# Contents

Preface xi

I Preliminaries 1

1 Survey and History of LCF 3
   1.1 The structure of LCF 4
   1.2 A brief history 7
   1.3 Further reading 10

2 Formal Proof in First Order Logic 13
   2.1 Fundamentals of formal logic 13
   2.2 Introduction to first order logic 14
   2.3 Conjunction 17
   2.4 Disjunction 19
   2.5 Implication 20
   2.6 Negation 22
   2.7 Understanding quantifiers 25
   2.8 The universal quantifier 29
   2.9 The existential quantifier 30
   2.10 Mathematical induction 33
   2.11 Equality 34
   2.12 A sequent calculus for natural deduction 38
   2.13 A sequent calculus for backwards proof 42
   2.14 Classical deduction in a sequent calculus 46
   2.15 How to find formal proofs 50
   2.16 Further reading 52

3 A Logic of Computable Functions 53
   3.1 The lambda calculus 53
   3.2 Semantic questions 57
   3.3 Computable functions 59
   3.4 Axioms and rules of the logic 68
   3.5 Fixed point induction 69
   3.6 Admissibility of fixed point induction 74
Contents

3.7 Further reading .............................................. 75

4 Structural Induction ........................................ 77
4.1 The Cartesian product of two types ......................... 78
4.2 The strict product of two types .............................. 83
4.3 The strict sum of two types ................................. 86
4.4 Lifted types ................................................. 90
4.5 Atomic types ................................................. 93
4.6 Recursive type definitions ................................. 94
4.7 The inverse limit construction for recursive domains .... 98
4.8 The type of lazy lists ........................................ 110
4.9 The type of strict lists ...................................... 114
4.10 Formal reasoning about types ............................. 118
4.11 Structural induction over lazy lists ....................... 125
4.12 Structural induction over strict lists ..................... 129
4.13 Automating the derivation of induction .................. 134
4.14 Further reading ............................................. 135

II Cambridge LCF .............................................. 137

5 Syntactic Operations for PPL ................................ 139
5.1 The syntax of PPL .......................................... 139
5.2 Quotations .................................................. 145
5.3 Primitive constructors and destructors .................... 150
5.4 Compound constructors and destructors ................... 154
5.5 Functions required for substitution ....................... 156
5.6 Pattern matching primitives ............................... 159
5.7 Terminal interaction and system functions ............... 160

6 Theory Structure ............................................. 163
6.1 Drafting a new theory ...................................... 164
6.2 Using a theory .............................................. 174
6.3 Inspecting a theory ........................................ 177
6.4 Limitations of theories .................................... 179

7 Axioms and Inference Rules ................................ 181
7.1 The representation of inference rules ..................... 181
7.2 First order logic ........................................... 183
7.3 Domain theory ............................................. 194
7.4 Forwards proof and derived rules ......................... 200
7.5 Discussion and further reading ............................ 207
# Contents

## 8 Tactics and Tacticals 209
  8.1 The validation of a backwards step 209
  8.2 Tactics for first order logic 213
  8.3 Domain theory 219
  8.4 Simple backwards proof 220
  8.5 Complex tactics 224
  8.6 The subgoal package 228
  8.7 Tacticals 234
  8.8 Discussion and further reading 244

## 9 Rewriting and Simplification 245
  9.1 The extraction of rewrite rules 246
  9.2 The standard rewriting strategy 248
  9.3 Top-level rewriting tools 249
  9.4 Conversions 257
  9.5 Implementing new rewriting strategies 262
  9.6 Further reading 263

## 10 Sample Proofs 265
  10.1 Addition of natural numbers 265
  10.2 Commutativity of addition 271
  10.3 Equality on the natural numbers 273
  10.4 A simple fixed point induction 278
  10.5 A mapping functional for infinite sequences 281
  10.6 Project suggestions 287

Bibliography 289

Index 296