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EIGENELEMENTS OF PERTURBED OPERATORS

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Abstract

Let λ_0 be a semisimple eigenvalue of an operator T_0. Let Γ_0 be a circle with centre λ_0 containing no other spectral value of T_0. Some lower bounds are obtained for the convergence radius of the power series for the spectral projection $P(t)$ (and for trace $T(t)P(t)$) associated with a linear perturbation family $T(t) = T_0 + tV_0$ and the circle Γ_0. They are useful when T_0 is a member of a sequence (T_n) which approximates an operator T in a collectively compact manner. These bounds result from a modification of Kato's method of majorizing series, based on an idea of Redont. If λ_0 is simple, it is shown that the same lower bounds are valid for the convergence radius of a power series yielding an eigenvector of $T(t)$.

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1. Introduction

Let T_0 and V_0 be closed (linear) operators on a domain D_0, which is a dense subspace of a Banach space X over the field $\mathbb{C}$ of complex numbers. For $t \in \mathbb{C}$, consider the operator

$$T(t) = T_0 + tV_0.$$

Then $T(0) = T_0$ is called the *unperturbed* operator, tV_0 the *perturbation* and $T(t)$ the *perturbed operator*. This situation typically arises in quantum mechanics as follows. Let T_0 be the Hamiltonian of a quantum mechanical system and V_0 be a potential energy operator. Then $T(1) = T_0 + V_0$ is the Hamiltonian of the perturbed system. Also, in operator approximation theory, one considers a sequence of operators (T_n) which approximates a given operator T. One can let $T_0 = T_n$ for some fixed integer n and $V_0 = T - T_0$, so as to have $T = T(1)$.

Let $\sigma(T_0)$ denote the spectrum of T_0 and λ_0 be an isolated point of it. Let Γ_0 be a simple closed curve which separates λ_0 from the rest of $\sigma(T_0)$. For $z \notin \sigma(T_0)$, let $R_0(z) = (T_0 - zI)^{-1}$. Consider the following open disk in $\mathbb{C}$:

$$\partial(\Gamma_0) = \left\{ t \in \mathbb{C} \colon |t| < \min_{z \in \Gamma_0} (1/r_\sigma(V_0 R_0(z)) \right\},$$

where r_σ denotes the spectral radius. For all $t \in \partial(\Gamma_0)$, it follows that $T(t)$ is a closed operator on the domain D_0, and Γ_0 does not intersect the spectrum $\sigma(T(t))$ of $T(t)$. For $z \notin \sigma(T(t))$, let $R(t, z) = (T(t) - zI)^{-1}$. Then the spectral projection

$$P(t) = -\frac{1}{2\pi i} \int_{\Gamma_0} R(t, z) dz,$$

associated with $T(t)$ and Γ_0, is an analytic function on $\partial(\Gamma_0)$. If the algebraic multiplicity of λ_0 is $m < \infty$, that is, if $P(0)$ has rank m, then for all $t \in \partial(\Gamma_0)$, $P(t)$ has rank m and Γ_0 contains precisely m eigenvalues of $T(t)$: $\lambda_1(t), \dots, \lambda_m(t)$, repeated according to their algebraic multiplicities. Also, the function

$$\hat{\lambda}(t) \equiv \frac{1}{m} \operatorname{trace} T(t) P(t) = (\lambda_1(t) + \cdots + \lambda_m(t))/m$$

is analytic for t in $\partial(\Gamma_0)$.

It is of interest to know for which values of the parameter t, the power series for $P(t)$ and $\hat{\lambda}(t)$ converge. The method of majorizing series developed by Kato [2, pages 89–91] allows one to find lower bounds for the convergence radii of these series. A modification of Kato's method yields improved results for the case where λ_0 is a semisimple eigenvalue of T_0 (Lemma 2.1 and Remark 2.2). This procedure also gives an upper bound for $|\lambda_i(t) - \lambda_0|$, $i = 1, \dots, m$, and a disk of isolation for the set $\{\lambda_1(t), \dots, \lambda_m(t)\}$ from the rest of the spectrum of $T(t)$ (Theorem 2.3). The basic idea behind these results is given in Redont's unpublished work [7, pages 79–80] (compare also [6, Proposition 2.1] and [4, Theorem 10.5]. The results are applicable when a compact operator T is approximated in a collectively compact manner (Remark 2.5).

In case λ_0 is a simple eigenvalue of T_0, let ϕ_0 (respectively ϕ_0^*) be an eigenvector of T_0 (respectively T_0^*) corresponding to λ_0 (respectively $\overline{\lambda_0}$) such that $\langle \phi_0, \phi_0^* \rangle = 1$, and consider the meromorphic function

$$\phi(t) = P(t)\phi_0 / \langle P(t)\phi_0, \phi_0^* \rangle.$$

If $t_0 \in \partial(\Gamma_0)$ is not a pole of the function ϕ, then $\phi(t_0)$ is an eigenvector of $T(t_0)$.

We describe a subregion of $\partial(\Gamma_0)$, which is devoid of any pole of the function ϕ (Proposition 3.2). This again yields a lower bound for the convergence radius of the power series for $\phi(t)$ about 0 (Theorem 3.3 and Remark 3.4).

2. Eigenvalues of $T(t)$ near λ_0

We assume that λ_0 is a semisimple eigenvalue of T_0 of algebraic multiplicity m, that is, the range of the spectral projection $P(0) = P_0$ coincides with the eigenspace of T_0 corresponding to λ_0 and is of dimension m. Let S_0 denote the reduced resolvent associated with T_0 and λ_0:

$$S_0 = \lim_{z \to \lambda_0} R_0(z)(I - P_0). \tag{1}$$

Then $S_{0|(I-P_0)D_0}$ is the inverse of $(T_0 - \lambda_0 I)_{|(I-P_0)D_0}$. Since $\sigma(T_{0|(I-P_0)D_0}) = \sigma(T_0)\backslash\{\lambda_0\}$, it follows that

$$\frac{1}{\operatorname{dist}(\lambda_0, \sigma(T_0)\backslash\{\lambda_0\})} = r_\sigma(S_0) \le \|S_0\|. \tag{2}$$

Let $0 < \varepsilon < 1$. Consider the circle Γ_ε with centre λ_0 and radius $\varepsilon/\|S_0\|$. The convergence radii of the series for $P(t)$ and $\hat{\lambda}(t)$ are greater than or equal to the radius of $\partial(\Gamma_\varepsilon)$. To find a lower bound for the radius of $\partial(\Gamma_\varepsilon)$, we introduce the following notation:

$$\begin{aligned} &s = \|S_0\|, \quad p = \|V_0 P_0\|, \quad q = \|V_0 S_0\|, \\ &r^2 = \sup\{\|V_0 S_0^{k+1} V_0 S_0\|/\|S_0\|^k,\ k = 0, 1, 2, \dots\}. \end{aligned} \tag{3}$$

Lemma 2.1. *Let $0 < \varepsilon < 1$. The radius of $\partial(\Gamma_\varepsilon)$ is at least*

$$\left[\frac{p^2 s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}\right]^{-1/2}.$$

The convergence radii of the power series for $P(t)$ and $\hat{\lambda}(t)$ about 0 are at least equal to

$$m_0 = \max_{0<\varepsilon<1} \left[\frac{p^2 s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}\right]^{-1/2}. \tag{4}$$

Proof. Since λ_0 is semisimple, we have $T_0 P_0 = \lambda_0 P_0$. Hence λ_0 is a pole of order 1 of $R_0(z)$. The Laurent expansion of $R_0(z)$ about λ_0 is given by

$$R_0(z) = \sum_{k=0}^{\infty} S_0^{k+1}(z - \lambda_0)^k - \frac{P_0}{z - \lambda_0}$$

for $0 < |z-\lambda_0| < \operatorname{dist}(\lambda_0, \sigma(T_0)\backslash\{\lambda_0\})$ (compare [2, (5.18)]. Writing $R_0(z) = R_0(z)P_0 + R_0(z)(I - P_0)$, we see that

$$[V_0R_0(z)]^2 = A(z) + B_1(z) + B_2(z) + C(z),$$

where

$$\begin{aligned} A(z) &= [V_0R_0(z)P_0]^2 = (V_0P_0)^2/(z-\lambda_0)^2, \\ B_1(z) &= V_0R_0(z)P_0V_0R_0(z)(I-P_0) \\ &= -\sum_{k=0}^{\infty} V_0P_0V_0S_0^{k+2}(z-\lambda_0)^k - (V_0P_0V_0S_0)/(z-\lambda_0), \\ B_2(z) &= V_0R_0(z)(I-P_0)V_0R_0(z)P_0 \\ &= -\sum_{k=0}^{\infty} V_0S_0^{k+2}V_0P_0(z-\lambda_0)^k - (V_0S_0V_0P_0)/(z-\lambda_0) \end{aligned}$$

and

$$C(z) = [V_0R_0(z)(I-P_0)]^2 = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} V_0S_0^{k+1}V_0S_0^{j+1}(z-\lambda_0)^{k+j}.$$

For $z \in \Gamma_\varepsilon$, we have $|z-\lambda_0| = \varepsilon/s$, so that

$$\|A(z)\| = \frac{p^2s^2}{\varepsilon^2}, \quad \|B_1(z)\|, \|B_2(z)\| \le pqs\left(\frac{1}{1-\varepsilon} + \frac{1}{\varepsilon}\right) = \frac{psq}{\varepsilon(1-\varepsilon)},$$

and

$$\begin{aligned} \|C(z)\| &\le \left(\sum_{k=0}^{\infty} \|V_0S_0^{k+1}V_0S_0\||z-\lambda_0|^k\right)\left(\sum_{j=0}^{\infty} s^j|z-\lambda_0|^j\right) \\ &\le r^2/(1-|z-\lambda_0|s)^2 = r^2/(1-\varepsilon)^2, \end{aligned} \tag{5}$$

as $\|V_0S_0^{k+1}V_0S_0\|/s^k \le r^2$ for $k = 0, 1, 2, \ldots$. Thus, it follows that for $z \in \Gamma_\varepsilon$

$$\|[V_0R_0(z)]^2\| \le \frac{p^2s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}.$$

Also, by the spectral radius formula,

$$r_\sigma(V_0R_0(z)) = \inf_{n=1,2,\ldots} \|[V_0R_0(z)]^n\|^{1/n} \le \|[V_0R_0(z)]^2\|^{1/2}.$$

As the radius of $\partial(\Gamma_\varepsilon)$ is $\min_{z\in\Gamma_\varepsilon} 1/r_\sigma(V_0R_0(z))$, we see that it must at least be

$$\left[\frac{p^2s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}\right]^{-1/2}.$$

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Since for every ε in (0, 1), the functions $P(t)$ and $\hat{\lambda}(t)$ are analytic on $\partial(\Gamma_\varepsilon)$, it follows that their power series converge if $|t|$ is less than m_0, given by (4).

REMARK 2.2. Letting $\varepsilon = \sqrt{ps}/(\sqrt{ps}+\sqrt{q})$, we see that

$$(\sqrt{ps}+\sqrt{q})^{-1}[ps + 2\sqrt{psq} + r^2/q]^{-1/2} \leq m_0. \tag{6}$$

Since $r^2 \leq \|V_0 S_0\|^2 = q^2$, it follows that (6) improves upon the lower bound $(\sqrt{ps}+\sqrt{q})^{-2}$ given by Kato [2, (3.21)]. This bound was obtained by majorizing the series

$$V_0 R_0(z) = \frac{-V_0 P_0}{z-\lambda_0} + \sum_{k=0}^{\infty} V_0 S_0^{k+1}(z-\lambda_0)^k$$

term by term by the power series for the function

$$\Phi_1(z-\lambda_0) = \frac{p}{z-\lambda_0} + \frac{q}{1-s(z-\lambda_0)}.$$

(See [2, (3.14)].) On the other hand, the lower bound m_0 given by (4) is obtained by majorizing the series for $[V_0 R_0(z)]^2$ by the power series for the function

$$\Phi_2(z-\lambda_0) = \frac{p}{(z-\lambda_0)^2} + \frac{2pq}{(z-\lambda_0)[1-s(z-\lambda_0)]} + \frac{r^2}{[1-s(z-\lambda_0)]^2}.$$

THEOREM 2.3. *The convergence radii of the power series for $P(t)$ and $\hat{\lambda}(t)$ are at least equal to $1/4u$, where*

$$u = \max\{ps, \sqrt{psq}, r\}. \tag{7}$$

If $|t| < 1/4u$, then the operator $T(t) = T_0 + tV_0$ has m eigenvalues $\lambda_1(t), \ldots, \lambda_m(t)$, counted according to their algebraic multiplicities, and they satisfy

$$|\lambda_i(t) - \lambda_0| \leq \frac{1-\sqrt{1-4u|t|}}{2s}, \qquad 1 \leq i \leq m. \tag{8}$$

Further, $\lambda_1(t), \ldots, \lambda_m(t)$ are the only spectral values of $T(t)$ lying in the open disk

$$\left\{ z \in \mathbb{C} : |z-\lambda_0| < \frac{1+\sqrt{1-4u|t|}}{2s} \right\}. \tag{9}$$

PROOF. Let $0 < \varepsilon < 1$. Then by Lemma 2.1, the radius of the disk $\partial(\Gamma_\varepsilon)$ is at least

$$\left[\frac{u^2}{\varepsilon^2} + \frac{2u^2}{\varepsilon(1-\varepsilon)} + \frac{u^2}{(1-\varepsilon)^2}\right]^{-1/2} = \frac{\varepsilon(1-\varepsilon)}{u}.$$

It can be easily seen that $|t| < \varepsilon(1-\varepsilon)/u$ if and only if $u_1(t) < \varepsilon < u_2(t)$, where

$$(10) \qquad u_1(t) = \frac{1-\sqrt{1-4u|t|}}{2} \quad \text{and} \quad u_2(t) = \frac{1+\sqrt{1-4u|t|}}{2}.$$

Hence for every ε satisfying $u_1(t) < \varepsilon < u_2(t)$, we see that $t \in \partial(\Gamma_\varepsilon)$. Thus, Γ_ε contains m eigenvalues $\lambda_1(t), \dots, \lambda_m(t)$ of $T(t)$ (counted according to their algebraic multiplicities) and does not contain any other spectral value of $T(t)$. In particular,

$$|\lambda_i(t) - \lambda_0| < \varepsilon/s, \qquad i = 1, \dots, m.$$

Letting $\varepsilon \to u_1(t)$, we have

$$|\lambda_i(t) - \lambda_0| \leq \frac{1-\sqrt{1-4u|t|}}{2s}, \qquad i = 1, \dots, m,$$

while letting $\varepsilon \to u_2(t)$, we find that the open disk

$$\left\{ z \in \mathbb{C} : |z - \lambda_0| < \frac{1+\sqrt{1-4u|t|}}{2s} \right\}$$

does not contain any other spectral value of $T(t)$.

Remark 2.4. The functions $u_1(t)$ and $u_2(t)$ introduced in (10) satisfy $0 \leq u_1(t) \leq \frac{1}{2} \leq u_2(t) \leq 1$ for $|t| < 1/4u$, and they depend only on $|t|$. As $|t| \downarrow 0$, $u_1(t)$ monotonically decreases to 0, while $u_2(t)$ monotonically increases to 1, and as $|t| \uparrow 1/4u$, $u_1(t)$ monotonically increases to $\frac{1}{2}$, while $u_2(t)$ monotonically decreases to $\frac{1}{2}$. Thus, a smaller absolute value of the parameter t (corresponding to a smaller perturbation tV_0) yields a better estimate for $|\lambda_i(t) - \lambda_0|$, $i = 1, \dots, m$, and at the same time a larger disk of isolation of $\{\lambda_1(t), \dots, \lambda_m(t)\}$ from the rest of $\sigma(T(t))$. For all t with $|t| < 1/4u$, we have

$$\left\{ z \in \mathbb{C} : |z - \lambda_0| < \frac{1}{2s} \right\} \cap \sigma(T(t)) = \{\lambda_1(t), \dots, \lambda_m(t)\}.$$

These phenomena are illustrated in Figure 1.

Remark 2.5. The lower bounds

$$(\sqrt{ps} + \sqrt{q})^{-1} \left[ps + 2\sqrt{psq} + \frac{r^2}{q} \right]^{-1/2} \qquad \text{and} \qquad \frac{1}{4\max\{ps, \sqrt{psq}, r\}}$$

for the convergence radii of the series for $P(t)$ and $\hat{\lambda}(t)$ obtained in Remark 2.2 and Theorem 2.3 are significant in the operator approximation theory. We note that both these bounds tend to infinity if p and r approach 0,

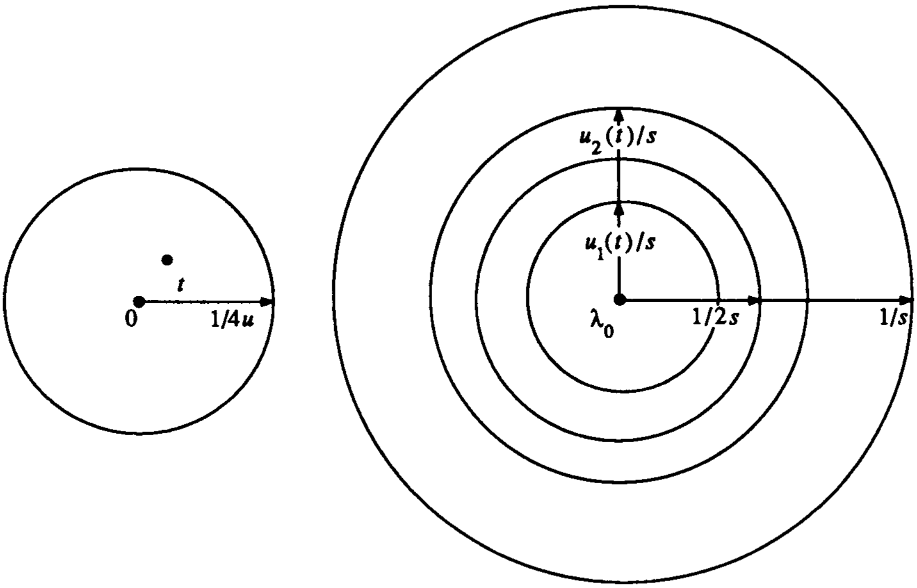

Figure 1

and if s and q remain bounded. Let a compact operator T on X be approximated by a sequence of bounded operators (T_n) in a collectively compact manner (that is, $T_n x \to Tx$ for every $x \in X$ and the set $\bigcup_{n=1}^{\infty}\{(T-T_n)x : x \in X, \|x\| \le 1\}$ is totally bounded). Let λ be a simple eigenvalue of T such that $|\lambda| > \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})$. It is shown in [3, Theorem 2.1] and [5, Theorem 3.4] that for all large n, T_n has a simple eigenvalue λ_n such that $\lambda_n \to \lambda$, and if we let $T_0 = T_n$, $V_0 = T - T_n$, $\lambda_0 = \lambda_n$, $s_n = \|S_n\|$, $p_n = \|V_n P_n\|$, $q_n = \|V_n S_n\|$ and

$$r_n^2 = \sup\{\|V_n S_n^{k+1} V_n S_n\| / \|S_n\|^k : k = 0, 1, 2, \dots\},$$

then $p_n \to 0$ and $r_n \to 0$, while (s_n) and (q_n) remain bounded as $n \to \infty$. The above lower bounds then guarantee that for a suitably large fixed n, the point 1 lies in the convergence disks of $P(t)$ and $\hat{\lambda}(t)$, and hence the spectral projection $P(1)$ and the eigenvalue $\lambda = \lambda(1)$ of $T(1) = T$ can be approximated by the partial sums of the power series for $P(t)$ and $\hat{\lambda}(t)$, respectively.

We note that since (q_n) may *not* tend to zero as $n \to \infty$, the lower bound $(\sqrt{p_n s_n} + \sqrt{q_n})^{-2}$ for the convergence radii of $P(t)$ and for $\lambda(t)$ obtained by Kato is not useful for ascertaining the convergence at $t = 1$. In case (T_n) converges to T in the norm, then (q_n) *does* tend to zero as $n \to \infty$, and Kato's lower bound becomes applicable. We mention that there are many important cases where (T_n) approximates T in the collectively compact manner, but not in the norm. For example, for a nonzero compact Fredholm integral operator T on $X = C([a, b])$, let (T_n) be a sequence of Nyström or Fredholm approximations of T, or if π_n is an interpolatory projection on X with $\pi_n x \to x$ for all $x \in X$, let $T_n = T\pi_n$ or $\pi_n T \pi_n$ ([1, pages 18–19] and [4, pages 292–294]).

In case T_0 is a normal operator on a Hilbert space X, then every eigenvalue of T_0 of finite algebraic multiplicity is semisimple. For this case, we refer to [4, Theorem 10.6] and [2, Theorem 3.9] for an especially elegant analogue of Theorem 2.3.

3. Convergence radius of an eigenvector series

Let λ_0 be a simple eigenvalue of T_0, that is, λ_0 is a semisimple eigenvalue and the associated eigenspace is one dimensional. Then for $t \in \partial(\Gamma_0)$, $P(t)$ is of rank 1, and the operator $T(t) = T_0 + tV_0$ has only one simple eigenvalue $\lambda(t)$ lying inside Γ_0, so $\hat{\lambda}(t) = \lambda(t)$ is analytic for $t \in \partial(\Gamma_0)$. Let ϕ_0 be an eigenvector of T_0 corresponding to λ_0. Then the adjoint operator T_0^* is closed and densely defined in the adjoint space $X^* = \{x^* : X \to \mathbb{C};\ x^* \text{ is conjugate linear and continuous}\}$. There is a unique eigenvector ϕ_0^* of T_0^* corresponding to $\overline{\lambda}_0$ such that $\langle \phi_0, \phi_0^* \rangle = 1$ and we have

$$P_0 x = \langle x, \phi_0^* \rangle \phi_0, \qquad x \in X.$$

As mentioned in the introduction, the function

$$\phi(t) = P(t)\phi_0 / \langle P(t)\phi_0, \phi_0^* \rangle \tag{11}$$

is meromorphic in $\partial(\Gamma_0)$ and $\phi(0) = \phi_0$. If $t_0 \in \partial(\Gamma_0)$ is not a pole of the function ϕ, then $\phi(t_0)$ is an eigenvector of $T(t_0)$ corresponding to $\lambda(t_0)$. We have not been able to decide whether the function ϕ ever has a pole in $\partial(\Gamma_0)$. Nevertheless, we shall obtain a subregion of $\partial(\Gamma_0)$ which contains 0 and which is free of any pole of the function ϕ.

LEMMA 3.1. *Let $t_0 \in \partial(\Gamma_0)$. Then there is a unique nonnegative integer k_0 such that*

$$\psi(t_0) = \lim_{t \to t_0} P(t)\phi_0 / (t - t_0)^{k_0} \tag{12}$$

is an eigenvector of $T(t_0)$ corresponding to $\lambda(t_0)$. Further, t_0 is a pole of the function ϕ if and only if $\langle \psi(t_0), \phi_0^* \rangle = 0$, and in this case, $\psi(t_0)$ is an eigenvector also of the operator $(I - P_0)T(t_0)_{|(I-P_0)D_0}$.

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PROOF. Let $t_0 \in \partial(\Gamma_0)$. Since $P(0)\phi_0 = \phi_0 \neq 0$, the X-valued analytic function $t \mapsto P(t)\phi_0$ is not identically zero on $\partial(\Gamma_0)$. Hence either it has an isolated zero of finite order at t_0 or it is nonzero at t_0. In any case, there is a unique nonnegative integer k_0 and an analytic function ψ in a neighbourhood U of t_0 such that $P(t)\phi_0 = (t-t_0)^{k_0}\psi(t)$ and $\psi(t_0) \neq 0$. For $t \in U$, we have

$$\langle P(t)\phi_0, \phi_0^*\rangle = (t-t_0)^{k_0}\langle \psi(t), \phi_0^*\rangle.$$

Hence the function

$$\phi(t) = \frac{\psi(t)}{\langle \psi(t), \phi_0^*\rangle}, \qquad t \neq t_0,$$

has a pole at t_0 if and only if $\langle \psi(t_0), \phi_0^*\rangle = 0$. Clearly, $\lim_{t\to t_0} \psi(t)$ exists and equals $\psi(t_0)$. Now,

$$\psi(t) = P(t)\phi_0/(t-t_0)^{k_0}, \qquad t \in U, \ t \neq t_0,$$

is an eigenvector of $T(t)$ corresponding to $\lambda(t)$. The continuity of the functions $t \mapsto P(t)$ and $t \mapsto \psi(t)$ at $t = t_0$ shows that $\psi(t_0)$ is an eigenvector of $T(t_0)$ corresponding to $\lambda(t_0)$.

Assume now that t_0 is a pole of the function ϕ. Then

$$P_0\psi(t_0) = \langle \psi(t_0), \phi_0^*\rangle\phi_0 = 0,$$

that is, $\psi(t_0) \in (I-P_0)D_0$. Since

$$(I-P_0)T(t_0)\psi(t_0) = \lambda(t_0)(I-P_0)\psi(t_0) = \lambda(t_0)\psi(t_0),$$

it is apparent that $\psi(t_0)$ is an eigenvector of $(I-P_0)T(t_0)_{|(I-P_0)D_0}$ corresponding to $\lambda(t_0)$.

PROPOSITION 3.2. *The meromorphic function ϕ given by* (11) *has no pole in the open set*

$$G = \{t \in \partial(\Gamma_0) : s|\lambda(t)-\lambda_0| + r|t| < 1\}.$$

For every $t \in G$, $\phi(t)$ is an eigenvector of $T(t)$ corresponding to $\lambda(t)$.

PROOF. Let $t \in G$. We show that $\lambda(t)$ is not in the spectrum of the operator $(I-P_0)(T_0+tV_0)_{|(I-P_0)D_0} = (I-P_0)T(t)_{|(I-P_0)D_0}$. Lemma 3.1 then implies the desired result.

Let us denote $(I-P_0)D_0$ by Z and $\lambda(t)$ by z for brevity. Since $t \in G$, we have $|z-\lambda_0| < 1/s$. By (2), λ_0 is the only spectral value of T_0 in the open disk with centre λ_0 and radius $1/s$, and $\sigma(T_{0|Z}) = \sigma(T_0)\backslash\{\lambda_0\}$. Hence it follows that $z \notin \sigma(T_{0|Z})$. In order to conclude $z \notin \sigma(I-P_0)(T_0+tV_0)_{|Z}$, it is enough to show that $r_\sigma(E(z)) < 1$, where

$$E(z) = [T_{0|Z} - (I-P_0)(T_0+tV_0)_{|Z}][T_{0|Z} - zI_{|Z}]^{-1}.$$

Now,

$$E(\lambda_0) = -t(I-P_0)V_0S_{0|Z},$$

and if $z \neq \lambda_0$, then

$$E(z) = -t(I-P_0)V_0R_0(z)_{|Z}.$$

In the first case,

$$\begin{aligned} r_\sigma(E(\lambda_0)) &\leq |t| r_\sigma((I-P_0)V_0S_0) = |t| r_\sigma(V_0S_0(I-P_0)) \\ &= |t| r_\sigma(V_0S_0) \leq |t|\,\|(V_0S_0)^2\|^{1/2} \leq |t| r, \end{aligned}$$

which is less than 1 since $t \in G$. In the second case, that is, when $z \neq \lambda_0$,

$$\begin{aligned} r_\sigma(E(z)) &\leq |t| r_\sigma((I-P_0)V_0R_0(z)) = |t| r_\sigma(V_0R_0(z)(I-P_0)) \\ &= |t|\,\|C(z)\|^{1/2} \leq |t| r/(1-s|z-\lambda_0|), \end{aligned}$$

by (5). Since $z = \lambda(t)$ and $t \in G$, it again follows that $r_\sigma(E(z)) < 1$.

THEOREM 3.3. *Let* $0 < \varepsilon < 1$ *and* $t \in \mathbb{C}$ *be such that*

$$|t| < \left[\frac{p^2s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}\right]^{-1/2}.$$

Then $\phi(t) = P(t)\phi_0/\langle P(t)\phi_0, \phi_0^*\rangle$ *is an eigenvector of* $T(t)$ *corresponding to* $\lambda(t)$ *such that* $\langle \phi(t), \phi_0^* \rangle = 1$. *The radius of convergence of the power series for* $\phi(t)$ *about* 0 *is at least equal to* m_0 *given by* (4).

PROOF. By Lemma 2.1, t belongs to $\partial(\Gamma_\varepsilon)$, so that $|\lambda(t) - \lambda_0| < \varepsilon/s$. Hence

$$s|\lambda(t)-\lambda_0| + r|t| < \varepsilon + \frac{r}{\left[\frac{p^2s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}\right]^{1/2}} < 1.$$

By Proposition 3.2, we see that $\phi(t)$ is an eigenvector of $T(t)$ corresponding to $\lambda(t)$. It is obvious that $\langle \phi(t), \phi_0^* \rangle = 1$. Thus, the function ϕ is analytic on

$$\left\{ t \in \mathbb{C} \colon |t| < \left[\frac{p^2s^2}{\varepsilon^2} + \frac{2psq}{\varepsilon(1-\varepsilon)} + \frac{r^2}{(1-\varepsilon)^2}\right]^{-1/2} \right\}$$

for every ε, $0 < \varepsilon < 1$. The desired result now follows from the very definition of m_0.

REMARK 3.4. As the quantities

$$(\sqrt{ps}+\sqrt{q})^{-1}\left[ps+2\sqrt{psq}+\frac{r^2}{q}\right]^{-1/2} \quad \text{and} \quad \frac{1}{4\max\{ps,\sqrt{psq},r\}}$$

are at most equal to m_0, they also give lower bounds for the convergence radius of the power series for $\phi(t)$, just as we had noted this for $P(t)$ and $\lambda(t)$. The lower bound $(\sqrt{ps}+\sqrt{q})^{-2}$ for $\phi(t)$ given in [2, Problem 3.7] (obtained by finding a majorizing series for $\phi(t)$) is thus improved to $(\sqrt{ps}+\sqrt{q})^{-1}[ps+2\sqrt{psq}+r^2/q]^{1/2}$. (Note $r \le q$.)

The lower bound $1/(4\max\{ps,\sqrt{psq},r\})$ for the convergence radii of the series for $\lambda(t)$ and for $\phi(t)$ is obtained in [5] by considering estimates for the iteratively defined nth terms of the power series for $\lambda(t)$ and $\phi(t)$, while in [7, pages 82 and 85], a similar lower bound is obtained by considering the power series for $P(t)$.

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