FINITE GROUPS WITH SOME $\mathcal{Z}$-PERMUTABLE SUBGROUPS

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Abstract. Let $\mathcal{Z}$ be a complete set of Sylow subgroups of a finite group $G$; that is to say for each prime $p$ dividing the order of $G$, $\mathcal{Z}$ contains one and only one Sylow $p$-subgroup of $G$. A subgroup $H$ of $G$ is said to be $\mathcal{Z}$-permutable in $G$ if $H$ permutes with every member of $\mathcal{Z}$. In this paper we characterise the structure of finite groups $G$ with the assumption that (1) all the subgroups of $G_p \in \mathcal{Z}$ are $\mathcal{Z}$-permutable in $G$, for all prime $p \in \pi(G)$, or (2) all the subgroups of $G_p \cap F^*(G)$ are $\mathcal{Z}$-permutable in $G$, for all $G_p \in \mathcal{Z}$ and $p \in \pi(G)$, where $F^*(G)$ is the generalised Fitting subgroup of $G$.

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1. Introduction and statements of results. All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [5]. Throughout this paper, $G$ stands for a finite group and $\pi(G)$ represents the set of distinct primes dividing $|G|$.

A subgroup of $G$ is called quasi-normal in $G$ if it permutes with every subgroup of $G$. We say, following Kegel [8], that a subgroup of $G$ is $S$-quasi-normal in $G$ if it permutes with every Sylow subgroup of $G$. Recently, Asaad and Heliel [1] introduced a new embedding property, namely the $3$-permutability of subgroups of a group; $\mathcal{Z}$ is called a complete set of Sylow subgroups of $G$ if for each prime $p \in \pi(G)$, $\mathcal{Z}$ contains exactly one Sylow $p$-subgroup of $G$, say $G_p$. A subgroup of $G$ is said to be $\mathcal{Z}$-permutable in $G$ if it permutes with every member of $\mathcal{Z}$. Obviously, every $S$-quasi-normal subgroup is $\mathcal{Z}$-permutable. In contrast to the fact that every $S$-quasi-normal subgroup is subnormal (see [8]), it does not hold in general that every $\mathcal{Z}$-permutable subgroup of $G$ is subnormal in $G$. It suffices to consider the alternating group of degree 4.

Many authors have investigated the structure of a group $G$ under the assumption that some subgroups of $G$ are well situated in $G$. Srinivasan [14] proved that a group $G$ is supersolvable if every maximal subgroup of any Sylow subgroup of $G$ is normal. Later on, Wall [15] gave a complete classification of finite groups under the assumption

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of Srinivasan. In [10], the authors obtained the following results ([10], Theorems 3.1 and 3.4): Let \( \mathfrak{Z} \) be a complete set of Sylow subgroups of a group \( G \) and \( p \) the smallest prime dividing \( |G| \). Then \( G \) is \( p \)-nilpotent if one of the following hold: (1) the maximal subgroups of \( G_p \in \mathfrak{Z} \) are \( 3 \)-permutable subgroups of \( G \); (2) \( G \) is \( A_4 \)-free and the \( 2 \)-maximal subgroups of \( G_p \) are \( 3 \)-permutable subgroups of \( G \). In [11], the authors obtained the following ([11], Theorems 3.1 and 3.4): Let \( \mathfrak{Z} \) be a complete set of Sylow subgroups of a group \( G \) and \( p \) the smallest prime dividing \( |G| \). Then \( G \) is \( p \)-nilpotent if one of the following holds: (1) every cyclic subgroup of prime order or order 4 (when \( p = 2 \)) of \( G_p \in \mathfrak{Z} \) is \( 3 \)-permutable in \( G \); (2) \( G \) is \( A_4 \)-free and every subgroup of prime square order of \( G_p \in \mathfrak{Z} \) is \( 3 \)-permutable in \( G \). We know that if every subgroup of \( G \) is normal in \( G \), then \( G \) is the Dedikind group [13]; every subgroup of \( G \) is quasi-normal in \( G \); then \( G \) is the quasi-Hamilton group [2]. It is easy to see that that \( G \) is nilpotent if and only if every subgroup of \( G \) of prime power order is \( 3 \)-permutable in \( G \), where \( \mathfrak{Z} \) is a complete set of Sylow subgroups of \( G \). In view of the above results, it is interesting to give the structure of \( G \) under the assumption that for any \( G_p \in \mathfrak{Z} \), every subgroup of \( G_p \) is \( 3 \)-permutable in \( G \). We get the following.

**Theorem 1.1.** Let \( G \) be a finite group and \( \mathfrak{Z} \) a complete set of Sylow subgroups of \( G \). Then every subgroup of \( G_p \in \mathfrak{Z} \), for any prime \( p \in \pi(G) \), is \( 3 \)-permutable in \( G \) if and only if there exists a normal subgroup \( L \) of \( G \) satisfying the following:

1. \( L \) is an abelian Hall subgroup of \( G \) and \( G/L \) is nilpotent;
2. the elements of \( G \) induce power automorphisms in \( L \);
3. for any two distinct primes \( p, q \notin \pi(L) \), \( [G_p, G_q] = 1 \), where \( G_p, G_q \in \mathfrak{Z} \).

It is interesting to limit the hypotheses to a smaller subgroup of \( G \). By [4] and [9], we know the following: Let \( G \) be a finite group and \( \mathfrak{Z} \) a complete set of Sylow subgroups of \( G \) and \( F^*(G) \) is the generalised Fitting subgroup of \( G \). Then \( G \) is supersolvable under one of the following assumptions: (1) the maximal subgroups of \( G_p \cap F^*(G) \) are \( 3 \)-permutable subgroups of \( G \), for all \( G_p \in \mathfrak{Z} \); (2) the cyclic subgroups of \( G_p \cap F^*(G) \) of prime order or order are \( 3 \)-permutable subgroups of \( G \), for all \( G_p \in \mathfrak{Z} \). Hence, it is interesting to investigate the structure of \( G \) under the assumption that all the subgroups of \( G_p \cap F^*(G) \) are \( 3 \)-permutable subgroups of \( G \), for all \( G_p \in \mathfrak{Z} \). Here we get the following.

**Theorem 1.2.** Let \( G \) be a finite group and \( \mathfrak{Z} \) a complete set of Sylow subgroups of \( G \), and \( F^*(G) \) is the generalised Fitting subgroup of \( G \). Then every subgroup of \( G_p \cap F^*(G) \), for any \( G_p \in \mathfrak{Z} \) and any \( p \in \pi(G) \), is \( 3 \)-permutable in \( G \) if and only if there exists a normal subgroup \( L \) of \( G \) satisfying the following:

1. \( L \) is abelian and \( G/L \) is nilpotent;
2. \( L \) is a Hall subgroup of \( F^*(G) \);
3. \( p' \)-elements of \( G \) induce power automorphisms in \( L_p \), the Sylow \( p \)-subgroup of \( L \).

**Corollary 1.3.** Let \( G \) be a finite group, and \( F^*(G) \) is the generalised Fitting subgroup of \( G \). Then every subgroup of \( F^*(G) \) is \( S \)-quasi-normal in \( G \) if and only if there exists a normal subgroup \( L \) of \( G \) satisfying the following:

1. \( L \) is abelian and \( G/L \) is nilpotent;
2. \( L \) is a Hall subgroup of \( F^*(G) \);
3. \( p' \)-elements of \( G \) induce power automorphisms in \( L_p \), the Sylow \( p \)-subgroup of \( L \).
Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$. If $N \triangleleft G$, we denote

\begin{align*}
\mathfrak{Z}N &= \{ G_pN : G_p \in \mathfrak{Z} \}, \\
\mathfrak{Z}N/N &= \{ G_pN/N : G_p \in \mathfrak{Z} \}, \\
\mathfrak{Z} \cap N &= \{ G_p \cap N : G_p \in \mathfrak{Z} \}.
\end{align*}

The generalised Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasi-nilpotent subgroup of $G$. Its important properties can be found in [7], Chapter X, Section 13.

Now, $G^V$ denotes the nilpotent residual of $G$, which some authors prefer to write as $K_\infty(G)$; it is the last term in the lower central series of $G$.

2. Preliminaries. The following lemmas will be used in the proofs of our results.

**Lemma 2.1.** ([1], Lemma 2.1) Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of $G$, $U$ a $\mathfrak{Z}$-permutable subgroup of $G$ and $N$ a normal subgroup of $G$. Then

1. $\mathfrak{Z} \cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of $N$ and $G/N$, respectively;
2. $UN/N$ is a $\mathfrak{Z}N/N$-permutable subgroup of $G/N$;
3. $U$ is a $\mathfrak{Z} \cap N$-permutable subgroup of $N$ if $U \leq N$.

**Lemma 2.2.** Let $G$ be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of $G$. Suppose $N$ is a normal $p$-subgroup of $G$; then every subgroup of $N$ is $\mathfrak{Z}$-permutable in $G$ if and only if every subgroup of $N$ is $S$-quasi-normal in $G$.

**Proof.** We only need to prove the necessity. Suppose any subgroup of $N$ is $\mathfrak{Z}$-permutable in $G$. Let $L$ be an arbitrary subgroup of $N$. Then $LG_{p_{\mathfrak{Z}}}^x$ is a subgroup of $G$ for every $G_{p_{\mathfrak{Z}}}^x \in \mathfrak{Z}$. Since $N \triangleleft G$, it follows that $L \leq N^x$ for all $x \in G$. Hence $L^{x^{-1}} \leq N$, and therefore $L^{x^{-1}}G_{p_{\mathfrak{Z}}}^x \leq G$. But $LG_{p_{\mathfrak{Z}}}^x = (L^{x^{-1}}G_{p_{\mathfrak{Z}}}^x)^x$ is a subgroup of $G$, then $L$ is $S$-quasi-normal in $G$. □

**Lemma 2.3.** ([7]; Chapter X, Section 13) Let $G$ be a group and $M$ a subgroup of $G$.

1. If $M$ is normal in $G$, then $F^*(M) \leq F^*(G)$.
2. If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

**Lemma 2.4.** Let $G$ be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of $G$. Suppose every subgroup of $F^*(G) \cap G_p$ is $\mathfrak{Z}$-permutable in $G$, for any $G_p \in \mathfrak{Z}$; then $G$ is supersolvable.

**Proof.** This is a corollary of results in [4] or [9]. □

**Lemma 2.5.** Suppose $G$ is a group and $P$ a normal $p$-subgroup of $G$. Then $P \leq Z_\infty(G)$ if and only if $C_G(P) \geq O^p(G)$.

**Proof.** If $C_G(P) \geq O^p(G)$, then $G/C_G(P)$ is a $p$-group; so $P \leq Z_\infty(G)$ by [16], p. 220, Theorem 6.3. The converse is [12], Lemma 2.8. □

**Lemma 2.6.** Suppose $P$ is a normal $p$-subgroup of $G$. If every subgroup of $P$ is $S$-quasi-normal in $G$, then every $p'$-element of $G$ induces a power automorphism in $P$. 


Proof. Take any \( a \in P \). Let \( x \) be a \( p' \)-element of \( G \). Then \( x \in G_p \) for some \( p' \)-Hall subgroup of \( G \). Then \( \langle a \rangle G_p \) is a group by hypotheses. Hence

\[
a^{(x)} = a^{(x)} \cap \langle a \rangle G_p = \langle a \rangle (a^{(x)} \cap G_p) \leq \langle a \rangle (P \cap G_p) = \langle a \rangle,
\]

i.e. \( a^{(x)} = \langle a \rangle \). Therefore \( x \) induces a power automorphism in \( \langle a \rangle \). \( \square \)

In the next lemma we collect some properties of power automorphism.

Lemma 2.7. Suppose \( N \) is a non-trivial normal \( p \)-subgroup of \( G \). Then

(i) \( p' \)-power automorphism of \( N \) is trivial if \( N \) is non-abelian;

(ii) there exists a positive integer \( n \) such that \( a^n = a \), for all \( a \in N \), if \( N \) is abelian and \( a \) is a power automorphism of \( N \);

(iii) the power automorphisms of \( N \) are in the centre of \( \text{Aut}(N) \) if \( N \) is abelian;

(iv) \( G/C_G(N) \) is nilpotent if all \( p' \)-elements of \( G \) induce power automorphisms in \( N \) by conjugate.

Proof. (1) It is [6], Hilfsatz 5.

(2) See [13], Chapter 13, Theorem 4.3.

(3) It is a direct corollary of (2).

(4) If \( N \) is non-abelian, then \( G/C_G(N) \) is a \( p \)-group by (1); hence \( G/C_G(N) \) is nilpotent. If \( N \) is abelian, then the power automorphisms are in the centre of \( \text{Aut}(N) \) by (3). It is easy to see that \( G/C_G(N) \) is nilpotent. \( \square \)

Lemma 2.8. Suppose \( L = K_{\infty}(G) \) is the nilpotent residual of \( G \). If \( L \) is nilpotent, then \( L_p = [L_p, G] \) for any \( p \in \pi(G) \).

Proof. By definition, \( L = [L, G] = [L_p \times L_{p'}, G] = [L_p, G] \times [L_{p'}, G] = L_p \times L_{p'} \), for any \( p \in \pi(G) \). Hence \( L_p = [L_p, G] \). \( \square \)

3. Proofs.

Proof of Theorem 1.1. We prove the necessity of this theorem in several steps.

(i) \( G \) is supersolvable. Hence \( F^*(G) = F(G) \).

By Lemmata 2.4 and 2.3.

(ii) If \( N \) is a normal \( p \)-subgroup of \( G \), then \( p' \)-elements of \( G \) induce power automorphisms in \( N \).

If \( N \) is a normal \( p \)-subgroup of \( G \), then \( N \leq G_p \leq \mathfrak{3} \). Thus every subgroup of \( N \) is \( \mathfrak{3} \)-permutable in \( G \) by hypotheses; then is \( S \)-quasi-normal in \( G \) by Lemma 2.2. Now applying Lemma 2.6, we get step (ii).

(iii) Pick \( L = G^N \); then \( G/L \) is nilpotent. Furthermore, \( L \) is abelian.

By (ii) and Lemma 2.7(4), for any \( p \in \pi(G) \), we know that \( G/C_G(O_p(G)) \) is nilpotent. So \( (G/C_G(O_p(G)))^N = 1 \), and it follows that \( G^N \leq C_G(O_p(G)) \). Therefore \( G^N \leq \bigcap_{p \in \pi(G)} C_G(O_p(G)) = C_G(F(G)) \leq F(G) \). Then \( L \leq Z(F(G)) \), Hence \( L \) is abelian.

(iv) \( L \) is a Hall subgroup of \( G \).

Let \( p \) be the largest prime dividing \( |G| \), and \( P \) is a Sylow \( p \)-subgroup of \( G \). Since \( G \) is supersolvable by step (i), we know that \( P \leq G \). Then \( P = G_p \in \mathfrak{3} \). Now, we consider the quotient group \( G/P \). By Lemma 2.1, all subgroups of every member in \( \mathfrak{3} P/P \) are \( \mathfrak{3} P/P \)-permutable in \( G/P \). By induction, \( (G/P)^N = G^N P/P = LP/P \) is a Hall subgroup of \( G/P \).
Suppose that every \( p' \)-element of \( G \) centralises \( P \). Let \( L_p \in Syl_p(L) \). If \( L_p \neq 1 \), then, by Lemma 2.8, \( L_p = [L_p, G] = [L_p, P] \leq P \) as \( L_p \leq P \), a contradiction. Hence \( L_p = 1 \) and \( L \) is a \( p' \)-group. Therefore, \( L \cong LP/P \) is a normal Hall subgroup of \( G \). Now suppose that there exists a \( p' \)-element \( x \) which induces a non-trivial power automorphism on \( P \). Hence \( P = [P, G] \leq L \). Therefore \( L \) is a Hall subgroup of \( G \).

(v) The elements of \( G \) induce power automorphisms in \( L \).

It is easy to see from (ii)–(iv).

(vi) For any two distinct primes \( p, q \notin \pi(L) \), \([G_p, G_q] = 1 \). Hence \([G_p, G_q] = 1 \).

By the hypotheses, \( G_pG_q \) is a group. Since \( G_pG_q \cong G_pG_qL/P \leq G/L \), \( G_pG_q \) is nilpotent by (iii). Hence \([G_p, G_q] = 1 \).

Conversely, it suffices to prove that \([x, y] = y \langle x \rangle \), for any \( p \)-element \( x \in G_p \) and \( q \)-element \( y \in G_q \). Let \( L \leq G \) be a normal abelian Hall subgroup of \( G \), we have that \( x, y \in L \) and \([x, y] = y \langle x \rangle \).

If \( p, q \in \pi(L) \), then \( x \leq G_p \) and \( y \leq G_q \). Therefore \( \langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle \).

Suppose \( p \in \pi(L) \) or \( q \in \pi(L) \). Without lose generality, let \( p \in \pi(L) \). Then \( x \leq L \) and \([x, y] \leq G \) by (ii). Hence \([x, y] = y \langle x \rangle \). Finishing the proof. \( \square \)

**Proof of Theorem 1.2.** We first prove the necessity of Theorem 1.2. With the same arguments as in the proof of Theorem 1.1, we get steps (i)–(iii).

(i) \( G \) is supersolvable. Hence \( F^*(G) = F(G) \).

(ii) If \( N \) is a normal \( p \)-subgroup of \( G \), then \( p' \)-elements of \( G \) induce power automorphisms in \( N \).

(iii) Pick \( L = G^N \); then \( G/L \) is nilpotent. Furthermore, \( L \) is abelian.

(iv) Denote \( F = F(G) \). Then \( F = C_G(L) \).

By the proof of (iii), we only need to prove \( C_G(L) \leq F \). We know that \( L \leq Z(C_G(L)) \), and hence \( C_G(L)/Z(C_G(L)) \leq G/Z(C_G(L)) \) is nilpotent. Hence \( C_G(L) \) is nilpotent. Therefore \( C_G(L) \leq F(G) \).

(v) \( F = Z_\infty(G) \times L \).

By [5], Chapter VI, Satz 7.15, \( G \) splits over \( L \), i.e. \( G = X \bowtie L \), for some subgroup \( X \) of \( G \). So \( F = F \cap (XL) = L(C_X(L) \times X) = C_X(L)L = C_X(L) \times L \).

Now we prove that \( C_X(L) = Z_\infty(G) \).

Notice that \([C_X(L), G] = [C_X(L), X] \leq C_X(L) \). Since \( X \leq G \) is nilpotent, there exists an integer \( n \) such that \( K_n(X) \leq K_\infty(X) = 1 \). Therefore \([C_X(L), G, \ldots, G] = [C_X(L), X, \ldots, X] \leq K_\infty(X) = 1 \). Therefore \( C_X(L) \leq Z_{n-2}(G) \leq Z_\infty(G) \). Thus \( Z_\infty(G) = Z_\infty(G) \cap F = C_X(L)(Z_\infty(G) \cap L) \).

Next we want to prove that \( Z_\infty(G) \cap L = 1 \). If \( Z_\infty(G) \cap L \neq 1 \), then there exists a prime \( p \in \pi(G) \) such that \( Z_\infty(G) \cap L_p \neq 1 \). If every \( p' \)-element of \( G \) centralises \( L_p \), then, by Lemma 2.8, \( L_p = [L_p, G] = [L_p, P] < L_p \), a contradiction. Hence there exists a \( p' \)-element \( x \) which induces a non-trivial power automorphism on \( L_p \), and so \([L_p, x] \neq 1 \).

On the other hand, we know that \([Z_\infty(G) \cap L_p, x] = 1 \) by Lemma 2.5. Hence \([L_p, x] = 1 \) by (iii) and Lemma 2.7(2), a contradiction. Hence \( Z_\infty(G) \cap L = 1 \). So \( Z_\infty(G) = C_X(L) \).

Thus \( F = Z_\infty(G) \times L \).

(vi) \( L \) is a Hall subgroup of \( F(G) \).

If \( L \) is not a Hall subgroup of \( F(G) \), then there is a prime \( p \in \pi(L) \cap \pi(F(G)/L) \).

Denote \( C = C_X(L) \).

By (iv), \( F = C_G(L) = Z_\infty(G) \times L = C_X(L) \times L = C \times L \). Since \( p \in \pi(C) \), we have \( C_p \neq 1 \); then \( C_p \cap Z(C) \neq 1 \). Therefore \( p \in \pi(Z(C)) \). For any \( p' \)-element \( x \in X \), \( x \in Z(C) \).

\[ x \in Z(C) \Rightarrow x \in C \Rightarrow x \in \text{Hall subgroup of } F(G) \]
induces a power automorphism on $F_p = C_p \times L_p$. By Lemma 2.5, $[C_p, x] = 1$. Hence $[Z(C_p), x] = 1$. By Lemma 2.7(2), $[Z(C_p) \times L_p, x] = 1$, and in particular, $[L_p, x] = 1$. Therefore $[L_p, X_p] = 1$. Then $L_p = [L_p, G] = [L_p, X] = [L_p, X_p] \vartriangleleft L_p$, a contradiction. Therefore $L$ is a Hall subgroup of $F(G)$.

Conversely, it is easy to see that $G$ is solvable; hence $F^*(G) = F(G) \neq 1$. It suffices to prove that every cyclic $p$-subgroup $< g >$ of $F(G)$ is $S$-quasi-normal in $G$, for any prime $p \in \pi(G)$.

Suppose $g \in O_p(G)$. If $P \in Syl_p(G)$, then $g \in O_p(G) \leq P$. Thus $< g > = P = P < g >= P$. Pick an arbitrary $Q \in Syl_p(G)$, where $g \neq p$. If $p \in \pi(L)$, then $g \in L_p$ by (iii). Then $Q$ normalises $< g >$ by (ii). Therefore $Q < g >= Q > Q$. Hence suppose that $p \notin \pi(L)$. Then $[< g > \vartriangleleft L/L, QL/L] = 1$ as $G/L$ is nilpotent by (i). So $[< g >, Q] \leq L$. It follows that $[Q, g] \leq L \cap g^G \leq L \cap O_p(G) = 1$. Hence $Q < g >= Q$. Completing the proof.

**Proof of Corollary 1.3.** If every subgroup of $F^*(G)$ is $S$-quasi-normal in $G$, then $G$ is supersolvable by the results in [4] or [9]. In particular, $F^*(G) = F(G)$. For any prime $p \in \pi(G)$ and any Sylow $p$-subgroup $G_p$ of $G$, every subgroup of $G_p \cap F^*(G) = G_p \cap F(G) = O_p(G)$ is $S$-quasi-normal in $G$ by the hypotheses. By Theorem 1.2, it is easy to see the necessity of Corollary 1.3 holds. Conversely, by the proof of Theorem 1.2, we know that every cyclic $p$-subgroup of $F(G) = F^*(G)$ is $S$-quasi-normal in $G$, for any prime $p \in \pi(G)$. It is easy to see that every subgroup of $F^*(G)$ is $S$-quasi-normal in $G$. Completing the proof. □

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