Canad. Math. Bull. Vol. **37** (1), 1994 pp. 46–53

SUBHARMONIC EXTENSIONS AND APPROXIMATIONS

P. M. GAUTHIER

ABSTRACT. In this note we extend subharmonic functions defined on closed sets.

Let Ω be a (C^∞, connected, orientable) Riemannian manifold and let $\Omega^* = \Omega \cup \{*\}$ denote the Alexandroff one-point compactification of Ω. In the sequel, all topological statements and notations not involving the ideal point $\{*\}$ will refer to the topology on Ω (rather than on Ω^*). Thus, if $A \subset \Omega$, $\bar{A}$ denotes the closure of A (in Ω) *etc.*

In this paper, we study the harmonic and (especially) the subharmonic functions on Ω. We shall assume that Ω is not compact (for Ω compact, our theorem is completely trivial). The sheaf of harmonic functions on Ω forms a harmonic space in the sense of Brelot (*cf.* [7]) and so harmonic and subharmonic functions on Ω share the most basic properties of classical harmonic and subharmonic functions respectively in Euclidean space. In particular, the Perron method for the (generalized) Dirichlet problem applies. If φ is a measurable function defined on the boundary ∂U of a relatively compact open set $U \subset \Omega$, we denote by h^U_φ the generalized solution of the Dirichlet problem on U for the given boundary values φ.

LEMMA 1 ([7, P. 131]). *Let V be a non-empty open subset of U and φ a resolutive function on ∂U. Then the function $\tilde{\varphi}$ defined on ∂V by*

$$\tilde{\varphi} = \begin{cases} \varphi & \text{on } \partial V \cap \partial U \\ h^U_\varphi & \text{on } \partial V \cap U \end{cases}$$

is resolutive with respect to V and $h^V_{\tilde{\varphi}} = h^U_\varphi|_V$.

By choosing for V charts on Ω, it follows from the lemma that the local properties of h^U_φ may be inferred from the theory of partial differential equations. For any point $a \in \Omega$, the Laplacian has a global fundamental solution $F = F_a$ with singularity at a. This follows, for example from the Principal Function Theorem (*e.g.* [25]). Harmonic functions on a Riemannian manifold also have the unique continuation property [5] which is intimately related to the possibility of harmonic approximation (*cf.* [20] and [19]).

Let W be an open set in Ω. We say that (W, Ω) is a subharmonic extension pair for *compact* sets if for every function u subharmonic on W and for each compact subset E of

This research was supported by NSERC (Canada) and FCAR (Québec).
Received by the editors May 7, 1992; revised October 27, 1992.
AMS subject classification: 31BO5, 31C12.
Key words and phrases: subharmonic extensions.

https://doi.org/10.4153/CMB-1994-008-3 Published online by Cambridge University Press

W, there is a function $\tilde{u}$, subharmonic on Ω, such that $\tilde{u} = u$ on E. We say that (W, Ω) is a subharmonic (respectively harmonic) Runge pair for *compact* sets if for every function u subharmonic (respectively harmonic) on W, for each compact subset E of W, and for each $\epsilon > 0$, there is a function u_ϵ, subharmonic (respectively harmonic) on Ω, such that $u - \epsilon \leq u_\epsilon \leq u + \epsilon$ on E. Similarly, we define the respective notions of subharmonic extension pair, subharmonic Runge pair, and harmonic Runge pair for *closed* sets by replacing compact subsets E of W, in the above definitions, by subsets E of W which are closed (in Ω, not only in W).

THEOREM 1. *The following are equivalent:*

a) (W, Ω) is a harmonic Runge pair for compact sets;

b) (W, Ω) is a harmonic Runge pair for closed sets;

c) (W, Ω) is a subharmonic Runge pair for compact sets;

d) (W, Ω) is a subharmonic Runge pair for closed sets;

e) (W, Ω) is a subharmonic extension pair for compact sets;

f) (W, Ω) is a subharmonic extension pair for closed sets;

g) $\Omega^* \setminus W$ is connected.

REMARKS. The theorem remains true if "subharmonic" is replaced by "continuous subharmonic" throughout.

Our main result is on subharmonic extension from closed sets, namely, g) $\longrightarrow$ f). The overall strategy of our proof is similar to that of [4], although we shall also introduce some ideas of our own. We have included other results in the formulation of the theorem in order to emphasize that all of these various problems depend on the same topological condition g). Moreover, in light of the above theorem, the problem of subharmonic Runge approximation (in the above form) becomes in a sense trivial. Indeed, whenever approximation is possible, it turns out that extension is possible and extension is, of course, the best conceivable approximation since the error function is identically zero. Thus, the implication g) $\longrightarrow$ d), which is also new, is a trivial consequence of the implication g) $\longrightarrow$ f).

The equivalence of a) and g) is classical (Walsh proved g) $\longrightarrow$ a) in the planar case [28]). See also [19]. The equivalence of b) and g) was treated in [16] in R^n, while on Riemannian manifolds, the implication g) $\longrightarrow$ b) follows from a recent result in [6] combined with our Lemma 2. Bliedtner and Hansen proved g) $\leftrightarrow$ c) in the context of axiomatic potential theory [9], at least in the continuous case.

Subharmonic extensions have been investigated in [1], [2], [3], [4], [18], and [24]. In the last of these, we may glean the implication g) $\longrightarrow$ e).

The statements of the theorem fall quite naturally into two parts: the harmonic part a) $\leftrightarrow$ b) $\leftrightarrow$ g) and the subharmonic part c) $\leftrightarrow$ d) $\leftrightarrow$ e) $\leftrightarrow$ f) $\leftrightarrow$ g). For compact sets we can infer the possibility of subharmonic approximation from the possibility of harmonic approximation and the Riesz representation theorem. The opposite inference is not clear, and for closed sets we do not see how to pass between the possibility of harmonic approximation and that of subharmonic approximation without going through condition g), which is equivalent to both.

PROOF. That c) $\longrightarrow$ g). Suppose $\Omega^* \setminus W$ is not connected. Then $\Omega \setminus W$ has a component X which is compact.

Let W_j be an exhaustion of W [22, p. 238]. Then, for some j, the component G of $\Omega \setminus \bar{W}_j$ containing X is relatively compact in Ω. That is, $G \subset\subset \Omega$. Let $x_0 \in \partial X$ and let $F(x)$ be a fundamental solution at x_0 for the Laplacian. Then $F(x_0) = +\infty$ (*cf.* [21, (8.4)]). The function F is harmonic on $\Omega \setminus \{x_0\}$ and in particular, on W. Choose $x_1 \in G \cap W$ so close to x_0 that

$$F(x_1) > \sup_{x \in \partial G} F(x).$$

Set $E = \partial G \cup \{x_1\}$. Then, by the maximum principle, F cannot be uniformly approximated with arbitrary precision on E by functions subharmonic in Ω. This proves c) $\longrightarrow$ g) (as well as a) $\longrightarrow$ g)).

The implications f) $\longrightarrow$ e) $\longrightarrow$ c) and f) $\longrightarrow$ d) $\longrightarrow$ c) are obvious. Since we have already shown that c) $\longrightarrow$ g), there remains only to establish our main result, g) $\longrightarrow$ f).

LEMMA 2. *Suppose $E \subset V \subset \Omega$, E is closed, V is open, and $\Omega^* \setminus V$ is connected. Then, there exists an open set U such that $E \subset U \subset \bar{U} \subset V$, ∂U is smooth, and $\Omega^* \setminus \bar{U}$ is connected and locally connected.*

PROOF. Using a partition of unity, we may construct a function $\varphi \in C^\infty(\Omega)$ such that $0 \le \varphi \le 1$, $\varphi = 0$ on E, and $\varphi = 1$ on $\Omega \setminus V$. By Sard's theorem, there is a value λ, $0 < \lambda < 1$, such that $\nabla\varphi \ne 0$ on the level set $\varphi^{-1}(\lambda)$. Let $U_\lambda = \{x : \varphi(x) < \lambda\}$ and let U be the union of U_λ and all compact components of $\Omega \setminus U_\lambda$. Then U satisfies the requirements of the lemma.

Indeed, since ∂U is smooth, the connectedness of $\Omega^* \setminus \bar{U}$ follows from that of $\Omega^* \setminus U$.

To see that $\Omega^* \setminus \bar{U}$ is locally connected, let K be an arbitrary compact set in Ω and let

$$K \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots$$

be an exhaustion of Ω by relatively compact open sets Ω_j, $j = 1, 2, \ldots$. We claim that for some j, each relatively compact component of $\Omega \setminus (\bar{U} \cup K)$ is contained in Ω_j. Suppose this were not the case. Then, there exists a sequence X_j of distinct components of $\Omega \setminus (\bar{U} \cup K)$ which meet $\Omega \setminus \bar{\Omega}_j$, $j = 1, 2, \ldots$. Since each X_j is connected and its closure meets K, there is a point $a_j \in X_j \cap \partial\Omega_1$, $j = 1, 2, \ldots$. We may assume (by choosing a subsequence, if necessary) that a_j converges to some point a. Since $a \in \partial\Omega_1$, it follows that $a \notin K$, and hence, since each a_j belongs to a different component of $\Omega \setminus (\bar{U} \cup K)$, we have that $a \in \partial U$. By continuity, $\varphi(a) = \lambda$ and so $\nabla\varphi(a) \ne 0$ by the choice of λ. Thus a has arbitrarily small neighbourhoods in which $\{x : \varphi(x) > \lambda\}$ is connected. But this contradicts the fact that each neighbourhood of a meets infinitely many X_j. Thus, there exists, as claimed, an Ω_j which contains each relatively compact component of $\Omega \setminus (\bar{U} \cup K)$. Let Q be the union of these components with $\bar{U} \cup K$. Then, $(\Omega^* \setminus \bar{U}) \setminus Q$ is a connected neighbourhood of the ideal point $*$ in the topological space $\Omega^* \setminus \bar{U}$. Moreover $(\Omega^* \setminus \bar{U}) \setminus Q$ is contained in $(\Omega^* \setminus \bar{U}) \setminus K$ which was an arbitrary neighbourhood of $*$ in $\Omega^* \setminus \bar{U}$. This shows that $\Omega^* \setminus \bar{U}$ is locally connected at the ideal point $*$. Since $\Omega^* \setminus \bar{U}$ is clearly locally connected at every other of its points, $\Omega^* \setminus \bar{U}$ is locally connected. This completes the proof of Lemma 2.

LEMMA 3. *Suppose* $E \subset V \subset \Omega$, *$E$ is closed, V is open, and* $\Omega^* \setminus V$ *is connected. If u is a subharmonic function on V and if K is a compact subset of* Ω, *then there exists a subharmonic function $\tilde{u}$ on a neighbourhood $\tilde{V}$ of $E \cup K$ such that* $\Omega^* \setminus \tilde{V}$ *is connected and $\tilde{u} = u$ on E.*

The lemma (and its proof) remains valid if "subharmonic" is replaced by "continuous subharmonic" throughout.

PROOF. Let U be a neighbourhood of E, constructed as in Lemma 2. Fix any smoothly bounded relatively compact neighbourhood G of K. We will specify G at the end of the proof.

Let X be a component of $G \setminus \bar{U}$. If $\partial X \cap \bar{U} = \emptyset$, we set $\tilde{u}$ equal to any subharmonic function on X.

Suppose now $\partial X \cap \bar{U} \neq \emptyset$, then $\partial X \not\subset \bar{U}$ since $\Omega^* \setminus U$ is connected. We may construct two relatively compact open sets A and B such that

$$\bar{A} \cap \bar{B} = \emptyset, \quad \partial X \cap \bar{U} \subset A \subset \bar{A} \subset V, \quad \partial X \cap B \neq \emptyset.$$

Each point of ∂X is regular for the Dirichlet problem on X. For points on $\partial X \setminus \partial G$ and $\partial X \setminus \partial U$, this is clear (*cf.* [21]) since ∂X is smooth at such points. The remaining points of ∂X are regular because (see [11, p. 116]) they are regular for the larger open set G. Let M be a positive number such that

$$M > \sup_{\bar{A}} u.$$

Set $\tilde{u} = u$ on $\bar{U}$, and on X let $\tilde{u}$ be the generalized solution of the Dirichlet problem on X for the boundary function which is equal to u on $\partial X \cap \bar{U}$ and equal to M on $\partial X \setminus \bar{U}$. Then $\tilde{u}$ is upper semicontinuous on $(\bar{U} \cup X)^0$ and

$$\tilde{u} \geq M \text{ on } \partial X \setminus \bar{U}. \tag{1}$$

There is a neighbourhood N of ∂X such that, for any such M

$$\tilde{u} \geq u \text{ on } \partial A \cap N \cap X. \tag{2}$$

Since $\partial X \cap B$ has positive harmonic measure [11, p. 67] and $(\partial A \setminus N) \cap X$ is compact, we may choose M so large that also

$$\tilde{u} \geq u \text{ on } (\partial A \setminus N) \cap X. \tag{3}$$

From (1), (2), (3), Lemma 1, the maximum principle, and the fact that

$$\tilde{u} = u \text{ on } \partial X \cap \bar{U}, \tag{4}$$

we have that

$$\tilde{u} \geq u \text{ on } A \cap X. \tag{5}$$

Since $\partial X \cap \bar{U} \subset A$, it follows from (4) and (5), that $\tilde{u}$ satisfies the mean value inequality for points of $\partial X \cap (\bar{U} \cup X)^0$ and sufficiently small regular neighbourhoods thereof. Thus, since $\tilde{u}$ is clearly subharmonic at other points of $U \cup X$, it follows [11, p. 72], that $\tilde{u}$ is a subharmonic function on all of $(\bar{U} \cup X)^0$.

Since any point of $\partial U \cap G$ lies on the boundary of one and only one such component X of $G \setminus \bar{U}$, we may apply the above extension procedure to all components X of $G \setminus \bar{U}$ and obtain a well-defined function $\tilde{u}$ which is subharmonic on $U \cup G$.

Since $\Omega^* \setminus \bar{U}$ is connected and locally connected, there is a compact set $Q \subset \Omega$ such that $K \subset Q^0$ and $\Omega^* \setminus (\bar{U} \cup Q)$ is connected. Set

$$\tilde{V} = \Omega \setminus \overline{[\Omega \setminus (\bar{U} \cup Q)]}.$$

Now choose G as at the beginning of this proof, but with the additional property that $Q \subset G$, and let $\tilde{u}$ be defined as above. Then, since

$$\Omega^* \setminus \tilde{V} = \{*\} \cup \overline{[\Omega \setminus (\bar{U} \cup Q)]},$$

$\tilde{u}$ and $\tilde{V}$ have the properties required by Lemma 3.

Now to prove g) $\longrightarrow$ f), suppose that $\Omega^* \setminus W$ is connected and that u is a subharmonic function on W. Let E be a subset of W which is closed in Ω. We wish to extend u to a function $\tilde{u}$ subharmonic on Ω such that $\tilde{u} = u$ on E. By Lemma 2, we may assume that $\Omega^* \setminus E$ is connected and locally connected. In particular, the ideal point $*$ has a neighbourhood base in $\Omega^* \setminus E$ consisting of connected sets. This means that there is a sequence of compact sets $K_1 \subset K_2 \subset \cdots$ in Ω such that each compact set $K \subset \Omega$ is contained in some K_j and for each j, $\Omega^* \setminus (E \cup K_j)$ is connected.

Set $u_0 = u$ and $K_0 = \emptyset$. By Lemma 3, we may inductively construct a sequence u_j of functions such that: for $j = 1, 2, \ldots$, u_j is a subharmonic function on a neighbourhood V_j of $E \cup K_j$; $\Omega^* \setminus V_j$ is connected; and $u_j = u_{j-1}$ on $E \cup K_{j-1}$. Hence, the function $\tilde{u}$ defined as u_j on each $E \cup K_j$ is a well-defined subharmonic function on all of Ω. This completes the proof of the theorem.

An application. Let $\tilde{\Omega}$ be a compactification of the Riemannian manifold Ω and let $\tilde{\partial}\Omega$ denote the ideal boundary $\tilde{\Omega} \setminus \Omega$. By a continuous path in Ω, we mean a continuous mapping $\sigma: [0, +\infty) \longrightarrow \Omega$. Such a continuous path is said to tend to a set F if, for each neighbourhood V of F, there is a $t_V \in [0, +\infty)$ such that $\sigma(t) \in V$, for all $t > t_V$. A subset F of the ideal boundary $\tilde{\partial}\Omega$ is said to be *accessible* from Ω if there is a continuous path in Ω which tends to F. The following maximum principle was stated by Chen Huaihui and the author. It gives a characterization of those boundary sets which can be ignored in the maximum principle (without even assuming boundedness of the functions).

THEOREM 2 ([12]). *Let $\tilde{\Omega}$ be a second countable compactification of the Riemannian manifold Ω and let F be a closed subset of the ideal boundary $\tilde{\partial}\Omega$. Then, a necessary and sufficient condition in order that*

$$\sup_{\Omega} u = \sup_{\tilde{\partial}\Omega\setminus F} u,$$

for every subharmonic function u on Ω, is that F not be accessible from Ω.

In [12] only the sufficiency was proved; for the necessity, the authors referred to a future paper (indeed, the present one). We now complete the proof of Theorem 2.

PROOF. Suppose $F \subset \tilde{\partial}\Omega$ is closed and accessible. Let σ be a continuous path tending to F. We may construct disjoint open neighbourhoods W_σ and W_0 of σ and $\tilde{\partial}\Omega \setminus F$ respectively, such that, setting

$$W = (W_0 \cap \Omega) \cup (W_\sigma \cap \Omega),$$

we have that $\Omega^* \setminus W$ is connected. Define u to be 1 on W_σ and 0 on W_0. Let $\tilde{E}$ be a closed neighbourhood of $F \cup \sigma$ contained in $W_0 \cup W_\sigma$, and set $E = \tilde{E} \cap \Omega$. Then, by Theorem 1, there is a (continuous) subharmonic function $\tilde{u}$ on Ω which agrees with u on E. This function fails to satisfy the maximum principle in Theorem 2. This proves the necessity of the condition in Theorem 2. In fact using the theorem of Bagby and Blanchet (g) $\longrightarrow$ b) of Theorem 1), we can even obtain a harmonic function $\tilde{u}$ for which the above maximum principle fails.

Related problems. We mention several problems regarding subharmonic extensions. For each of these problems, there is an analogous approximation problem, not necessarily open, which we also state (in parentheses).

1. The concept of a subharmonic extension (respectively Runge) pair was defined for W an open subset of Ω. Similarly, if E is a closed or compact subset of Ω, we may say that (E, Ω) is a subharmonic extension (resp. Runge) pair if each function u subharmonic on a neighbourhood of E can be extended (resp. approximated) by a function $\tilde{u}$ subharmonic on Ω. From the discussion in the present note, it follows that if $\Omega^* \setminus E$ is connected and locally connected, then (E, Ω) is an extension (thus, trivially, a Runge) pair. At this time, however, we are unable to characterize such pairs.

2. Let us say that a closed set E is a subharmonic extension (resp. Walsh) set if each function upper-semicontinuous on E and subharmonic on E^0 can be extended (resp. approximated) by a function subharmonic on a neighbourhood of E. The approximation version of this problem was considered by Shirinbekov [26] who claimed that a compact set $E \subset R^n$ is a subharmonic Walsh set if and only if it is a harmonic Walsh set. So once again we would have the same solution for the subharmonic problem as for the harmonic problem. The proof in [26] is not quite complete; however, Shirinbekov has subsequently given a complete proof (in the continuous case) [27]. This result is also a corollary of the solution to problem 2I below.

The above problems on classes of functions also have individual-function versions which we briefly state.

0I. The analog of the theorem of the present paper, for individual functions, is the following problem. Given W open in Ω, which are the functions on W which can be extended (resp. approximated) from (resp. on) compact or closed subsets of W by subharmonic functions on Ω?

1I. A similar question can be raised replacing the open set W by a closed or compact set E. The question is then to describe which functions on E can be extended (resp. approximated) by subharmonic functions on Ω. Sufficient conditions can be inferred from solutions to problems 2I and 0I.

2I. Given a closed or compact subset E, which functions on E can be extended (resp. approximated) by functions subharmonic on a neighbourhood of E? The approximation version of this problem was solved (at least in the continuous case) by Bliedtner and Hansen for the case that E is compact [10]. Once again, the solution is completely analogous to the corresponding problem on harmonic approximation. Namely, a function u given on a compact set E can be uniformly approximated by functions continuous and subharmonic (resp. harmonic) on neighbourhoods of E if and only if u is continuous on E and finely subharmonic (resp. harmonic) on the fine interior of E (see [9], [10], and [13]). Such results also hold on closed sets for harmonic [17] approximations. Analogous questions for subharmonic functions are presently being investigated by Charaf Bensouda [8].

For applications, it would be desirable to investigate the possibility of extending smooth subharmonic functions smoothly. The analogous approximation problems should perhaps be treated using appropriate C^m-norms. Such problems for harmonic approximation have been treated in depth and may suggest the appropriate direction for the subharmonic case. See the recent paper of Paramonov [23] for harmonic C^1-approximation and for a survey of the present state of harmonic C^m-approximation.

One can of course consider plurisubharmonic (rather than subharmonic) functions. Recently, interesting results have been obtained on approximation by plurisubharmonic functions [14] (see also [15]).

This note was written while the author was visiting the Steklov Institute, via the Queen's-Steklov exchange, and the University of Warszawa. The author thanks all of these institutions.

NOTE. Recently, Professor Stephen Gardiner has independently obtained interesting results related to, but distinct from, ours in a preprint entitled "Superharmonic extension and harmonic approximation". Therein he solves problem 1 as well as the harmonic analog thereof.

In closing, I wish to thank P. V. Paramonov as well as the referee (presumably not the same person) for their helpful suggestions.

REFERENCES

1. V. Anandam, *Théorie du potentiel dans un espace harmonique sans potentiel positif*, Mathématiques, Bull. Sect. Sci. **3**(1981), 7–36.
2. ———, *Subharmonic functions outside a compact set in R^n*, Proc. Amer. Math. Soc. **84**(1982), 52–54.
3. ———, *Superharmonic extensions and the flux*, Bull. Sci. Math. **106**(1982), 85–92.
4. D. H. Armitage, *On the extension of superharmonic functions*, J. London Math. Soc. (2) **4**(1971), 215–230.
5. N. Aronsjajn, *A unique continuation theorem for solutions of elliptic equations or inequalities of second order*, J. Math. Pure Appl. **36**(1957), 235–249.
6. T. Bagby and P. Blanchet, *Uniform harmonic approximation on Riemannian manifolds*, J. d'Analyse Math., to appear.
7. H. Bauer, *Harmonische Räume und ihre Potentialtheorie*, SLN **22**, Springer-Verlag, Heidelberg, 1966.
8. Ch. Bensouda, Thèse, Université de Montréal, en rédaction.
9. J. Bliedtner and W. Hansen, *Simplicial cones in potential theory*, Invent. Math. **29**(1975), 83–110.
10. ———, *Simplicial cones in potential theory II*, Invent. Math. **46**(1978), 255–275.
11. M. Brelot, *Lectures on Potential Theory*, Tata Institute, Bombay, 1960.
12. H. Chen and P. M. Gauthier, *A maximum principle for subharmonic and plurisubharmonic functions*, Canad. Math. Bull **35**(1992), 1–6.
13. A. Debiard and B. Gaveau, *Potentiel fin et algèbres de fonctions analytiques I*, J. Funct. Anal. **16**(1974), 289–304.
14. J. E. Fornaess and J. Wiegerinck, *Approximation of plurisubharmonic functions*, Ark. Mat. **27**(1989), 257–272.
15. P. M. Gauthier, *Approximation by (pluri)subharmonic functions: fusion and localization*, Canad. J. Math. **44**(1992), 941–950.
16. P. M. Gauthier, M. Goldstein and W. H. Ow, *Uniform approximation on closed sets by harmonic functions with Newtonian singularities*, J. London Math. Soc. (2) **28**(1983), 71–82.
17. P. M. Gauthier and S. Ladouceur, *Uniform approximation and fine potential theory*, J. Approx. Theory **72**(1993), 138–140.
18. H. Guessous, *Prolongement surharmonique dans un espace harmonique*, Notion du flux. Potential Theory, Copenhagen 1979, SLN **787**, 121–143, Springer-Verlag, Heidelberg, 1980.
19. A. de la Pradelle, *Approximation et caractère de quasi-analyticité dans la théorie axiomatique des fonctions harmoniques*, Ann. Inst. Fourier **17**(1967), 383–399.
20. P. Lax, *A stability theorem for abstract differential equations and its application to the study of the local behavior of solutions of elliptic equations*, Commun. Pure Appl. Math. **9**(1956), 747–766.
21. C. Miranda, *Partial Differential Equations of Elliptic Type*, 2nd. ed., Springer-Verlag, Heidelberg, 1970.
22. R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland, Amsterdam, 1968.
23. P. V. Paramonov, *On harmonic approximation in the C^1-norm*, Mat. Sb. **181**(1990), 1341–1365; English transl. in Math. USSR Sbornik **71**(1992), 183–207.
24. M. Premalatha, *On a superharmonic extension*, Rev. Roum. Math. Pures Appl. **26**(1981), 631–640.
25. L. Sario, M. Nakai, C. Wang and L. O. Chung, *Classification Theory of Riemannian Manifolds*, SLN **605**, Springer-Verlag, Heidelberg, 1977.
26. M. Shirinbekov, *On Hartogs compacts of holomorphy*, Mat. Sb. **115**(1981), 453–462; English transl. in Math. USSR Sbornik **43**(1982), 403–411.
27. ———, *Private communication*, 1990.
28. J. L. Walsh, *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions*, Bull. Amer. Math. Soc. **35**(1929), 499–544.

Département de mathématiques et de statistique et
Centre de recherches mathématiques
Université de Montréal, CP 6128-A
Montréal, Québec H3C 3J7
e-mail: gauthier@ere.umontreal.ca