A NEW APPROACH TO THE DISTRIBUTION OF THE DURATION OF THE BUSY PERIOD FOR A \(G/G/1\) QUEUEING SYSTEM

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Abstract

For a \(G/G/1\) queueing system let \(X_t\) be the number of customers present at time \(t\) and \(Y_t(Z_t)\) be the time elapsed since the last arrival of a customer (the last completion of a service) at time \(t\). Let \(\tau_t\) be the time until the number of customers in the system is reduced from \(j\) to \(j - l\), given that \(X_0 = j \geq l\), \(Y_0 = y\), \(Z_0 = z\). For the joint distribution of \(\tau_1\) and \(Y_{\tau_1}\) and the Laplace transforms of the \(\tau_t\) integral equations are derived. Under slight conditions these integral equations have unique solutions which can be determined by standard methods. Our results offer a method for calculating the busy period distribution which is completely different from the usual fluctuation theoretic approach.


1. Introduction

For the \(G/G/1\) queueing system the distribution of the duration of a busy period has been derived by Finch (1961) and Kingman (1961). The transform of the joint distribution of the number \(N\) of customers served during a busy period, its duration \(\tau\) and the length of the subsequent idle period \(I\) is given in several textbooks (for example, Prabhu (1980)): for all \(z \in (0, 1)\), \(\theta_1 \geq 0\), \(\theta_2 \geq 0\),

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(1.1) \[ E(z^N e^{-\theta_1 \tau - \theta_2 I}) = 1 - \exp \left\{ - \sum_{n=1}^{\infty} \frac{z^n}{n} \int \int_{v-u \leq 0} e^{-\theta_1 v + \theta_2 (v-u)} dA^n(u) dB^n(v) \right\} , \]

where \( A \) and \( B \) are the distribution functions of the interarrival times and the service times, and \( A^n \) denotes \( n \)-fold convolution of \( A \). The derivation of (1.1) is based on fluctuation theory applied to the underlying random walk of the queueing system.

In this paper a different approach is developed. For \( t \geq 0 \) let \( X_t \) be the number of customers in the system at time \( t \), \( Y_t \) be the time elapsed at time \( t \) since the last arrival of a customer and \( Z_t \) be the time elapsed between \( t \) and the last completion of a service before \( t \). Let the system start at time 0 with the condition \( X_0 = j, Y_0 = y, Z_0 = z \) for some \( j \in \mathbb{N} \) and \( y, z \geq 0 \).

Clearly \( (X_t, Y_t, Z_t) \) is a Markov process. For \( 1 \leq j \) let \( \tau_l \) be the time passing until the number of customers in the system is reduced from \( j \) to \( j - l \). We shall derive an integral equation for

(1.2) \[ \Psi_{y,z}(\alpha, E) := E(e^{-\alpha \tau_l} 1_{\{Y_t \in E\}}|X_0 = j, Y_0 = y, Z_0 = z) \]

(\( \alpha > 0, E \) a Borel subset of \([0, \infty)\)) by some rather simple arguments based on the Markov character of the process \((X_t, Y_t, Z_t)\). Under a slight condition this integral equation is seen to determine \((\alpha, E) \rightarrow \Psi_{y,z}(\alpha, E)\) uniquely and, moreover, turns out to be solvable by the method of successive approximations. This method provides a sequence \( \Psi_{y,z}^{(n)} \) tending to \( \Psi_{y,z} \) at an exponential rate of convergence uniformly with respect to \((y, z)\). This will be useful for a numerical determination of the joint distribution of \( \tau_1 \) and \( Y_{\tau_1} \). Note that \( \tau_1 \), the time for decreasing the numbers of customers from \( j \) to \( j - 1 \), is for \( j = 1 \) simply the ordinary busy period duration, while \( Y_{\tau_1} \) is, for \( j = 1 \), the waiting time of the last customer served in the busy period under consideration.

The process \((X_t, Y_t, Z_t)\) has also been studied by Keilson and Kooharian (1960, 1962) who rely on rather involved Wiener-Hopf techniques. The fairly straightforward approach given here is however sufficient to derive the conditional joint distribution of \((\tau_1, Y_{\tau_1})\) for an arbitrary initial condition on \((X_0, Y_0, Z_0)\). Further it will be seen in Section 3 that our method can be applied to determine the Laplace transforms \( \varphi_j(\cdot|y, z) \) of the \( \tau_j \), conditional on \( X_0 = l \geq j, Y_0 = y, Z_0 = z \). Define their joint generating function by

(1.3) \[ \Phi(x, u|y, z) = \sum_{j=1}^{\infty} \varphi_j(u|y, z)x^j, \quad |x| < 1, \ u, y, z \geq 0. \]

We obtain a system of two Fredholm integral equations of the second kind for the functions \( y \rightarrow \Phi(x, u|y, 0) \) and \( z \rightarrow \Phi(x, u|0, z) \) and an equation which gives \( \Phi \) in terms of these two functions. Under weak conditions this
system of integral equations has the corresponding Neumann series as its unique solution.

In the concluding Section 4 we use our technique to calculate $E(\exp\{-ut_j\} | X_0 = l, Z_0 = z)$ for the bulk-arrival queue $M^X/G/1$. The distribution of the busy period duration (i.e. of $\tau_1$ given that $X_0 = 1, Z_0 = 0$) has already been derived in Cohen (1980, Chapter III, 2.3) using a different method.

Throughout the paper we assume that the distribution functions of the interarrival times and the service times possess densities $a(x)$ and $b(y)$.

2. The joint distribution of $(\tau_1, Y_{\tau_1})$ in a $G/G/1$ queueing system

For $y, z \geq 0$ and $j = 1, 2$ let $Q_{y,z,j}$ be the joint conditional distribution of $(\tau_j, Y_{\tau_j})$ given that $X_0 = j, Y_0 = y, Z_0 = z$. Let

$$
(2.1) \quad \Psi_{y,z}(\alpha, E) := \int_0^\infty e^{-\alpha t} Q_{y,z,1}(dt, E),
$$

where $\alpha > 0$ and $E$ is a Borel subset of $[0, \infty)$. The duration of a busy period initiated by one customer arriving in the system at time 0 then has the Laplace transform $\alpha \rightarrow \Psi_{0,0}(\alpha, [0, \infty))$. We shall now derive an integral equation for $\Psi_{y,z}$.

**Theorem 1.** For all $u, y, z \geq 0$ and all $\alpha > 0$ we have

$$
(2.2) \quad \Psi_{y,z}(\alpha, [0, u]) = \int_0^\infty e^{-\alpha t} \frac{(1 - A(y + t))b(z + t)}{(1 - A(y))(1 - B(z))} 1_{[0, \infty)}(u - y - t) dt
$$

$$
+ \int_{t \geq 0} \int_{w \geq 0} e^{-\alpha t} \frac{(1 - B(z + t))a(y + t)}{(1 - B(z))(1 - A(y))} \Psi_{0,z+t}(\alpha, dw') \Psi_{w',0}(\alpha, [0, u]) dt.
$$

**Proof.** Let $X_0 = j, Y_0 = y, Z_0 = z$. The first change of the queue size occurs at the time $\min(S_y, T_z)$, where $S_y$ and $T_z$ are independent random variables with distributions given by

$$
(2.3) \quad P(S_y \geq v) = \frac{1 - A(y + v)}{1 - A(y)}, \quad v \geq 0,
$$

$$
(2.4) \quad P(T_z \geq w) = \frac{1 - B(z + w)}{1 - B(z)}, \quad w \geq 0.
$$

($S_y(T_z)$ is the first positive arrival (departure) time.) If $S_y = v < T_z$, we have $X_t = j$ for $0 \leq t < v$ and $X_v = j + 1$, $Y_v = 0, Z_v = z + v$. The time remaining thereafter up to $\tau_j$ has the conditional distribution of $\tau_{j+1}$, given that $X_0 = j + 1, Y_0 = 0, Z_0 = z + v$. 


If \( T_z = w < S_y \), the analogous relations are \( X_t = j \) for \( 0 \leq t < w \), \( X_w = j - 1 \), \( Y_w = y + w \), \( Z_w = 0 \), so that the remaining time up to \( \tau_j \) has the same distribution as \( \tau_{j-1} \), given that \( X_0 = j - 1 \), \( Y_0 = y + w \), \( Z_0 = 0 \).

Using these ideas for \( j = 1 \) it is seen that \( Q_{y,z,1} \) satisfies

\[ Q_{y,z,1}([0, t] \times [0, u]) = \int_{0 \leq s < t} \frac{(1 - A(y + s))b(z + s)}{(1 - A(y))(1 - B(z))} \, ds \]

\[ + \int \int \int_{s + v \leq t} \frac{(1 - B(z + s))a(y + s)}{(1 - B(z))(1 - A(y))} Q_{0,z+s,2} (dv, dw) \, ds. \]

Next we shall use the following relation between \( Q_{y,z,1} \) and \( Q_{y,z,2} \):

\[ Q_{y,z,1} (E \times F) = \int \int \int_{w \geq 0} Q_{y,z,1} (dv', dw') Q_{w',0,1} (dv'', dw) \]

for all Borel subsets \( E, F \) of \([0, \infty)\). To see (2.6), note that in order to reduce the queue size from 2 to 0, it must be first decreased to 1 which happens at some time \( v' \), say, and the time which is then elapsed since the last arrival can be any \( w' \in [0, \infty) \). Thereafter the queue size has to be decreased from 1 to 0 after some time \( v'' \). Integrating with respect to \( (v', w') \) and \( v'' \) yields (2.6).

Inserting (2.6) into (2.5) we obtain

\[ Q_{y,z,1} ([0, t] \times [0, u]) = \int_{0}^{t} \frac{(1 - A(y + s))b(z + s)}{(1 - A(y))(1 - B(z))} \, ds \]

\[ + \int \int_{s + v, v' \leq t} \int_{w \leq u} \frac{(1 - B(z + s))a(y + s)}{(1 - B(z))(1 - A(y))} \, dw \]

\[ \times Q_{0,z+s,1} (dv', dw') Q_{w',0,1} (dv'', dw) \, ds. \]

Finally one has to take the Laplace transform of the measure

\[ B \to Q_{y,z,1} (B \times [0, u]), \]

where \( u \) is fixed, to complete the proof.

For fixed \( \alpha > 0 \) the function \((y, z, E) \to \Psi_{y,z}(\alpha, E)\) is uniquely determined by equation (2.2) and the condition that \( \Psi_{y,z}(\alpha, \cdot) \) is a subprobability measure, if

\[ G(\alpha) := \sup_{y \geq 0} \int_{0}^{\infty} \frac{a(y + t)}{1 - A(y)} e^{-\alpha t} \, dt < \frac{1}{2}. \]

For let \( \mathcal{B} \) be the Banach space of all functions \( \rho(y, z, E) \) such that \( \rho(\cdot, \cdot, E) : [0, \infty)^2 \to \mathbb{R} \) is measurable for each Borel subset \( E \) of \([0, \infty) \), \( \rho(y, z, \cdot) \) is a signed measure for each \((y, z) \in [0, \infty)^2 \) and

\[ \| \rho \| := \sup_{y,z \geq 0} |\rho(y, z, \cdot)| < \infty \]
(|ν| denotes the total variation of a signed measure ν). Let \( \mathcal{H} := \{ \rho \in \mathcal{B} \| \rho \| \leq 1 \} \) and define the operator \( U_\alpha : \mathcal{H} \rightarrow \mathcal{H} \) by

\[
(2.10) \quad (U_\alpha \rho)(y, z, E) := \int_{\mathbb{R}_+} \int_{w' > 0} e^{-\alpha t} \frac{(1 - B(z + t))a(y + t)}{(1 - B(z))(1 - A(y))} \times \rho(0, z + t, dw') \rho(w', 0, E) dt.
\]

If \( \rho, \tilde{\rho} \in \mathcal{H} \), we have

\[
(2.11) \quad \| U_\alpha \rho - U_\alpha \tilde{\rho} \| \leq \sup_{y, z \geq 0} \int_{t \geq 0} e^{-\alpha t} \frac{(1 - B(z + t))a(y + t)}{(1 - B(z))(1 - A(y))} \times \left\{ \left| \int [\rho(0, z + t, dw') \rho(w', 0, dw) - \rho(0, z + t, dw') \tilde{\rho}(w', 0, dw)\right| + \left| \rho(0, z + t, dw') \tilde{\rho}(w', 0, dw) - \rho(0, z + t, dw') \tilde{\rho}(w', 0, dw)\right| \right\} dt
\]

\[
\leq \sup_{y, z \geq 0} 2 \| \rho - \tilde{\rho} \| \int_{t \geq 0} e^{-\alpha t} \frac{a(y + t)}{1 - A(y)} dt
\]

\[
= 2G(\alpha) \| \rho - \tilde{\rho} \|.
\]

Thus if \( G(\alpha) < 1/2 \) and \( \rho, \tilde{\rho} \) are two solutions of (2.2) satisfying \( \| \rho \|, \| \tilde{\rho} \| \leq 1 \), (2.2) and (2.11) entail that

\[
(2.12) \quad \| \rho - \tilde{\rho} \| = \| U_\alpha \rho - U_\alpha \tilde{\rho} \| \leq 2G(\alpha) \| \rho - \tilde{\rho} \|
\]

so that \( \rho = \tilde{\rho} \).

Especially if

\[
(2.13) \quad G(\alpha) < \frac{1}{2} \quad \text{for all} \quad \alpha \geq \alpha_0
\]

for some \( \alpha_0 > 0 \), equation (2.2) uniquely determines the Laplace transform of the measure \( Q_{y, z, 1}(\cdot, E) \) for every fixed triple \( (y, z, E) \). Moreover, the above considerations show that the method of successive approximations yields a sequence of \( (\Psi^{(n)})_{n \geq 0} \) which converges to \( \Psi \) in the total variation distance with respect to \( E \) and uniformly with respect to \( y \) and \( z \): we have

\[
(2.14) \quad \sup_{y, z} |\Psi^{(n)}_{y, z}(\alpha, \cdot) - \Psi_{y, z}(\alpha, \cdot)| \leq \frac{[2G(\alpha)]^n}{1 - 2G(\alpha)} \sup_{y, z} |\Psi^{(0)}_{y, z}(\alpha, \cdot) - \Psi^{(0)}_{y, z}(\alpha, \cdot)|.
\]

We can take an arbitrary \( \Psi^{(0)}_{y, z}(\alpha, E) \) belonging to \( \mathcal{H} \) and then, for \( n \geq 1 \), have to find \( \Psi^{(n)}_{y, z}(\alpha, E) \) recursively by

\[
(2.15) \quad \Psi^{(n)}_{y, z}(\alpha, E) := \int_0^\infty e^{-\alpha t} \frac{(1 - A(y + t))b(z + t)}{(1 - A(y))(1 - B(z))} I_{[0, \infty)}(u - y - t) dt
\]

\[
+ \int_{t \geq 0} \int_{w' \geq 0} e^{-\alpha t} \frac{(1 - B(z + t))a(y + t)}{(1 - B(z))(1 - A(y))} \Psi^{(n-1)}_{0, z+t}(\alpha, dw') \Psi^{(n-1)}_{w', 0}(\alpha, E) dt.
\]
Condition (2.13) is for example satisfied, if \( A \) has a bounded hazard rate \( a(y)/(1 - A(y)) \). For if \( a/(1 - A) \leq K \),
\[
G(\alpha) \leq \int_0^\infty \frac{a(y + t)}{1 - A(y + t)} e^{-\alpha t} dt \leq K/\alpha.
\]

3. The Laplace transform of \( \tau_j \)

Next we consider
\[
\varphi_j(u|y, z) := E(\exp\{-u\tau_j\}|X_0 = l, Y_0 = y, Z_0 = z),
\]
where \( u, y, z \geq 0 \) and \( l \geq j \geq 1 \). Using the argument already employed at the beginning of the proof of Theorem 1, but now for arbitrary \( j \geq 1 \) and for the Laplace transforms instead of the distributions themselves, we obtain
\[
\varphi_j(u|y, z) = \int_0^\infty e^{-uw} \varphi_{j+1}(u|0, z + v) \frac{(1 - B(z + v))a(y + v)}{(1 - B(z))(1 - A(y))} dv
\]
\[
+ \int_0^\infty e^{-uw} \varphi_{j-1}(u|y + w, 0) \frac{(1 - A(y + w))b(z + w)}{(1 - A(y))(1 - B(z))} dw, \quad j \geq 2,
\]
and
\[
\varphi_1(u|y, z) = \int_0^\infty e^{-uw} \frac{(1 - A(y + w))b(z + w)}{(1 - A(y))(1 - B(z))} dw
\]
\[
+ \int_0^\infty e^{-uw} \varphi_2(u|0, z + v) \frac{(1 - B(z + v))a(y + v)}{(1 - B(z))(1 - A(y))} dv.
\]

To solve this system we introduce the generating function
\[
\Phi(x, u|y, z) := \sum_{j=1}^\infty \varphi_j(u|y, z)x^j, \quad |x| < 1.
\]

Summing (3.2) and (3.3) over \( j \) yields after some simple manipulations
\[
x\Phi(x, u|y, z) = x^2 \int_0^\infty e^{-uw} \frac{(1 - A(y + w))b(z + w)}{(1 - A(y))(1 - B(z))} dw
\]
\[
+ \int_0^\infty e^{-uw} \Phi(x, u|0, z + w) \frac{(1 - B(z + w))a(y + w)}{(1 - B(z))(1 - A(y))} dw
\]
\[
+ x^2 \int_0^\infty e^{-uw} \Phi(x, u|y + w, 0) \frac{(1 - A(y + w))b(z + w)}{(1 - A(y))(1 - B(z))} dw
\]
\[
- x \int_0^\infty e^{-uw} \varphi_1(u|0, z + w) \frac{(1 - B(z + w))a(y + w)}{(1 - B(z))(1 - A(y))} dw.
\]
Note that

\[(3.6) \quad \varphi_1(u|y, z) = \Psi_{y,z}(u, [0, \infty)),\]

and \(\Psi\) has been determined in the previous section. We arrive at the following result.

**Theorem 2.** Let \(u \geq 0\) and \(x \in \mathbb{R}, |x| < 1\), be fixed. The functions \(\Phi(x, u|0, \cdot)\) and \(\Phi(x, u|1, 0)\) satisfy the following system of Fredholm integral equations of the second kind:

\[(3.7) \quad (1 - B(z))x\Phi(x, u|0, z) = x^2 \int_0^\infty e^{-uw}(1 - A(w))b(z + w) \, dw
+ \int_0^\infty e^{-uw}\Phi(x, u|0, z + w)(1 - B(z + w))a(w) \, dw
+ x^2 \int_0^\infty e^{-uw}\Phi(x, u|0, 0)(1 - A(w))b(z + w) \, dw
-x \int_0^\infty e^{-uw}\varphi_1(u|0, z + w)(1 - B(z + w))a(w) \, dw\]

\[(3.8) \quad (1 - A(y))x\Phi(x, u|y, 0) = x^2 \int_0^\infty e^{-uw}(1 - A(y + w))b(w) \, dw
+ \int_0^\infty e^{-uw}\Phi(x, u|0, w)(1 - B(w))a(y + w) \, dw
+ x^2 \int_0^\infty e^{-uw}\Phi(x, u|y + w, 0)(1 - A(y + w))b(w) \, dw
-x \int_0^\infty e^{-uw}\varphi_1(u|0, w)(1 - B(w))a(y + w) \, dw.\]

For arbitrary \(y, z \geq 0\) the function \(\Phi(x, u|y, z)\) is then connected with \(\Phi(x, u|0, \cdot), \Phi(x, u|1, 0)\) and \(\varphi_1(u|0, z)\) by (3.5).

Equations (3.7) and (3.8) can be written in the form

\[(3.9) \quad f = g + Kf,\]

where

\[(3.10) \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad Kf = \begin{pmatrix} K_{11}f_1 + K_{12}f_2 \\ K_{21}f_1 + K_{22}f_2 \end{pmatrix}\]
and \( f_1, f_2, g_1, g_2 \) are defined as follows:

\[
\begin{align*}
(3.11) \quad f_1(y) &= (1 - B(y))\Phi(x, u|0, y), \\
(3.12) \quad f_2(y) &= (1 - A(y))\Phi(x, u|y, 0), \\
(3.13) \quad g_1(y) &= x \int_{0}^{\infty} e^{-uw}(1 - A(w))b(y + w) \, dw \\
&\quad - \int_{0}^{\infty} e^{-uw} \varphi_1(u|0, y + w)(1 - B(y + w))a(w) \, dw, \\
(3.14) \quad g_2(y) &= x \int_{0}^{\infty} e^{-uw}(1 - A(y + w))b(w) \, dw \\
&\quad - \int_{0}^{\infty} e^{-uw} \varphi_1(u|0, w)(1 - B(w))a(y + w) \, dw.
\end{align*}
\]

The definition of the integral operators \( K_{ij}, i, j = 1, 2 \) is clear from (3.7) and (3.8); for instance,

\[
(3.15) \quad (K_{11}h)(z) = x^{-1} \int_{0}^{\infty} e^{-uv}h(z + v)a(v) \, dv.
\]

For \( y > 0 \) let \( \hat{a}_y \) and \( \hat{b}_y \) be the Laplace transform of the functions \( v \to a(y + v) \) and \( v \to b(y + v) \). Then we have, for arbitrary bounded measurable functions \( h: [0, \infty) \to \mathbb{R} \),

\[
(3.16) \quad |K_{11}h(z)| = x^{-1} \int_{0}^{\infty} e^{-uv}h(z + v)a(v) \, dv \leq |x|^{-1} \hat{a}_0(u)\|h\|_{\infty},
\]

where \( \|h\|_{\infty} := \sup_{z \geq 0} |h(z)| \), and similarly

\[
(3.17) \quad |K_{12}h(z)| = x \int_{0}^{\infty} e^{-uv}h(v)b(z + v) \, dv \leq |x|\hat{b}_z(u)\|h\|_{\infty},
\]

\[
(3.18) \quad |K_{21}h(z)| \leq |x|^{-1} \hat{a}_z(u)\|h\|_{\infty},
\]

\[
(3.19) \quad |K_{22}h(z)| \leq |x|\hat{b}_0(u)\|h\|_{\infty}.
\]

If we define \( \|h\| := \|h_1\|_{\infty} + \|h_2\|_{\infty} \) for

\[
h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : [0, \infty) \to \mathbb{R}^2,
\]

(3.16)-(3.18) yield

\[
(3.20) \quad \|Kh\| \leq |x|^{-1} \hat{a}_0(u)\|h_1\|_{\infty} + |x|\sup_{z \geq 0} \hat{b}_z(u)\|h_2\|_{\infty} \\
+ |x|^{-1} \sup_{z \geq 0} \hat{a}_z(u)\|h_1\|_{\infty} + |x|\hat{b}_0(u)\|h_2\|_{\infty}.
\]

Let us assume that

\[
(3.21) \quad \lim_{u \to \infty} \sup_{z \geq 0} \hat{a}_z(u) = \lim_{u \to \infty} \sup_{z \geq 0} \int_{0}^{\infty} e^{-uv}a(z + v) \, dv = 0
\]
and

\[(3.22) \lim_{u \to \infty} \sup_{z \geq 0} \delta_z(u) = 0.\]

Equations (3.21) and (3.22) are not very restrictive conditions; they are for example satisfied, if \(a\) and \(b\) are monotone on \([T, \infty)\) for some \(T \geq 0\). If (3.21) and (3.22) are valid, some standard arguments using (3.20) now show that, for sufficiently large \(u\), (3.9) possesses a unique continuous solution which is given by the uniformly convergent Neumann series

\[(3.23) f = g + Kg + K^2g + \cdots.\]

Thus for large \(u\) the functions \(\Phi(x, u|0, \cdot)\) and \(\Phi(x, u|\cdot, 0)\) are uniquely determined by (3.7) and (3.8), and the series (3.23) gives a way to approximate them exponentially fast. For arbitrary \(\varepsilon \in (0, 1/2)\) this convergence is uniform with respect to \(x \in (\varepsilon, 1-\varepsilon)\), if \(u \geq u_0 = u_0(\varepsilon)\).

4. The bulk queue \(M^X/G/1\)

For the queueing system \(M^X/G/1\) the above technique can also be applied to determine the conditional Laplace transform

\[(4.1) \varphi_j(u|z) := E(\exp\{-u \tau_j\}|X_0 = l, Z_0 = z), \quad u, z \geq 0, \ l \geq j \geq 1,\]

of the first time instant \(\tau_j\) at which the queue size is decreased from \(l\) to \(l-j\). Let \(A(x) = 1 - e^{-\lambda x}, \ x \geq 0, \ \text{where some} \ \lambda > 0\). At the time of the \(i\)th arrival in \((0, \infty)\) a group of \(A_i\) customers enters the system, where \(A_1, A_2, \ldots\) are assumed to be independent random variables having the common distribution \(P(A_i = n) = p_n, \ n = 1, 2, \ldots\), and the generating function \(p(s) := \sum_{n=1}^{\infty} p_n s^n\).

For \(M^X/G/1\) obviously \((X_t, Z_t)\) is a Markov process. It is not difficult to see that

\[(4.2) \varphi_j(u|z) = \varphi_1(u|z) \varphi_1(u|0)^{j-1}\]

so that it suffices to compute \(\varphi_1(u|z)\). Now given that \(X_0 = 1, \ Z_0 = z\), the following possibilities can be distinguished. If no new customers enter before the next service is completed at time \(x\), say (an event of probability \([b(z + x)/(1 - B(z))]e^{-\lambda x} \ dx\), we have \(\tau_1 = x\). If \(j \geq 1\) arrivals take place before the next service completion time \(x\) and the number of new customers entering the system in \((0, x]\) is equal to \(n\), we have \(\tau_1 = x + \tilde{\tau}_n\), where \(\tilde{\tau}_n\) has the same distribution as \(\tau_n\), given that \(X_0 = n, \ Z_0 = 0\). This possibility occurs with probability

\[e^{-\lambda x} \frac{(\lambda x)^j}{j!} P\left(\sum_{i=1}^{j} A_i = n\right) \frac{b(z + x)}{1 - B(z)} \ dx.\]
For $\varphi_1(u|z)$ these considerations yield

\begin{equation}
\varphi_1(u|z) = \int_0^\infty e^{-ux} \frac{b(z+x)}{1-B(z)} e^{-\lambda x} \, dx \\
+ \sum_{j=1}^\infty \sum_{n=j}^\infty \left[ \int_0^\infty e^{-ux} e^{-\lambda x} \frac{\lambda x^j}{j!} p \left( \sum_{i=1}^j A_i = n \right) \frac{b(z+x)}{1-B(z)} \, dx \right] \varphi_1(u|0)^n \\
= \frac{1}{1-B(z)} \int_0^\infty e^{-(u+\lambda)x} b(z+x) \, dx \\
+ \sum_{j=1}^\infty \int_0^\infty e^{-(u+\lambda)x} b(z+x) \frac{\lambda x^j}{j!} p(\varphi_1(u|0))^j \, dx \\
= \frac{1}{1-B(z)} \int_0^\infty b(z+x) \exp\{\lambda x p(\varphi_1(u|0)) - (u+\lambda)x\} \, dx.
\end{equation}

Equations (4.3) show how to compute $\varphi_1(u|z)$, if $\varphi_1(u|0)$ is known. For $z = 0$, (4.3) can be written as

\begin{equation}
\varphi_1(u|0) = f(u + \lambda - \lambda p(\varphi_1(u|0))),
\end{equation}

where $f$ is the Laplace transform of $b(x)$.

Equation (4.4) is a generalization of the well-known Takács equation (Feller (1971), pages 441–442 and 473) which comes out for $p(x) = x$. As in the classic case the following lemma is easily proved.

**Lemma.** Assume that $1/\mu := \int_0^\infty xb(x) \, dx < \infty$ and $\nu := \sum_{n=1}^\infty np_n < \infty$. The equation

\begin{equation}
\varphi(u) = f(u + \lambda - \lambda p(\varphi(u))), \quad u > 0,
\end{equation}

possesses a unique solution $\varphi(u)$ which is the Laplace transform of a distribution which is proper if $\nu/\mu \leq 1$ and defective otherwise.

As an example, let us consider the case when $B(x) = 1 - e^{-\mu x}$, $x \geq 0$, for some $\mu > 0$. Equation (4.4) for $\varphi = \varphi_1(\cdot|0)$ takes the form

\begin{equation}
\varphi(u) = \frac{\mu}{\mu + \lambda + u - \lambda p(\varphi(u))} = \frac{\mu}{\mu + \lambda + u} + \frac{\lambda}{\mu + \lambda + u} \varphi(u)p(\varphi(u)).
\end{equation}

We note that $\varphi$ can be expanded into ascending powers of $(\lambda + \mu + u)^{-1}$ in the form

\begin{equation}
\varphi(u) = \sum_{n=1}^\infty q_n[(\lambda + \mu)/(\lambda + \mu + u)]^n, \quad u \geq 0,
\end{equation}

where $q_n \geq 0$ for all $n \geq 1$. To derive (4.7), let $X_i$ be the time between the $(i-1)$th and the $i$th jump of the queue size and let $Y_i$ be the size of the $i$th
jump. Then if \( j_1, j_2, \ldots, j_n \in \{-1, 1, 2, 3, \ldots\} \) satisfy \( j_1 + \cdots + j_m > -1 \) for \( m = 1, \ldots, n - 1 \) and \( j_1 + \cdots + j_n = -1 \), it is easily seen that

\[
\int_{\{Y_1 = j_1, \ldots, Y_n = j_n\}} e^{-u(x_1 + \cdots + x_n)} dP = \prod_{m=1}^{n} \int_{\{Y_m = j_m\}} e^{-uX_m} dP
\]

and

\[
\int_{\{Y_m = -1\}} e^{-uX_m} dP = \frac{\mu}{\lambda + \mu + u},
\]

\[
\int_{\{Y_m = j_m\}} e^{-uX_m} dP = \frac{\lambda}{\lambda + \mu + u}, \quad \text{if } j_m \geq 1,
\]

since \( X_m \) can be represented as the minimum of two exponential variables \( S_m \) and \( T_m \), say, with means \( 1/\lambda \) and \( 1/\mu \), respectively, and \( Y_m = 1 \) if and only if \( S_m < T_m \). Obviously \( \phi(u) \) can be written as a series of terms of the form (4.8). Inserting (4.9) and (4.10) into (4.8) shows (4.7).

Let \( v := (\lambda + \mu)/(\lambda + \mu + u) \), \( \tilde{\phi}(v) := \phi(u) \) if \( v \in (0, 1] \) and \( \tilde{\phi}(0) := 0 \). From (4.6) it follows that

\[
\tilde{\phi}(v) = \frac{\mu}{\mu + \lambda} v + \frac{\lambda}{\mu + \lambda} v \tilde{\phi}(v) p(\tilde{\phi}(v)).
\]

Inserting (4.7) into (4.11) and comparing the coefficients at both sides gives the following recursive relation for the \( q_n \):

\[
q_1 = \frac{\mu}{(\mu + \lambda)}, \quad q_2 = 0,
\]

\[
q_{n+1} = \frac{\lambda}{\mu + \lambda} \sum \left( \frac{i_1 + \cdots + i_n}{i_1, \ldots, i_n} \right) p_{i_1 + \cdots + i_n - 1} q_1^{i_1} \cdots q_n^{i_n}, \quad n > 1,
\]

where the sum is taken over all \( n \)-tuples \( (i_1, \ldots, i_n) \) of nonnegative integers for which \( \sum_{j=1}^{n} j i_j = n \). To check (4.9), it is convenient to use the formula

\[
\frac{d^n}{dv^n} (F \circ \phi)(v) = \sum \frac{n!}{i_1! \cdots i_n!} F^{(i_1 + \cdots + i_n)}(\phi(v)) \prod_{j=1}^{n} \left( \frac{\phi^{(j)}(v)}{j!} \right)^{i_j}
\]

where the sum is extended over the same set of \( n \)-tuples as in (4.12) (see Gradshteyn and Ryzhik (1980), page 19, formulae 0.430).

Equation (4.7) can be inverted term-by-term. Thus the density of \( \tau_1 \) is given by

\[
e^{-\left(\lambda + \mu\right)t} \sum_{n=1}^{\infty} \frac{(\mu + \lambda)^n}{(n - 1)!} q_n t^{n-1}.
\]
References


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