A NOTE ON DUNFORD–PETTIS OPERATORS

by J. R. HOLUB

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Talagrand has shown [**4**, p. 76] that there exists a continuous linear operator from $L^1[0, 1]$ to c_0 which is not a Dunford–Pettis operator. In contrast to this result, Gretsky and Ostroy [**2**] have recently proved that every positive operator from $L^1[0, 1]$ to c_0 is a Dunford–Pettis operator, hence that every regular operator between these spaces (i.e. a difference of positive operators) is Dunford–Pettis. In this note we prove that the converse is also true, thereby characterizing the Dunford–Pettis operators from $L^1[0, 1]$ to c_0 as follows:

THEOREM. *An operator $T: L^1[0, 1] \to c_0$ is a Dunford–Pettis operator if and only if $T = T_1 - T_2$ where T_1 and T_2 are positive operators.*

In order to prove this result we need the following simple representation for bounded linear operators from $L^1[0, 1]$ to c_0. As usual, $\{e_n\}_{n=1}^{\infty}$ denotes the basis for c_0 and l^p, $p \geq 1$, defined by $e_n = \{\delta_{ni}\}_{i=1}^{\infty}$.

LEMMA. *A linear operator $W: L^1[0, 1] \to c_0$ is continuous if and only if there is a sequence $\{H_n\}_{n=1}^{\infty}$ in $L^{\infty}[0, 1]$ which converges to zero in w*-topology and for which $W(f) = \sum_{n=1}^{\infty} \langle H_n, f \rangle e_n$ for all f in $L^1[0, 1]$.*

Proof. ($\Rightarrow$): If W is continuous from $L^1[0, 1]$ to c_0 then for every f in $L^1[0, 1]$ we have $W(f) = \sum_{n=1}^{\infty} \langle Wf, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle W^* e_n, f \rangle e_n$, where $\{W^* e_n\}_{n=1}^{\infty} = \{H_n\}_{n=1}^{\infty}$ is a sequence in $L^{\infty}[0, 1]$ which is w*-convergent to zero since W^* is w*-continuous from l^1 to $L^{\infty}[0, 1]$.

($\Leftarrow$): If such a sequence $\{H_n\}_{n=1}^{\infty}$ exists then $\{\langle H_n, f \rangle\}_{n=1}^{\infty} \in c_0$ for all f in $L^1[0, 1]$, implying that $\sum_{n=1}^{\infty} \langle H_n, f \rangle e_n$ converges in c_0 for all f in $L^1[0, 1]$. Hence by the Uniform Boundedness Principle the operator W defined by $W(f) = \sum_{n=1}^{\infty} \langle H_n, f \rangle e_n$ is continuous.

Proof of Theorem. ($\Leftarrow$): If T is the difference of positive operators then T is a Dunford–Pettis operator by the theorem of Gretsky and Ostroy [**2**].

($\Rightarrow$): Conversely, suppose $T: L^1[0, 1] \to c_0$ is a Dunford–Pettis operator. Let $\{H_n\}_{n=1}^{\infty}$ be the corresponding sequence in $L^{\infty}[0, 1]$ (equal to $\{T^* e_n\}_{n=1}^{\infty}$) for which $T(f) = \sum_{n=1}^{\infty} \langle H_n, f \rangle e_n$ for all f in $L^1[0, 1]$, which exists by the preceding Lemma. Since T is Dunford–Pettis it is well-known that if $i: L^{\infty}[0, 1] \to L^1[0, 1]$ denotes the canonical injection map then the operator $T \circ i: L^{\infty}[0, 1] \to L^1[0, 1] \to c_0$ is compact [**1**]. Hence applying the Banach–Steinhaus theorem [**3**, p. 86] to the partial sum operators associated with the basis $\{e_n\}_{n=1}^{\infty}$ for c_0 gives

$$\lim_{N\to\infty} \sup_{\substack{\|g\|\leq 1\\ g\in L^\infty[0,1]}} \left\| \sum_{n=N}^{\infty} \langle T\circ i(g), e_n\rangle e_n \right\| = 0,$$

or

$$\lim_{N\to\infty} \sup_{\substack{\|g\|\leq 1\\ g\in L^\infty[0,1]}} \left\| \sum_{n=N}^{\infty} \langle i(T^*e_n), g\rangle e_n \right\| = 0.$$

Since $T^*e_n = H_n$ it follows from the definition of the norm in c_0 that $\lim_{n\to\infty} \sup_{\substack{\|g\|\leq 1\\ g\in L^\infty[0,1]}} |\langle i(H_n), g\rangle| = 0$, and hence that $\lim_{n\to\infty} \|i(H_n)\|_1 = 0$, where $\|\cdot\|_1$ denotes the norm in $L^1[0, 1]$.

Now for each $n = 1, 2, \ldots$, $H_n = H_n^+ - H_n^-$ where each of these is non-negative a.e. on $[0, 1]$. Moreover since $\|i(H_n)\|_1 \to 0$ it follows that $\|i(H_n^+)\|_1 \to 0$ and $\|i(H_n^-)\|_1 \to 0$ as well. But then since $\{H_n^+\}$ and $\{H_n^-\}$ are bounded sequences of non-negative functions in $L^\infty[0, 1]$ it is clear that for any f in $L^1[0, 1]$ we have $\lim_{n\to\infty} \int_0^1 H_n^+(t)f(t)\,dt = \lim_{n\to\infty} \int_0^1 H_n^-(t)f(t)\,dt = 0$.

That is, $\{H_n^+\}_{n=1}^{\infty}$ and $\{H_n^-\}_{n=1}^{\infty}$ both converge to zero in the w*-topology on $L^\infty[0, 1]$, implying, by the Lemma, that the operators T_1 and T_2 from $L^1[0, 1]$ to c_0 defined by $T_1(f) = \sum_{n=1}^{\infty} \langle H_n^+, f\rangle e_n$ and $T_2(f) = \sum_{n=1}^{\infty} \langle H_n^-, f\rangle e_n$ are both continuous. Since T_1 and T_2 are clearly positive and $T = T_1 - T_2$, the proof is complete.

Remarks. It turns out that the result of Gretsky and Ostroy actually holds for a positive operator $T: L^1(\mu) \to c_0$, where μ is any σ-finite measure. The proof in this general case is, of necessity, very different from that given by Gretsky and Ostroy for the case $L^1[0, 1]$ owing to the fact that the characterization of Dunford–Pettis operators given in [**1**] and used in their proof (and in that of the Theorem above) does not apply in the more general case. This result and others related to the ideas in this note (e.g., if X is any separable Banach space in which weak and norm convergence of sequences are not the same then there is a non-Dunford–Pettis operator $T: X \to c_0$) will be given in a later paper devoted to a more comprehensive study of Dunford–Pettis operators then we have attempted here.

REFERENCES

1. J. Bourgain, Dunford–Pettis operators on L^1 and the Radon–Nikodym property, *Israel J. Math.* **37** (1980), 27–34.

2. N. Gretsky and J. Ostroy, The compact range property and c_0, *Glasgow Math. J.* **28** (1986), 113–114.

3. H. Schaefer, *Topological vector spaces* (Macmillan, 1966).

4. M. Talagrand, Pettis integral and measure theory, *Mem. Amer. Math. Soc.* **307** (1984).

VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY
BLACKSBURG VIRGINIA 24061