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On the Radius of Comparison of a Commutative C*-algebra

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Abstract. Let X be a compact metric space. A lower bound for the radius of comparison of the C^*-algebra $C(X)$ is given in terms of $\dim_{\mathbb{Q}} X$, where $\dim_{\mathbb{Q}} X$ is the cohomological dimension with rational coefficients. If $\dim_{\mathbb{Q}} X = \dim X = d$, then the radius of comparison of the C^*-algebra $C(X)$ is $\max\{0, (d-1)/2 - 1\}$ if d is odd, and must be either $d/2 - 1$ or $d/2 - 2$ if d is even (the possibility $d/2 - 1$ does occur, but we do not know if the possibility $d/2 - 2$ can also occur).

1 Introduction

The radius of comparison of a C^*-algebra A, denoted by $\mathrm{rc}(A)$, was introduced by Andrew Toms in [11] in order to measure the perforations in the Cuntz semigroup of A. Let X be a compact metric space and consider the C^*-algebra $C(X)$. It is known that the radius of comparison of $C(X)$ is always dominated by one half of the covering dimension of X (see [1, 4.1] or (2.1)). Moreover, if X is a finite CW-complex, the radius of comparison is approximately equal to one half of the covering dimension of X (see [11, Theorem 6.6]).

In this note, we shall show that when X is a general compact metric space, there is a lower bound for the radius of comparison which can be expressed in terms of the cohomological dimension with coefficients in $\mathbb{Q}$. More precisely, we have the following theorem.

Theorem 1.1 *Consider $A = C(X)$, where X is a compact metrizable space. Then*

$$\mathrm{rc}(A) \geq \begin{cases} (\dim_{\mathbb{Q}} X - 1)/2 - 1 & \text{if } \dim_{\mathbb{Q}} X \text{ is odd,} \\ \dim_{\mathbb{Q}} X/2 - 2 & \text{if } \dim_{\mathbb{Q}} X \text{ is even,} \\ \infty & \text{if } \dim_{\mathbb{Q}} X = \infty, \end{cases}$$

where $\dim_{\mathbb{Q}}$ is the cohomological dimension with coefficients in $\mathbb{Q}$.

If, in addition, $\dim_{\mathbb{Q}} X = \dim X$ (as in [11] in the finite CW-complex case), then, as in the finite CW-complex case, the radius of comparison is again approximately one half of the covering dimension. More precisely, applying the known upper bound of $\mathrm{rc}(A)$ (see (2.1)), one has the following corollary.

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Corollary 1.2 *With the assumptions of Theorem 1.1, if* $\dim_{\mathbb{Q}} X = \dim X$*, then*

(i) *if* $\dim X$ *is odd, one has* $\mathrm{rc}(A) = \max\{0, (\dim X - 1)/2 - 1\}$;
(ii) *if* $\dim X$ *is even, one has* $\dim X/2 - 2 \leq \mathrm{rc}(A) \leq \dim X/2 - 1$.

2 Preliminaries and the Proof of the Main Result

Let τ be a lower semicontinuous 2-quasitrace on a unital C*-algebra A. Then the formula

$$d_\tau(a) := \sup \tau(a^{\frac{1}{n}}),\ 0 \leq a \leq 1,$$

defines a functional on $\mathrm{Cu}(A)$ in the sense of [5], where $\mathrm{Cu}(A)$ is the stabilized Cuntz semigroup of A (see [2]). (In fact, by [5, Proposition 4.2], this map induces a bijection between functionals on $\mathrm{Cu}(A)$ and lower semicontinuous 2-quasitraces on A.)

Definition 2.1 ([11, Definition 6.1]; see [1, Section 3.1]) The radius of comparison of A, denoted by $\mathrm{rc}(A)$, is the infimum of the set of real numbers $r > 0$ with the property that $a, b \in \bigcup \mathrm{M}_n(A)$ satisfy $a \preceq b$ whenever

$$d_\tau(a) + r < d_\tau(b), \quad \tau \in \mathrm{QT}_2^1(A),$$

where $\mathrm{QT}_2^1(A)$ denotes the set of normalized 2-quasitraces on A, and $\preceq$ denotes the Cuntz pre-order relation.

Remark 2.2 In Definition 2.1, the radius of comparison is defined in the context of the original definition of the Cuntz semigroup. If one considers the radius of comparison in the context of the modified definition of the Cuntz semigroup $\mathrm{Cu}(A)$ introduced in [2] (in which one first passes to the stabilization of the C*-algebra), then, as follows from [1, Proposition 3.2.3], these two notions of radius of comparison agree with each other for commutative C*-algebras.

In fact, for exact C*-algebras, one only has to consider extremal tracial states.

Lemma 2.3 *Assume that A is a unital exact C*-algebra. Denote by* $\partial\mathrm{T}(A)$ *the set of extremal tracial states. Then the radius of comparison of A can be obtained as*

$$\inf\{r > 0; d_\tau(a) + r < d_\tau(b), \tau \in \partial\mathrm{T}(A),\ \textit{implies}\ a \preceq b\}.$$

Proof Set

$$S_0 := \{r > 0; d_\tau(a) + r < d_\tau(b), \tau \in \mathrm{T}(A),\ \text{implies}\ a \preceq b\},$$

and

$$S_1 := \{r > 0; d_\tau(a) + r < d_\tau(b), \tau \in \partial\mathrm{T}(A),\ \text{implies}\ a \preceq b\}.$$

Note that $\mathrm{rc}(A) = \inf S_0$.

It is clear that $S_1 \subseteq S_0$, and hence $\inf S_0 \leq \inf S_1$. Let $r \in S_0$, and let $\varepsilon > 0$ be arbitrary. Consider $r + \varepsilon$. Then, if

$$d_\tau(a) + r + \varepsilon < d_\tau(b), \quad \tau \in \partial\mathrm{T}(A),$$

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by the Krein–Milman theorem, which is applicable to $\mathrm{T}(A)$ because it is a compact convex subset of a locally convex topological vector space, one has (as can be seen be replacing d_τ by τ and a by $a^{\frac{1}{n}}$, and using that the map $\tau \mapsto \tau(a)$ is continuous, and that (hence) the map $\tau \mapsto d_\tau(b)$ is lower semicontinuous)

$$d_\tau(a) + r + \varepsilon \leq d_\tau(b), \quad \tau \in \mathrm{T}(A),$$

and therefore (by the definition of S_0) $a \preceq b$. This shows that in fact $r + \varepsilon \in S_1$. Since ε is arbitrary, one has $\inf S_0 \geq \inf S_1$, and so

$$\mathrm{rc}(A) = \inf S_0 = \inf S_1,$$

as desired. ■

Corollary 2.4 *The radius of comparison of a commutative C*-algebra is always either an integer or ∞.*

Proof Any extremal tracial state τ on a commutative C*-algebra is induced by a Dirac measure δ_x. It follows that $d_\tau(a)$ is the rank of $a(x)$, which is an integer. Therefore, the real numbers r in S_1 may be chosen to be integers (they can be chosen as the integer parts of real numbers in the set S_1), and hence their infimum must be an integer or (in the case of the empty set) ∞. ■

For commutative C*-algebras, as a special case of [13, Theorem 4.6], one has the following theorem. (An early result was obtained in [12] with $(\dim X - 1)/1$ replaced by $9 \dim X$.)

Theorem 2.5 *Let a and b be positive elements of a matrix algebra over $\mathrm{C}(X)$ with X a compact metric space. If*

$$\mathrm{Rank}(a(x)) + \frac{\dim X - 1}{2} \leq \mathrm{Rank}(b(x)), \quad x \in X,$$

then $a \preceq b$.

Since the radius of comparison of a commutative C*-algebra must be an integer, one has as a consequence (this formula also appears in [1, 4.1])

$$\mathrm{rc}(\mathrm{C}(X)) \leq \begin{cases} \max\{0, (\dim X - 1)/2 - 1\} & \text{if } \dim X \text{ is odd,} \\ \dim X/2 - 1 & \text{if } \dim X \text{ is even.} \end{cases} \tag{2.1}$$

To get the lower bound asserted in Theorem 1.1, we need to recall some facts from dimension theory and homotopy theory. Let G be an abelian group. Denote by $K(n, G)$ for each $n = 0, 1, 2, \ldots$ the Eilenberg–MacLane space satisfying

$$\pi_k(K(n, G), *) = \begin{cases} G, & k = n, \\ \{0\}, & k \neq n. \end{cases}$$

The spaces $K(n, G)$ are the classifying spaces for Čech cohomology with coefficients in G; that is, $\check{H}^n(X; G)$ is isomorphic to $[X, K(n, G)]$ naturally for any compact metrizable space X.

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Remark 2.6 The spaces $K(n,G)$ are unique up to homotopy equivalence. They may be chosen to be CW-complexes, and then are absolute neighbourhood extensors for metric spaces ([9]). Moreover, the skeletons of $K(n,G)$ may be chosen so that every cell has dimension at least n.

Theorem 2.7 (Homotopy Extension Theorem) *Let A be a closed subset of a space X, and let L be an absolute neighbourhood extensor. Then any map $H\colon (A\times I)\cup(X\times\{0\})\to L$ extends to a homotopy $\widetilde{H}\colon X\times I\to L$.*

Definition 2.8 A compact metrizable space X has *cohomological dimension at most n with coefficients in G*, written as $\dim_G X\leq n$, provided that for each closed subset A, every continuous map $\alpha\colon A\to K(n,G)$ extends to a continuous map $\widetilde{\alpha}\colon X\to K(n,G)$.

In fact, as stated in the following theorem, if $\dim_G(X)\leq n$, then $\dim_G X\leq m$ for any $m\geq n$. A proof can be found in [3, Theorem 1.1].

Theorem 2.9 *A compact metrizable space X has cohomological dimension at most n with coefficients in G if and only if for each closed subset A and each $m\geq n$, every continuous map $\alpha\colon A\to K(m,G)$ extends to a continuous map $\widetilde{\alpha}\colon X\to K(m,G)$. In particular, for any natural number $d<\dim_G X$, there exist a closed subset A and a continuous map $\alpha\colon A\to K(d,G)$ that cannot be extended to X.*

It is interesting to compare the definition of cohomological dimension to the following characterization of covering dimension for a normal space (see [10, 9-9]), which is also formulated in terms of extensions.

Theorem 2.10 *A normal space X has (covering) dimension at most n, written as $\dim X\leq n$, provided that for any closed subset A, any map $\alpha\colon A\to S^n$ extends to a map $\widetilde{\alpha}\colon X\to S^n$.*

The cohomological dimension and the covering dimension are closely related (a proof may be found in [10]):

Theorem 2.11 *For any compact metrizable space X, one has*

$$\dim_{\mathbb{Q}} X\leq\dim_{\mathbb{Z}} X\leq\dim X.$$

If $\dim X<\infty$*, then* $\dim_{\mathbb{Z}} X=\dim X$*.*

Remark 2.12 Since $K(1,\mathbb{Z})\cong S^1$, one has that $\dim X\leq 1$ if and only if $\dim_{\mathbb{Z}} X\leq 1$. However, there exists a compact metric space X that is an inverse limit of finite CW-complexes with $\dim X=\infty$, but $\dim_{\mathbb{Z}} X=2$ and $\dim_{\mathbb{Q}} X=1$. See [4] for more details.

For a real number x, denote by $\langle x\rangle$ the smallest integer n with $x\leq n$.

Lemma 2.13 *Let n be an even number, and let A be a compact metrizable space with covering dimension at most $n+d$. Assume that there is a continuous map*

$$\alpha\colon A\longrightarrow K(n,\mathbb{Q})$$

that is not homotopic to a constant map. Then there is a complex vector bundle E on A with rank at least $n/2$ such that E does not contain any trivial sub-bundle of rank strictly greater than $\operatorname{rank}(E) - n/2$.

Moreover, if $d < \infty$, the vector bundle E can be chosen to have rank $n/2 + \langle (d-1)/2 \rangle$ and not to contain any trivial sub-bundle of rank strictly greater than $\langle (d-1)/2 \rangle$.

Proof Since $\alpha(A)$ is compact, there is a finite sub-complex $K' \subseteq K(n, \mathbb{Q})$ such that $\alpha(A) \subseteq K'$ (see, for instance, [6, Proposition A.1]). In particular, K' is compact as a topological space. Note that every cell of K' has dimension at least n.

Denote by $g \in \check{H}^n(K'; \mathbb{Q}) \cong \check{H}^n(K') \otimes \mathbb{Q}$ the element corresponding to the embedding

$$\iota\colon K' \hookrightarrow K(n, \mathbb{Q}),$$

and note that

$$\alpha^*(g) = [\alpha] \in \check{H}^n(A, \mathbb{Q}).$$

Since α is not homotopic to a constant map, one has that $[\alpha] \neq 0$. (Recall that $K(n, \mathbb{Q})$ is the classifying space for $\check{H}^n(\,\cdot\,, \mathbb{Q})$.)

Since the rationalized Chern character $\operatorname{Ch}\colon \mathrm{K}_{\mathbb{C}}(K') \otimes \mathbb{Q} \to \bigoplus_{i=0}^{\infty} \check{H}^{2i}(K', \mathbb{Q})$ is a vector space isomorphism (see [8, Theorem V.3.25]), there are complex vector bundles $E_1', \dots, E_s'$ and $F_1', \dots, F_s$ on K' such that

$$\operatorname{Ch}_{n/2}\Big(r_1\big([E_1'] - [F_1']\big) + \cdots + r_s\big([E_s'] - [F_s']\big) \Big) = g \in \check{H}^n(K'; \mathbb{Q})$$

for some $r_1, \dots, r_s \in \mathbb{Q}$.

Since the finite CW-complex K' has no cell with dimension strictly less than n, one has that $\check{H}^i(K', \mathbb{Z}) = \{0\}$ for $i = 1, \dots, n-1$. In particular,

$$c_1(E_i') = c_2(E_i') = \cdots = c_{n/2-1}(E_i') = 0, \quad 1 \leq i \leq s,$$

and

$$c_1(F_i') = c_2(F_i') = \cdots = c_{n/2-1}(F_i') = 0, \quad 1 \leq i \leq s.$$

Consider the pull-backs of E_i' and F_i' on A, and denote them by E_i and F_i respectively. One has

$$\operatorname{Ch}_{n/2}\Big(r_1\big([E_1] - [F_1]\big) + \cdots + r_s\big([E_s] - [F_s]\big) \Big) = \alpha^*(g) = [\alpha] \neq 0, \tag{2.2}$$

and

$$c_1(E_i) = c_2(E_i) = \cdots = c_{n/2-1}(E_i) = 0, \quad 1 \leq i \leq s,$$
$$c_1(F_i) = c_2(F_i) = \cdots = c_{n/2-1}(F_i) = 0, \quad 1 \leq i \leq s.$$

It is then clear that if E_i or F_i has rank at most $n/2 - 1$, one has that $c_k(E_i) = 0$, $k \in \mathbb{N}$, or $c_k(F_i) = 0$, $k \in \mathbb{N}$. If all of E_i and F_i with rank at least $n/2$ had trivial sub-bundles of rank strictly larger than $\operatorname{rank}(E_i) - n/2$ and $\operatorname{rank}(F_i) - n/2$ respectively, one would have

$$c_{\operatorname{rank}(E_i)}(E_i) = \cdots = c_{n/2+1}(E_i) = c_{n/2}(E_i) = \cdots = 0,$$

and

$$c_{\operatorname{rank}(F_i)}(F_i) = \cdots = c_{n/2+1}(F_i) = c_{n/2}(F_i) = \cdots = 0.$$

Hence (by definition of Ch) $\mathrm{Ch}_{n/2}([E_i]) = 0$ and $\mathrm{Ch}_{n/2}([F_i]) = 0$, which contradicts (2.2). Hence at least one of the vector bundles in $\{E_i, F_i,\ i = 1, \dots, s\}$, say E, does not contain a trivial sub-bundle with rank strictly larger than $\operatorname{rank}(E) - n/2$.

If A has covering dimension at most $n + d$ with $d < \infty$, by factoring out or adding trivial sub-bundles, one may then assume that the rank of E is $n/2 + \langle (d-1)/2 \rangle$, and this proves the additional statement of the lemma in the case that $d < \infty$. ■

Lemma 2.14 *Let X be a compact metrizable space.*

(i) *If $\dim_{\mathbb{Q}} X$ is odd, there exist a closed subset $A \subseteq X$ and a complex vector bundle E on A with rank at least $(\dim_{\mathbb{Q}} X - 1)/2$ such that E does not contain any trivial sub-bundle of rank strictly greater than $\operatorname{rank}(E) - (\dim_{\mathbb{Q}} X - 1)/2$.*

(ii) *If $\dim_{\mathbb{Q}} X$ is even, there exist a closed subset $A \subseteq X$ and a complex vector bundle E on A with rank at least $(\dim_{\mathbb{Q}} X - 2)/2$ such that E does not contain any trivial sub-bundle of rank strictly greater than $\operatorname{rank}(E) - (\dim_{\mathbb{Q}} X - 2)/2$.*

(iii) *If $\dim_{\mathbb{Q}} X = \infty$, then, for any even number m, there exist a closed subset $A \subseteq X$ and a complex vector bundle E on A with rank at least $m/2$ such that E does not contain any trivial sub-bundle of rank strictly greater than $\operatorname{rank}(E) - m/2$.*

Proof Assume that $\dim_{\mathbb{Q}} X$ is odd. By definition, there is a closed subset $A \subseteq X$ and a map

$$\alpha \colon A \longrightarrow K(\dim_{\mathbb{Q}} X - 1, \mathbb{Q})$$

that cannot be extended to X. In particular, the map α is not homotopic to a constant map; otherwise, since the constant maps are always extendible, the map α must be extendible from A to X by the Homotopy Extension Theorem (Theorem 2.7). Then Lemma 2.13 applies.

If $\dim_{\mathbb{Q}} X$ is even, by Theorem 2.9, there is a closed subset $A \subseteq X$ and a map

$$\alpha \colon A \longrightarrow K(\dim_{\mathbb{Q}} X - 2, \mathbb{Q})$$

that cannot be extended to X. Applying the result above to the even number $\dim_{\mathbb{Q}} X - 2$, one shows the statement (ii).

If $\dim_{\mathbb{Q}} X = \infty$, then, by Theorem 2.9 again, for any even number m, there is a closed subset $A \subseteq X$ and a map $\alpha \colon A \to K(m, \mathbb{Q})$ that cannot be extended to X. Thus, applying the result above to the even number m, one obtains the desired conclusion. ■

We are now ready to prove the main theorem.

Proof of Theorem 1.1 Let $m \in \mathbb{N}$ be an arbitrary even number. Let A denote the closed subset of X and E the complex vector bundle over A assured by Lemma 2.14. Choose a projection $p \in \mathrm{M}_k(\mathrm{C}(A))$ (for a suitable k) representing E. Lift p to a positive element p' of $\mathrm{M}_k(\mathrm{C}(X))$. Choose a continuous positive function $h \in \mathrm{C}(X)$ such that $h(x) = 0$ if and only if $x \in A$.

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Set

$$C = \begin{cases} \operatorname{rank}(E) - (\dim_{\mathbb{Q}} X - 1)/2 + 1 & \text{if } \dim_{\mathbb{Q}} X \text{ is odd,} \\ \operatorname{rank}(E) - \dim_{\mathbb{Q}} X/2 + 2 & \text{if } \dim_{\mathbb{Q}} X \text{ is even,} \\ \operatorname{rank}(E) - m/2 + 1 & \text{if } \dim_{\mathbb{Q}} X = \infty. \end{cases}$$

Choose a constant projection a of rank C in $\mathrm{M}_k(\mathrm{C}(X))$ and set

$$b := p' + h1 \in \mathrm{M}_k^+(\mathrm{C}(X)).$$

It is clear that

$$\operatorname{Rank}(b(x)) - \operatorname{Rank}(a(x)) > \begin{cases} (\dim_{\mathbb{Q}} X - 1)/2 - 2 & \text{if } \dim_{\mathbb{Q}} X \text{ is odd,} \\ \dim_{\mathbb{Q}} X/2 - 3 & \text{if } \dim_{\mathbb{Q}} X \text{ is even,} \\ m/2 - 2 & \text{if } \dim_{\mathbb{Q}} X = \infty, \end{cases}$$

for any $x \in X$. It follows (as for any extreme tracial state τ concentrated at $x \in X$, the value $d_{\tau_x}(f)$ is exactly the rank of $f(x)$ for any positive element $f \in \mathrm{M}_k(\mathrm{C}(X))$) that

$$d_\tau(b) - d_\tau(a) > \begin{cases} (\dim_{\mathbb{Q}} X - 1)/2 - 2 & \text{if } \dim_{\mathbb{Q}} X \text{ is odd,} \\ \dim_{\mathbb{Q}} X/2 - 3 & \text{if } \dim_{\mathbb{Q}} X \text{ is even,} \\ m/2 - 2 & \text{if } \dim_{\mathbb{Q}} X = \infty, \end{cases} \tag{2.3}$$

for any extremal tracial state τ on $\mathrm{C}(X)$.

However, a is not Cuntz dominated by b. If it were, then the restriction of a to A would be dominated by the restriction of b to A, which is the projection p. Then p (*i.e.*, E) would contain a trivial sub-projection of rank C, which contradicts Lemma 2.14.

Therefore, by Lemma 2.3, the radius of comparison of $\mathrm{C}(X)$ is strictly larger than the integer on the right-hand side of (2.3). Using the facts that $\mathrm{rc}(\mathrm{C}(X))$ is an integer (Corollary 2.4) and m is arbitrary, one has

$$\mathrm{rc}(\mathrm{C}(X)) \geq \begin{cases} (\dim_{\mathbb{Q}} X - 1)/2 - 1 & \text{if } \dim_{\mathbb{Q}} X \text{ is odd,} \\ \dim_{\mathbb{Q}} X/2 - 2 & \text{if } \dim_{\mathbb{Q}} X \text{ is even,} \\ \infty & \text{if } \dim_{\mathbb{Q}} X = \infty, \end{cases}$$

as asserted. ∎

Remark 2.15 When $\dim_{\mathbb{Q}} X = \dim X = d$ is even, the integer $d/2 - 1$ can be the radius of comparison of the C*-algebra $\mathrm{C}(X)$. The following is such an example.

Consider the canonical line bundle E' over S^2. Then $E := \pi_1^*(E') \oplus \pi_2^*(E')$ is a rank-two vector bundle over $S^2 \times S^2$, where π_1 and π_2 are the coordinate projections of $S^2 \times S^2$. Then the Euler class of E is nonzero, and hence E does not contain any nonzero trivial sub-bundles. Now let p be a projection corresponding to E in a matrix algebra over $A = \mathrm{C}(S^2 \times S^2)$, and put $e = 1_A$; then $e \not\precsim p$. However, for any $\tau \in \mathrm{T}(A)$, one has

$$d_\tau(e) = 1 < 2 = d_\tau(p),$$

and therefore, $\mathrm{rc}(A) = 1$ (since $\mathrm{rc}(A)$ is either 0 or 1 by Theorem 1.1).

But we do not know whether $d/2 - 2$ can be the radius of comparison of the C^*-algebra $\mathrm{C}(X)$.

Remark 2.16 It would be interesting to determine whether the statement of Theorem 1.1 is still true if one replaces $\dim_{\mathbb{Q}} X$ by $\dim_{\mathbb{Z}} X$ (or $\dim X$). Moreover, consider a space with infinite covering dimension but only finite cohomological dimension (with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$). It is even unclear to us whether the radius of comparison of this space should be infinite or finite; or, if the radius of comparison is finite, whether it should be roughly equal to one half of the cohomological dimension or not.

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