Proceedings of the Edinburgh Mathematical Society (1993) 37, 53–55 ©

A NOTE ON THE WIMAN–VALIRON METHOD

by P. C. FENTON

(Received 19th June 1992)

Without prior assumptions about growth, fundamental inequalities for the Taylor series of an entire function are obtained, valid outside a certain exceptional set. The results are vacuous or not depending on the estimate for the exceptional set. Only then does the growth of the function enter.

1991 *Mathematics subject classification*: 30D20.

1. Introduction

W. K. Hayman [3] has given a comprehensive survey of what might be called the "comparison" version of the Wiman–Valiron theory of entire functions, an approach originally made systematic by Kövari [4]. Having established certain fundamental estimates for the terms in the Taylor series, Hayman gave illustrations of the method in various applications to local growth, behaviour of derivatives and so on. Subsequent work by the author [1, 2] extended the estimates, and by implication the applications, to entire functions about which something is known of the lower growth: in general Hayman's results are applicable only when the upper growth, however it is measured, is finite. The purpose of this note is to free the method from any prior assumption about growth. Thus the fundamental estimates are established for an entire function outside a certain exceptional set. To obtain non-vacuous conclusions for any given function the exceptional set must not be too large and it is at that point, but only then, that the growth of the function figures. The details are essentially the same as in the author's earlier versions, but in suppressing preliminary considerations of growth there is a considerable gain in clarity and simplicity.

In the Wiman–Valiron method the main parts are played by $\mu(r)$, the *maximum term* of the Taylor series, and $N(r)$, the *central index*, which is the largest index of those terms (should there be more than one) whose modulus is $\mu(r)$. We have $\mu(r)=|a_N|r^N$.

We shall prove:

Theorem. *Suppose that $f(z)=\sum a_n z^n$ is an entire function and that $\alpha(t)$ is a negative-valued, strictly decreasing function on $[0,\infty)$. Suppose that*

$$A_n=\exp\left(\int_0^n \alpha(t)\,dt\right),\ \rho_n=\exp(-\alpha(n)). \tag{1}$$

Given $R>0$ and $K>1$, we have

$$|a_n|r^n \leqq \mu(r)\frac{A_n}{A_N}\rho_N^{n-N} \quad for \quad 0 \leqq n \leqq KN_0 \tag{2}$$

and

$$|a_n|r^n \leqq \mu(r)\left(\frac{\rho_N}{\rho_{KN}}\right)^{(1-K^{-1})n} \quad for \quad n>KN_0, \tag{3}$$

for all r in the interval $(0,R)$ outside a subset of logarithmic measure at most $-\alpha(KN_0)+\alpha(0)$, where $N=N(r)$ and $N_0=N(R)$.

The dependence of (2) and (3) on N_0 can be eliminated if necessary. For the inequality in (2) certainly holds for $0 \leqq n \leqq KN$, since $N \leqq N_0$, and for $n>KN$ we have, taking (2) and (3) together,

$$|a_n|r^n \leqq \mu(r)\max\left\{\left(\frac{\rho_N}{\rho_{KN}}\right)^{(1-K^{-1})n}, \frac{A_n}{A_N}\rho_N^{n-N}\right\}, \tag{4}$$

which is an alternative to (3).

2. Estimate of the terms

Suppose that $K>1$ and $R>0$ are given, and that $N_0=N(R)$, as in the Theorem. For t satisfying $0<t \leqq R/\rho_{KN_0}$, let $M=M(t)$ be the index of the largest element in the set $\{|a_n|t^n/A_n : 0 \leqq n \leqq KN_0\}$, or the largest among the indices of such largest elements, in case there are several.

We have $M \leqq N_0$. For if $N_0<n \leqq KN_0$ then

$$\frac{|a_n|}{A_n}\frac{A_{N_0}}{|a_{N_0}|}t^{n-N_0}=\left(\frac{|a_n|}{|a_{N_0}|}R^{n-N_0}\right)\left(\frac{A_{N_0}}{A_n}\rho_n^{N_0-n}\right)\left(\frac{t\rho_n}{R}\right)^{n-N_0},$$

and the first factor on the right-hand side is less than 1 since $N_0=N(R)$, the second is also less than 1—this follows for $n \neq N_0$ from (1) and the fact that α is strictly decreasing—and the third is no greater than 1 since, for these values of n, $\rho_n \leqq \rho_{KN_0}$. Thus the N_0th element in the set exceeds subsequent elements and therefore the index of the largest element is no greater than N_0.

It follows from the definition of M that, with $r=t\rho_M$,

$$\frac{|a_n|}{|a_M|}r^{n-M} \leqq \frac{A_n}{A_M}\rho_M^{n-M} \quad \text{for} \quad 0 \leqq n \leqq KN_0. \tag{5}$$

The right-hand side of (5) is less than 1 for $n \neq M$, as we observed. Moreover $r = t\rho_M \leqq t\rho_{N_0} \leqq R$, and therefore $N(r) \leqq N(R) = N_0$. Thus $M = N(r)$. With this substitution (5) becomes (2).

It remains to establish (3). For $n > KN_0$ we have, with $r = t\rho_M$ and $0 < t \leqq R/\rho_{KN_0}$,

$$|a_n|r^n = |a_n|R^n\left(\frac{r}{R}\right)^n \leqq |a_{N_0}|R^{N_0}\left(\frac{r}{R}\right)^n = |a_{N_0}|r^{N_0}\left(\frac{r}{R}\right)^{n-N_0}$$

$$\leqq |a_N|r^N\left(\frac{r}{R}\right)^{n-N_0} \leqq |a_N|r^N\left(\frac{\rho_N}{\rho_{KN_0}}\right)^{n-N_0} \leqq |a_N|r^N\left(\frac{\rho_N}{\rho_{KN}}\right)^{(1-K^{-1})n}$$

since $M = N \leqq N_0$. This gives (3).

3. Estimate of the exceptional set

The set of r for which the estimates of (2) and (3) apply is $S = \{r\rho_{M(t)} : 0 < t \leqq T\}$, where $T = R/\rho_{KN_0}$, which is, as we saw in the previous section, a subset of $(0, R]$.

Suppose that $M(t)$ has jumps at points T_j, $j = 1$ to q, where $0 < T_1 < T_2 < \cdots < T_q \leqq T$. Then S is the union of the intervals

$$(0, T_1\rho_{M(0+)}], [T_1\rho_{M(T_1)}, T_2\rho_{M(T_1)}], \ldots, [T_q\rho_{M(T_q)}, T\rho_{M(T_q)}],$$

and thus the exceptional values lie in the complementary intervals

$$(T_1\rho_{M(0+)}, T_1\rho_{M(T_1)}), \ldots, (T_q\rho_{M(T_{q-1})}, T_q\rho_{M(T_q)}), (T\rho_{M(T_q)}, R),$$

the sum of whose logarithmic measures is evidently

$$\log(R/T\rho_{M(0+)}) = \log(\rho_{KN_0}/\rho_{M(0+)}) \leqq \log(\rho_{KN_0}/\rho_0) = -\alpha(KN(R)) + \alpha(0).$$

REFERENCES

1. P. C. Fenton, Some results of Wiman–Valiron type for integral functions of finite lower order, *Ann. of Math.* **103** (1976), 237–252.

2. P. C. Fenton, Wiman–Valiron theory for entire functions of finite lower growth, *Trans. Amer. Math. Soc.* **252** (1979), 221–232.

3. W. K. Hayman, The local growth of power series: a survey of the Wiman–Valiron method, *Canad. Math. Bull.* **17** (1974), 317–358.

4. T. Kővari, On the Borel exceptional values of lacunary integral functions, *J. Analyse Math.* **9** (1961), 71–109.

University of Otago
New Zealand