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SMOOTHNESS OF CONVOLUTION PRODUCTS OF ORBITAL MEASURES ON RANK ONE COMPACT SYMMETRIC SPACES

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Abstract

We prove that all convolution products of pairs of continuous orbital measures in rank one, compact symmetric spaces are absolutely continuous and determine which convolution products are in L^2 (meaning that their density function is in L^2). We characterise the pairs whose convolution product is either absolutely continuous or in L^2 in terms of the dimensions of the corresponding double cosets. In particular, we prove that if G/K is not $\mathrm{SU}(2)/\mathrm{SO}(2)$, then the convolution of any two regular orbital measures is in L^2, while in $\mathrm{SU}(2)/\mathrm{SO}(2)$ there are no pairs of orbital measures whose convolution product is in L^2.

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1. Introduction

Let G/K be an irreducible, simple, simply connected, compact symmetric space. By an orbital measure, μ_z, we mean the K-bi-invariant, singular measure on G supported on the double coset KzK. We prove that, in any rank one symmetric space, the convolution product of two orbital measures, $\mu_{z_1} * \mu_{z_2}$, is absolutely continuous if and only if

$$\dim Kz_1K + \dim Kz_2K \geq \dim G/K \tag{1.1}$$

and if and only if μ_{z_1} and μ_{z_2} are both continuous. For short, we write $\mu_{z_1} * \mu_{z_2} \in L^1(G)$ because being absolutely continuous is equivalent to the density function belonging to L^1. Furthermore, we prove that $\mu_{z_1} * \mu_{z_2} \in L^2(G)$ if and only if the inequality (1.1) is strict. We show that only four of the infinite number of rank one symmetric spaces admit a pair of continuous orbital measures whose convolution is not in L^2.

It was previously shown in [1] that there are continuous orbital measures in the rank one symmetric space $\mathrm{SU}(2)/\mathrm{SO}(2)$ whose convolution is in L^1, but not in L^2.

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This came as a surprise because, in the special case that the symmetric space is $(H \times H)/H \sim H$ for a compact Lie group H, it is known that $\mu_z^p \in L^1(H)$ if and only if $\mu_z^p \in L^2(H)$ for all integers p, where the exponent p means the p-fold convolution product [8]. One consequence of our characterisation is that there are no pairs of orbital measures for $\mathrm{SU}(2)/\mathrm{SO}(2)$ whose convolution is in L^2.

The continuous orbital measures on $\mathrm{SU}(2)/\mathrm{SO}(2)$ are all examples of what are called 'regular' orbital measures. (For the definition, see Section 2.) Previously, it was shown that, in any symmetric space, the convolution of two regular orbital measures is in L^1 [7]. Here we see that, in any rank one symmetric space, the convolution of any two continuous orbital measures is in L^1 and if G/K is any rank one symmetric space other than $\mathrm{SU}(2)/\mathrm{SO}(2)$, then the convolution of any two regular orbital measures is in $L^2(G)$. We also prove that, in any rank one symmetric space, the product of any three continuous orbital measures belongs to L^2. Previously it was known that such a three-fold product was in L^1 [9], with the sharper L^2 result known only for $\mathrm{SU}(2)/\mathrm{SO}(2)$ [1].

The problem of establishing the absolute continuity of convolution products of orbital measures was originally studied by Ragozin in [14]. Extensive treatment of the absolute continuity problem in the noncompact case has been carried out by Graczyk and Sawyer (see [5, 6]).

2. Absolutely continuous convolution products

2.1. Notation and terminology. If G is a compact group and K is a compact connected subgroup fixed by an involution θ, then G/K is called a compact symmetric space. We will assume that G/K is an irreducible, simple, simply connected, compact symmetric space of Cartan type I. Of primary interest are the spaces of rank one; see the appendix for a complete list. Let $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ (the Lie algebra of G), let $\mathfrak{a}$ denote a maximal abelian subalgebra of $\mathfrak{p}$ and assume that $\mathfrak{t}$ is a torus of $\mathfrak{g}$ that contains $\mathfrak{a}$. Then $K = \exp(\mathfrak{k})$ and, if we let $A = \exp(i\mathfrak{a})$, $G = KAK$. Hence every double coset, KzK, contains an element z in A.

We denote the set of positive roots of $(\mathfrak{g}, \mathfrak{t})$ by Σ^+ and let

$$\Phi^+ = \{\alpha = \beta|_{\mathfrak{a}} : \beta \in \Sigma^+, \beta|_{\mathfrak{a}} \neq 0\}$$

be the set of (positive) *restricted roots*. When G/K is rank one, there is either one positive restricted root, α, or there are two, α and 2α. We write m_β for the multiplicity of the restricted root β, that is, the dimension of the restricted root space $\mathfrak{g}_\beta$. We remark that, in the rank one spaces, the dimension of $G/K = \dim \mathfrak{p} = m_\alpha + m_{2\alpha} + 1$ and it is always the case that $m_\alpha \geq 1 + m_{2\alpha}$. For the convenience of the reader, we list important facts about these spaces and their restricted root systems in the appendix. Further information about these spaces can be found, for example, in [2, 10, 12].

By an *orbital measure* on the compact symmetric space G/K, we mean the probability measure denoted by μ_z, for $z \in G$, defined by

$$\int_G f \, d\mu_z = \int_K \int_K f(k_1 z k_2) \, dm_K(k_1) \, dm_K(k_2)$$

for all continuous functions f on G. The orbital measure is K-bi-invariant and it is singular because it is supported on the double coset KzK, which is a set of Haar measure zero. Since every double coset contains an element of A, there is no loss of generality in assuming that $z \in A$. The measure μ_z is continuous (that is, nonatomic) if and only if $z \notin N_G(K)$, the normaliser of K in G.

It was shown in [9], that if $r = \operatorname{rank} G/K$ and μ_{x_j} are continuous for $j = 1, \ldots, 2r+1$, then $\mu_{x_1} * \cdots * \mu_{x_{2r+1}}$ is absolutely continuous with respect to Haar measure, meaning that its density function (or Radon–Nikodym derivative) is in $L^1(G)$. In particular, the convolution product of any three continuous orbital measures, on any rank one symmetric space, has density function in L^1. This improved upon the much earlier work of Ragozin [14] who had shown that any product of $\dim G/K$, continuous, orbital measures is absolutely continuous. If, instead, $x \in N_G(K)$, then μ_x^p (the p-fold convolution of μ_x) is singular with respect to Haar measure for all p since, in this case, μ_x^p is supported on the subset $(KxK)^p = x^p K$, which is a set of Haar measure zero.

Given $z \in A$, say, $z = e^{iZ}$ for $Z \in \mathfrak{a}$, we let

$$\Phi_z = \{\alpha \in \Phi^+ : \alpha(Z) = 0 \ (\mathrm{mod}\ \pi)\}$$

be the set of *annihilating roots* of z. The annihilating roots are very important in questions about orbital measures and double cosets. For instance, the dimension of KzK equals $\sum_{\beta\in\Phi^+\setminus\Phi_Z} m_\beta$. It is known that $\Phi_z = \Phi^+$ if and only if $z \in N_G(K)$. In particular, if $z \notin N_G(K)$, then $\dim KzK \geq m_\alpha$.

We call z *regular* if Φ_z is empty and then we will also call μ_z regular. In this case $\dim KzK = m_\alpha + m_{2\alpha}$. If G/K has only one restricted root, then every $z \in A$ is either regular or belongs to $N_G(K)$, equivalently, μ_z is either regular or not continuous. This is the situation with SU(2)/SO(2). If a rank one symmetric space G/K has two positive restricted roots, then a continuous orbital measure μ_z is not regular if and only if $2\alpha(Z) \equiv 0 \ (\mathrm{mod}\ \pi)$, but $\alpha(Z) \not\equiv 0 \ (\mathrm{mod}\ \pi)$ and then $\dim KzK = m_\alpha$.

2.2. Absolute continuity of convolution products. In [7] it was shown that if z_1, z_2 are both regular, then $\mu_{z_1} * \mu_{z_2}$ is absolutely continuous. Similar arguments show that the same conclusion is true if z_1 is regular and $z_2 \notin N_G(K)$ or *vice versa*. Our first result is to prove that the same conclusion holds for the convolution of any two continuous orbital measures in a rank one symmetric space.

Theorem 2.1. *If G/K is a rank one symmetric space, then $\mu_{z_1} * \mu_{z_2}$ is absolutely continuous if $z_1, z_2 \notin N_G(K)$.*

Proof. We will write E_β for any restricted root vector in $\mathfrak{g}_\beta$. To simplify notation, we will write E_β^- for $E_\beta - \theta E_\beta$ and E_β^+ for $E_\beta + \theta E_\beta$. Note that $E_\beta^+ \in \mathfrak{k}$ and $E_\beta^- \in \mathfrak{p}$.

Given $z \in A$, let

$$\mathcal{N}_z = sp\{E_\beta^- : \text{restricted root}\ \beta \notin \Phi_z\} \subseteq \mathfrak{p},$$

where sp denotes the real span. It was shown in [9] that $\mu_{z_1} * \mu_{z_2}$ is absolutely continuous if and only if there is some $k \in K$ such that

$$\mathfrak{p} = sp\{\mathcal{N}_{z_1}, Ad(k)(\mathcal{N}_{z_2})\}. \tag{2.1}$$

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As remarked above, the result is already known if, in addition, either z_1 or z_2 is regular. So assume otherwise. In particular, we can assume that G/K has two positive restricted roots, $\Phi_{z_1} = \Phi_{z_2} = \{2\alpha\}$ and $\mathcal{N}_{z_1} = \mathcal{N}_{z_2} = sp\{E_\alpha^{(j)-} : E_\alpha^{(j)}$ is a basis for $\mathfrak{g}_\alpha\}$. Put $E_\alpha^{(1)} = E_\alpha$ and let $k_t = \exp tE_\alpha^+ \in K$ for small $t > 0$.

Standard facts about root vectors and the Lie bracket imply that

$$\begin{aligned}[E_\alpha^+, E_\alpha^{(j)-}] &= [E_\alpha, E_\alpha^{(j)}] - \theta[E_\alpha, E_\alpha^{(j)}] + [\theta E_\alpha, E_\alpha^{(j)}] - \theta[\theta E_\alpha, E_\alpha^{(j)}] \\ &= [E_\alpha, E_\alpha^{(j)}] - \theta[E_\alpha, E_\alpha^{(j)}] + H_j,\end{aligned}$$

where $H_j \in \mathfrak{a}$. Since $[g_\alpha, g_{2\alpha}] = 0 = [g_{2\alpha}, g_{2\alpha}]$ and $[g_\alpha, g_\alpha] \subseteq g_{2\alpha}$, it follows, from [3], that there is a scalar $c > 0$ such that, for every $Z \in \mathfrak{g}_{2\alpha}$, there is some $J_Z = J_Z(E_\alpha) \in \mathfrak{g}_\alpha$ with

$$[E_\alpha, J_Z] = cZ.$$

Temporarily fix $Z = E_{2\alpha}^{(j)}$, and assume that $J_Z = \sum d_i E_\alpha^{(i)}$. With $J_Z^- = J_Z - \theta J_Z$, the observations above imply that

$$[E_\alpha^+, J_Z^-] = \sum d_i([E_\alpha, E_\alpha^{(i)}] - \theta[E_\alpha, E_\alpha^{(i)}]) + H_Z$$

for some $H_Z \in \mathfrak{a}$. Thus

$$[E_\alpha^+, J_Z^-] = [E_\alpha, J_Z] - \theta[E_\alpha, J_Z] + H_Z = c(Z - \theta Z) + H_Z = cE_{2\alpha}^{(j)-} + H_Z.$$

Since $E_\alpha^-, J_Z^- \in \mathcal{N}_{z_2}$, and $[E_\alpha^+, E_\alpha^-]$ is a nonzero element (and hence generator) of $\mathfrak{a}$,

$$sp\{\mathcal{N}_{z_1}, ad(E_\alpha^+)(\mathcal{N}_{z_2})\} = \mathfrak{p}.$$

As $\exp tE_\alpha^+ = \mathrm{Id} + t \cdot ad(E_\alpha^+) + P_t$ for some operator P_t with norm $O(t^2)$, it follows that, for small enough $t > 0$, $sp\{\mathcal{N}_{z_1}, Ad(k_t)(\mathcal{N}_{z_2})\} = \mathfrak{p}$. (We refer the reader to [7] for the details of a similar argument.) This completes the proof. □

Remark 2.2. Observe that $\dim KzK = \dim \mathcal{N}_z$.

Corollary 2.3. *For a rank one symmetric space G/K, the following are equivalent:*

(1) $\mu_{z_1} * \mu_{z_2}$ *is absolutely continuous;*
(2) μ_{z_1} *and* μ_{z_2} *are continuous measures; and*
(3) $\dim Kz_1K + \dim Kz_2K \geq G/K$.

Proof. Theorem 2.1 gives that (2) implies (1) since μ_z is continuous if and only if $z \notin N_G(K)$.

Since $\dim sp\{\mathcal{N}_{z_1}, Ad(k)(\mathcal{N}_{z_2})\} \leq \dim \mathcal{N}_{z_1} + \dim \mathcal{N}_{z_2}$, it is immediate from (2.1) and Remark 2.2 that if $\dim Kz_1K + \dim Kz_2K < \dim \mathfrak{p} = \dim G/K$, then $\mu_{z_1} * \mu_{z_2}$ is not absolutely continuous. Thus (1) implies (3).

Lastly, we observe that if, say, μ_{z_1} is not continuous, then $\dim Kz_1K = 0$. Thus $\dim Kz_1K + \dim Kz_2K < \dim G/K$, so (3) implies (2). □

Remark 2.4. It follows, from [14], that the absolute continuity of $\mu_{z_1} * \mu_{z_2}$ is also equivalent to Kz_1Kz_2K having nonempty interior.

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3. Convolution products that are in L^2

In the remainder of this paper, we study when the convolution product of orbital measures belongs to the smaller space $L^2(G)$. We will do this by estimating the decay in the Fourier transform of orbital measures. For this, we introduce further notation.

NOTATION 3.1. An irreducible, unitary representation (π, V_π) of G is called *spherical* if there exists a K-invariant vector in V_π. It is known that the dimension of the K-invariant subspace, V_π^K, is one (see [1]). We will let $X_1 = X_\pi, X_2, \ldots, X_{\dim V_\pi}$ be an orthonormal basis for V_π, where we suppose that V_π^K is spanned by X_π.

The following facts can essentially be found in [1] (and are valid in any compact symmetric space, not just those of rank one).

LEMMA 3.2. *For any $x, y \in G$, $\langle \widehat{\mu_x}(\pi)\widehat{\mu_y}(\pi)X_i, X_j\rangle = 0$ if $(i, j) \neq (1, 1)$ and*

$$\langle \widehat{\mu_x}(\pi)\widehat{\mu_y}(\pi)X_\pi, X_\pi\rangle = \langle \pi(x^{-1})X_\pi, X_\pi\rangle\langle \pi(y^{-1})X_\pi, X_\pi\rangle.$$

PROOF. It is shown in [1] that, for all i, $\widehat{\mu_x}(\pi)X_i \in V_\pi^K = spX_\pi$, $\widehat{\mu_x}(\pi)X_i = 0$ for all $i \neq 1$ and $\langle \widehat{\mu_x}(\pi)X_\pi, X_\pi\rangle = \langle \pi(x^{-1})X_\pi, X_\pi\rangle$. □

PROPOSITION 3.3. *For all $x, y \in G$,*

$$\|\mu_x * \mu_y\|_2^2 = \|\widehat{\mu_x * \mu_y}\|_2^2 = \sum_{\pi \text{ spherical}} \dim V_\pi |\langle \pi(x)X_\pi, X_\pi\rangle\langle \pi(y)X_\pi, X_\pi\rangle|^2.$$

PROOF. By the Peter–Weyl theorem,

$$\|\mu_x * \mu_y\|_2^2 = \sum_{\pi \text{ spherical}} \dim V_\pi \|\widehat{\mu_x * \mu_y}(\pi)\|_{HS}^2.$$

The previous lemma and orthogonality give

$$\begin{aligned}\|\widehat{\mu_x * \mu_y}(\pi)\|_{HS}^2 &= \sum_{1\le i\le \dim V_\pi} \|\widehat{\mu_x * \mu_y}(\pi)X_i\|^2 = \sum_i \sum_j |\langle \widehat{\mu_x}(\pi)\widehat{\mu_y}(\pi)X_i, X_j\rangle|^2 \\ &= |\langle \widehat{\mu_x}(\pi)\widehat{\mu_y}(\pi)X_\pi, X_\pi\rangle|^2 = |\langle \pi(x)X_\pi, X_\pi\rangle\langle \pi(y)X_\pi, X_\pi\rangle|^2.\end{aligned}$$ □

We will let

$$\phi_\pi(x) = \langle \pi(x)X_\pi, X_\pi\rangle.$$

These are called *spherical functions* and have been well studied (see [11, Ch. IV, V], [12, Ch. III]), particularly in the rank one case which we will assume for the remainder of this section. The following result is critical for us.

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Theorem 3.4 [11, Ch. V, Theorem 4.5]. *Let G/K be a simply connected, compact symmetric space of rank one and let β denote the larger element in Φ^+. Let π be a spherical representation of G and let λ denote the restriction of the highest weight of π to $\mathfrak{a}$. Then $\lambda = n\beta$, where n is a positive integer. The spherical function, ϕ_π, is given by the hypergeometric function,*

$$\phi_\pi(x) = {}_2F_1(\tfrac{1}{2}m_{\beta/2} + m_\beta + n, -n; \tfrac{1}{2}(m_{\beta/2} + m_\beta + 1); \sin^2(\beta(X)/2)),$$

where $x = \exp iX$, $X \in \mathfrak{a}$. Moreover, there is such a spherical representation for each positive integer n.

By β the 'larger element', we mean $\beta = 2\alpha$ if there are two restricted roots and $\beta = \alpha$ otherwise. Here $m_{\beta/2}$ should be understood as zero if Φ^+ has only one element.

Using the symmetry of the first two arguments of the hypergeometric function and the relationship between the hypergeometric functions and the Jacobi polynomials $P_n^{(a,b)}(x)$, namely,

$$\frac{\Gamma(n+1)\Gamma(a+1)}{\Gamma(a+n+1)} P_n^{(a,b)}(x) = {}_2F_1\Big(-n, n+a+b+1, a+1; \frac{1}{2}(1-x)\Big),$$

(see [13]), we obtain the following expression for the spherical functions.

Proposition 3.5. *Let π_n be the spherical representation of G with highest weight restricted to $\mathfrak{a}$ equal to $n\beta$. Assume that $z = e^{iZ}$ with $Z \in \mathfrak{a}$. Then*

$$\phi_{\pi_n}(z) = \frac{\Gamma(n+1)\Gamma(a+1)}{\Gamma(a+n+1)} P_n^{(a,b)}(\cos\beta(Z)),$$

where

$$a = \tfrac{1}{2}(m_{\beta/2} + m_\beta - 1), \quad b = \tfrac{1}{2}(m_\beta - 1).$$

For the remainder of the paper, π_n will denote the spherical representation of G with highest weight restricted to $\mathfrak{a}$ equal to $n\beta$, where $\beta = 2\alpha$ if there are two positive restricted roots and $\beta = \alpha$ otherwise.

The asymptotic dimension formula for the spherical representations π_n can be derived using the Weyl dimension formula. Complicated explicit formulas are also known (see [4]).

Proposition 3.6. *There are constants $c_1, c_2 > 0$ such that, for any n, the spherical representation π_n has dimension bounded by*

$$c_1 n^{m_\alpha + m_{2\alpha}} \le \dim V_{\pi_n} \le c_2 n^{m_\alpha + m_{2\alpha}}.$$

Proof. Suppose that π_n has highest weight λ_n, where $\lambda_n|_{\mathfrak{a}} = n\beta$ with β the largest restricted root. As shown in [11, Ch. V, Theorem 4.1], λ_n vanishes on $\mathfrak{t} \cap \mathfrak{k}$, and thus

$$\langle \lambda_n, \gamma \rangle = n\langle \beta, \gamma|_{\mathfrak{a}} \rangle.$$

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It follows that the Weyl dimension formula implies that

$$\dim V_{\pi_n} = \prod_{\gamma\in\Sigma^+} \frac{\langle \lambda_n + \rho, \gamma\rangle}{\langle \rho, \gamma\rangle} = \prod_{\gamma\in\Sigma^+} \left(\frac{n\langle \beta, \gamma|_{\mathfrak{a}}\rangle}{\langle \rho, \gamma\rangle} + 1 \right).$$

This expression is polynomial in n, with each factor being either linear or one depending on whether $\langle \beta, \gamma|_{\mathfrak{a}}\rangle$ is nonzero. As $\mathfrak{a}$ is one-dimensional, $\gamma|_{\mathfrak{a}} = c\beta$ for some scalar c and thus $n\langle \beta, \gamma|_{\mathfrak{a}}\rangle \neq 0$ if and only if $\gamma|_{\mathfrak{a}} \neq 0$. Hence the degree of the polynomial is the number of $\gamma \in \Sigma^+$ with $\gamma|_{\mathfrak{a}} \neq 0$, namely, $m_\alpha + m_{2\alpha}$. □

We next recall the well-known asymptotic estimate for the Jacobi polynomials which can be found, for example, in [15]. With these we can easily obtain asymptotic estimates on the size of the spherical functions.

Lemma 3.7. *Let $a, b \in \mathbb{R}$. Then*

$$P_n^{(a,b)}(-1) = \binom{n+b}{n}(-1)^n \quad \text{and} \quad P_n^{(a,b)}(1) = \binom{n+a}{n},$$

while, if $\theta \in (0, \pi)$,

$$P_n^{(a,b)}(\cos\theta) = k(\theta) n^{-1/2}\cos(N\theta + \gamma) + O(n^{-3/2}),$$

where

$$N = n + \tfrac{1}{2}(a + b + 1), \quad \gamma = -\tfrac{1}{2}\pi(a + \tfrac{1}{2})$$

and

$$k(\theta) = \pi^{-1/2}(\sin(\tfrac{1}{2}\theta))^{-a-1/2}(\cos(\tfrac{1}{2}\theta))^{-b-1/2} > 0.$$

Corollary 3.8.

(a) *If $z \in A$ is regular, then*

$$|\phi_{\pi_n}(z)| \leq C n^{-(m_\alpha + m_{2\alpha})/2}.$$

(Here $m_{2\alpha}$ should be understood as zero in the single root case.)

(b) *If $z \in A \setminus N_G(K)$, but is not regular, there are positive constants C_1, C_2 such that*

$$C_1 n^{-m_\alpha/2} \leq |\phi_{\pi_n}(z)| \leq C_2 n^{-m_\alpha/2}.$$

Proof. These estimates follow directly from the previous result since the gamma function is known to satisfy

$$\frac{\Gamma(n+1)}{\Gamma(a+n+1)} = O(n^{-a}) \quad \text{and} \quad \binom{n+s}{n} = O(n^s) \quad \text{for } s > 0.$$ □

We are now ready to prove our main result. Note that, throughout the proof, C will denote a constant that can vary from one line to the next.

Theorem 3.9. *Assume that G/K is a rank one, simple, simply connected, compact symmetric space not isomorphic to* SU(2)/SO(2). *Assume that $z_1, z_2 \in A \setminus N_G(K)$.*

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(a) *If either z_1 or z_2 is regular, then $\mu_{z_1} * \mu_{z_2} \in L^2(G)$.*
(b) *If G/K is not type AIII or CII with $q = 2$, or type FII, then $\mu_{z_1} * \mu_{z_2} \in L^2(G)$.*
(c) *If G/K is type AIII or CII with $q = 2$, or type FII, and neither z_1 nor z_2 is regular, then $\mu_{z_1} * \mu_{z_2} \notin L^2(G)$.*

Remark 3.10. The compact symmetric spaces of type *AIII* or *CII* with $q = 2$ are those isomorphic to $\mathrm{SU}(3)/(S(U(2) \times U(1)))$ or $\mathrm{Sp}(6)/(\mathrm{Sp}(4) \times \mathrm{Sp}(2))$, and those of type *FII* are $F_4/\mathrm{SO}(9)$. The significance of these is that they are the rank one spaces with two positive restricted roots that satisfy $m_\alpha = m_{2\alpha} + 1$.

Proof. (a) From Proposition 3.3,

$$\|\mu_{z_1} * \mu_{z_2}\|_2^2 = \sum_n \dim V_{\pi_n} |\phi_{\pi_n}(z_1)\phi_{\pi_n}(z_2)|^2.$$

If both z_1 and z_2 are regular, then Corollary 3.8(i) gives

$$|\phi_{\pi_n}(z_j)| \le Cn^{-(m_\alpha + m_{2\alpha})/2}$$

for both $j = 1, 2$. Combining this with $\dim V_{\pi_n} \le O(n^{m_\alpha + m_{2\alpha}})$ yields the bound

$$\|\mu_{z_1} * \mu_{z_2}\|_2^2 \le C \sum_n \dim V_{\pi_n} n^{-2(m_\alpha + m_{2\alpha})} \le C \sum_n n^{-(m_\alpha + m_{2\alpha})}.$$

For all rank one symmetric spaces other than $\mathrm{SU}(2)/\mathrm{SO}(2)$, $m_\alpha + m_{2\alpha} \ge 2$ (see the appendix) and hence this sum converges. Thus $\mu_{z_1} * \mu_{z_2} \in L^2$.

Next, suppose z_1, but not z_2, is regular. Then $|\phi_{\pi_n}(z_2)| \le Cn^{-m_\alpha/2}$, by Corollary 3.8(ii), and similar arguments to the first case show that

$$\|\mu_{z_1} * \mu_{z_2}\|_2^2 \le C \sum_n \dim V_{\pi_n} n^{-(m_\alpha + m_{2\alpha})} n^{-m_\alpha} \le C \sum_n n^{-m_\alpha}.$$

This sum is finite since $m_\alpha \ge 2$ whenever there are two positive restricted roots, as must be the case if there is such an element z_2.

(b) We can assume that neither z_1 nor z_2 are regular for otherwise we could simply apply (a). Arguing as above gives

$$\|\mu_{z_1} * \mu_{z_2}\|_2^2 \le C \sum_n n^{m_\alpha + m_{2\alpha}} n^{-2m_\alpha} \le C \sum_n n^{-m_\alpha + m_{2\alpha}}.$$

But $m_\alpha - m_{2\alpha} \ge 2$ in all the rank one symmetric spaces other than those we have listed.

(c) If neither z_1 nor z_2 are regular, then the lower bounds of Proposition 3.6 and Corollary 3.8(b) give

$$\|\mu_{z_1} * \mu_{z_2}\|_2^2 \ge C' \sum_n n^{-m_\alpha + m_{2\alpha}}$$

for some constant $C' > 0$. Thus $\mu_{z_1} * \mu_{z_2} \notin L^2$ whenever $m_\alpha - m_{2\alpha} = 1$, as is the case for these symmetric spaces. □

The final result of this section shows that any three-fold convolution of continuous orbital measures is in L^2.

PROPOSITION 3.11. *If G/K is any rank one compact symmetric space and $z_j \notin N_G(K)$, $j = 1, 2, 3$, then $\mu_{z_1} * \mu_{z_2} * \mu_{z_3} \in L^2$.*

PROOF. First, assume that G/K is not $SU(2)/SO(2)$. By (a) of the previous theorem we can assume that none of z_j, $j = 1, 2, 3$, are regular. Then, as above,

$$\|\mu_{z_1} * \mu_{z_2} * \mu_{z_3}\|_2^2 \leq C \sum_n n^{m_\alpha + m_{2\alpha}} n^{-3m_\alpha} \leq C \sum_n n^{-2m_\alpha + m_{2\alpha}}$$

and this is finite for all these symmetric spaces.

A proof of this result for $SU(2)/SO(2)$ is given in [1]. It can also be shown by a similar argument to the above, noting that all $z_j \notin N_{SU(2)}(SO(2))$ are regular and using the asymptotic formula from Corollary 3.8(a). □

4. Convolution products in SU(2)/SO(2)

In this section we will show that *no* product of two orbital measures in $SU(2)/SO(2)$ has an L^2 density function. We will make use of the following well-known facts.

LEMMA 4.1. *Consider the trigonometric series for $x \in [0, \pi]$*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos nx}{n}.$$

The first converges pointwise to the odd, 2π-periodic extension of $(\pi - x)/2$. The second converges pointwise to the even, 2π-periodic extension of $-\log(2\sin(x/2))$, except at $x = 0$.

We note that every double coset of $SU(2)/SO(2)$ contains an element $z \in A$ of the form $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$, where $\theta \in [0, \pi/2]$. We will abuse notation and let z also denote the angle θ. There is only one restricted root, α, of multiplicity $m_\alpha = 1$ and $\alpha(z) = 2z$.

THEOREM 4.2. *If $z_1, z_2 \in SU(2)$, then $\mu_{z_1} * \mu_{z_2} \notin L^2$.*

PROOF. Throughout the proof, C will denote a (strictly) positive constant that can change from line to another.

Without loss of generality, we can assume $z_j \in A$. Furthermore, we can assume both (the angles) $z_j \in (0, \pi/2)$, because if $z_j = 0$ or $\pi/2$, then $\alpha(z_j) = 2z_j = 0 \pmod{\pi}$ so $z_j \in N_G(K)$ and, in this case, even $\mu_{z_1} * \mu_{z_2} \notin L^1$.

The spherical representation π_n of $SU(n)$, with highest weight restricted to $\mathfrak{a}$ equal to $n\alpha$, is known to have dimension $2n + 1$ (see [1] or [12, page 322]). From Proposition 3.5,

$$\phi_{\pi_n}(z) = P_n^{(0,0)}(\cos \alpha(z)) = P_n^{(0,0)}(\cos 2z).$$

The asymptotic estimates for Jacobi polynomials give

$$\phi_{\pi_n}(z) = Cn^{-1/2}(\cos((n + 1/2)2z - \pi/4) + O(n^{-1})).$$

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Squaring gives

$$\begin{aligned}(\phi_{\pi_n}(z))^2 &= Cn^{-1}(\cos^2((2n+1)z-\pi/4)+O(n^{-1}))\\ &= Cn^{-1}(\tfrac{1}{2}(\cos(2z(2n+1)-\pi/2)+1)+O(n^{-1}))\\ &= Cn^{-1}(\sin(2z(2n+1))+1+O(n^{-1})).\end{aligned}$$

Hence

$$\begin{aligned}\|\mu_{z_1}*\mu_{z_2}\|_2^2 &= \sum_n \dim V_{\pi_n}(\phi_{\pi_{2n}}(z_1))^2(\phi_{\pi_{2m}}(z_2))^2\\ &= \sum_n C\frac{(2n+1)}{n^2}(\sin(2z_1(2n+1))+1+O(n^{-1}))(\sin(2z_2(2n+1))+1+O(n^{-1}))\\ &= \sum_n \frac{C}{n}((\sin(2z_1(2n+1))+1)(\sin(2z_2(2n+1))+1)+O(n^{-1})).\end{aligned}$$

We claim that this sum diverges. Of course, the convergence or divergence of the sum depends only on the convergence or divergence of

$$\sum_n \frac{1}{n}(\sin(2z_1(2n+1))+1)(\sin(2z_2(2n+1))+1)$$

and therefore upon the sum

$$\sum_n \frac{1}{n}(\sin(2z_1(2n+1))\sin(2z_2(2n+1))+\sin(2z_1(2n+1))+\sin(2z_2(2n+1))+1). \tag{4.1}$$

As $\sin((2n+1)\theta)=\sin 2n\theta\cos\theta+\sin\theta\cos 2n\theta$, Lemma 4.1 implies that

$$\sum_n \frac{\sin((2n+1)\theta)}{n}$$

converges for any $\theta = 2z_1, 2z_2$ as $4z_j \neq 0 \pmod{2\pi}$. Thus (4.1) converges if and only if

$$\sum_n \frac{1}{n}(\sin(2z_1(2n+1))\sin(2z_2(2n+1))+1)<\infty. \tag{4.2}$$

Another application of basic trigonometric identities shows that there are scalars $c_j = c_j(z_1, z_2)$ such that

$$\begin{aligned}\sin(2z_1(2n+1))\sin(2z_2(2n+1)) &= \cos(4n(z_1-z_2))c_1-\sin(4n(z_1-z_2))c_2\\ &\quad -\cos(4n(z_1+z_2))c_3+\sin(4n(z_1+z_2))c_4.\end{aligned}$$

It follows, from the above lemma, that

$$\sum_n \frac{1}{n}\sin(2z_1(2n+1))\sin(2z_2(2n+1))$$

converges if $4(z_1 \pm z_2) \neq 0 \pmod{2\pi}$. Of course, in this case (4.2) diverges.

It remains to consider the two possibilities $z_1 \pm z_2 \equiv 0 \pmod{\pi/2}$.

Case 1. $z_1 - z_2 \equiv 0 \pmod{\pi/2}$. As $z_1, z_2 \in (0, \pi/2)$, this can only happen if $z_1 = z_2$. Then

$$\begin{aligned}1 + \sin(2z_1(2n+1))\sin(2z_2(2n+1)) &= 1 + \sin^2(2z_1(2n+1)) \\ &= \tfrac{1}{2}(3 - (\cos 8nz_1 \cos 4z_1 - \sin 8nz_1 \sin 4z_1)).\end{aligned}$$

If $z_1 \neq \pi/4$, then $\sum(\cos 8nz_1)/n$ converges and hence (4.2) diverges. If $z_1 = z_2 = \pi/4$, then direct substitution shows that (4.2) diverges.

Case 2. $z_1 + z_2 \equiv 0 \pmod{\pi/2}$. As $z_1, z_2 \in (0, \pi/2)$, we must have $z_1 + z_2 = \pi/2$ and so $\sin(2z_2(2n+1)) = \sin(2z_1(2n+1))$ and hence the arguments are the same. □

To conclude, we summarise our results. We will say that G/K satisfies the $L^1 \longleftrightarrow L^2$ dichotomy if $\mu_x * \mu_y \in L^1$ implies that $\mu_x * \mu_y \in L^2$. The following is an immediate consequence of Theorems 3.9 and 4.2.

Corollary 4.3. *The rank one symmetric space G/K satisfies the $L^1 \longleftrightarrow L^2$ dichotomy if and only if $m_\alpha - m_{2\alpha} > 1$. More specifically:*

(i) *when $G/K = \mathrm{SU}(2)/\mathrm{SO}(2)$ ($m_\alpha = 1$, $m_{2\alpha} = 0$), then $\mu_x * \mu_y \notin L^2$ for any x, y; and*
(ii) *if G/K is type AIII or CII with $q = 2$, or type FII (the other symmetric spaces with $m_\alpha = m_{2\alpha} + 1$) and $\mu_x * \mu_y \in L^1$, then $\mu_x * \mu_y \notin L^2$ if and only if neither x nor y is regular.*

Corollary 4.4. *If G/K is a rank one symmetric space, then $\mu_{z_1} * \mu_{z_2} \in L^2$ if and only if* $\dim Kz_1K + \dim Kz_2K > G/K$.

Proof. First, suppose that G/K is not type *AIII* or *CII* with $q = 2$, not type *FII* or (isomorphic to) $\mathrm{SU}(2)/\mathrm{SO}(2)$. Then Theorem 3.9 says $\mu_{z_1} * \mu_{z_2} \in L^2$ if and only if μ_{z_1}, μ_{z_2} are continuous. In this case, $\dim Kz_jK \geq m_\alpha$ and, as $m_\alpha \geq 2 + m_{2\alpha}$ for these symmetric spaces, it follows that $\dim Kz_1K + \dim Kz_2K \geq m_\alpha + 2 + m_{2\alpha} > \dim G/K$.

If, instead, $\dim Kz_1K + \dim Kz_2K \leq \dim G/K$, then $\dim Kz_jK < m_\alpha$ for some j. But that means that μ_{z_j} is not continuous, and hence $\mu_{z_1} * \mu_{z_2}$ is not even in L^1.

If G/K is type *AIII* or *CII* with $q = 2$ or type *FII*, then $\mu_{z_1} * \mu_{z_2} \in L^2$ if and only if both μ_{z_1} and μ_{z_2} are continuous and at least one is regular. But then, since $m_\alpha \geq 2$, $\dim Kz_1K + \dim Kz_2K \geq m_\alpha + m_\alpha + m_{2\alpha} > \dim G/K$. On the other hand, if neither μ_{z_1} nor μ_{z_2} is regular, then $\dim Kz_1K + \dim Kz_2K \leq 2m_\alpha = m_\alpha + 1 + m_{2\alpha} = \dim G/K$.

Finally, if G/K is isomorphic to $\mathrm{SU}(2)/\mathrm{SO}(2)$, then $\mu_{z_1} * \mu_{z_2} \notin L^2$ for any z_1, z_2, while $\dim Kz_1K + \dim Kz_2K \leq 2 = \dim G/K$ for all z_1, z_2. □

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Appendix

We list here the families of compact symmetric spaces of rank one, along with the multiplicities of $\alpha, 2\alpha$ and restricted root system Φ^+.

Type	G/K	Φ^+	m_α	$m_{2\alpha}$
AI	$\mathrm{SU}(2)/\mathrm{SO}(2)$	A_1	1	–
AII	$\mathrm{SU}(4)/\mathrm{Sp}(4)$	A_1	4	–
$AIII$	$\mathrm{SU}(q+1)/S(U(q)\times U(1))$ $q>1$	BC_1	$2(q-1)$	1
BII	$\mathrm{SO}(q+1)/S(O(q)\times O(1))$ $q>2$	A_1	$q-1$	–
CII	$\mathrm{Sp}(2q+2)/\mathrm{Sp}(2q)\times\mathrm{Sp}(2)$ $q>1$	BC_1	$4(q-1)$	3
FII	$F_4/\mathrm{SO}(9)$	BC_1	8	7

These facts can be found in [10, Ch. X]. We have excluded BII with $q = 2$ as this is isomorphic to $\mathrm{SU}(2)/\mathrm{SO}(2)$. Similarly, the only simple, rank one symmetric space of type $DIII$ is isomorphic to $\mathrm{SU}(4)/S(U(3)\times U(1))$, that is, type $AIII$ with $q = 3$.

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