suggests the problem: If $\Delta > R^2$, prove that the triangle must be acute-angled. Also, if $\Delta = R^2$ and the triangle is not acute-angled, then it must be both right-angled and isosceles.

I am grateful to my colleagues in the Mathematics Department for helpful suggestions and also to Mr. Ross Peters for preparing the diagram. My thanks are due to the referee for suggesting improved presentation for the details of the proof of Theorem 2.

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