TRAVELLING WAVE SOLUTIONS IN NONLOCAL REACTION–DIFFUSION SYSTEMS WITH DELAYS AND APPLICATIONS

ZHI-XIAN YU*1 and RONG YUAN1

(Received 17 April, 2009; revised 14 August, 2009)

Abstract

This paper deals with two-species convolution diffusion-competition models of Lotka–Volterra type with delays which describe more accurate information than the Laplacian diffusion-competition models. We first investigate the existence of travelling wave solutions of a class of nonlocal convolution diffusion systems with weak quasimonotonicity or weak exponential quasimonotonicity by a cross-iteration technique and Schauder’s fixed point theorem. When the results are applied to the convolution diffusion-competition models with delays, we establish the existence of travelling wave solutions as well as asymptotic behaviour.


Keywords and phrases: travelling wave solutions, convolution, cross-iteration, Schauder’s fixed point theorem.

1. Introduction

The theory of travelling wave solutions for Laplacian diffusion equations has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics (see [7, 13, 22, 24, 26]). Recently, many researchers have focused on the existence of travelling wave solutions for Laplacian diffusion equations with discrete time delays (see [2, 10, 11, 18, 19]) and spatiotemporal or nonlocal delays (see [6, 15, 23, 25] and the references therein). Despite the popularity of Laplacian diffusion models, they have some drawbacks. One important shortcoming for ecological and epidemiological models is that Laplacian diffusion is a local operator where individuals in the population can only influence their immediate neighbours. With diffusion models there is some disconnection between the experimentally collected data and the limited number of parameters that are available to fit that data. One

1School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China; e-mail: yuzx@mail.bnu.edu.cn, ryuan@bnu.edu.cn.

© Australian Mathematical Society 2010, Serial-fee code 1446-1811/2010 $16.00
method in overcoming these problems with the Laplacian operator is to describe these models concerning spatial migration by integral equations, such as the diffusion system with the convolution operator

$$\frac{\partial u(x, t)}{\partial t} = D[(J \ast u)(x, t) - u(x, t)] + f(u(x, t)), \quad (1.1)$$

where \( \ast \) is the spatial convolution operator and

$$(J \ast u)(x, t) = \int_{\mathbb{R}} J(y)u(x - y, t) \, dy.$$ \hfill \text{Lee et al. [9]}$$

argue that, for processes where the spatial scale for movement is large in comparison with its temporal scale, nonlocal models using integro-differential equations may allow for a better estimation of the parameters from the data and provide more insight into the biological system. In (1.1) if the diffusion kernel \( J(x) = \delta(x) + \delta''(x) \), where \( \delta \) is the Dirac delta (see Medlock and Kot [12]), then (1.1) reduces to the traditional reaction-diffusion model

$$\frac{\partial u(x, t)}{\partial t} = D \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)). \quad (1.2)$$

For the nonlocal convolution diffusion model (1.1) without delays, one can refer to [1, 3–5] and the references cited therein. Delays are incorporated into (1.1) by the authors in [17] and [16] with the quasimonotone condition and the exponential quasimonotone condition, respectively. Unfortunately, it is quite common for the reaction terms in some systems arising from a practical problem to not satisfy the above conditions; for example, this is true for the following nonlocal convolution diffusion competitive models with delays:

$$\begin{align*}
\frac{\partial}{\partial t} u_1(x, t) &= d_1[(J_1 \ast u_1)(x, t) - u_1(x, t)] \\
&\quad + r_1 u_1(x, t) \left[ 1 - a_1 u_1(x, t) - b_1 u_2(x, t - \tau_1) \right], \\
\frac{\partial}{\partial t} u_2(x, t) &= d_2[(J_2 \ast u_2)(x, t) - u_2(x, t)] \\
&\quad + r_2 u_2(x, t) \left[ 1 - b_2 u_1(x, t - \tau_2) - a_2 u_2(x, t) \right]
\end{align*} \quad (1.3)$$

and

$$\begin{align*}
\frac{\partial}{\partial t} u_1(x, t) &= d_1[(J_1 \ast u_1)(x, t) - u_1(x, t)] \\
&\quad + r_1 u_1(x, t) \left[ 1 - a_1 u_1(x, t - \tau_1) - b_1 u_2(x, t - \tau_2) \right], \\
\frac{\partial}{\partial t} u_2(x, t) &= d_2[(J_2 \ast u_2)(x, t) - u_2(x, t)] \\
&\quad + r_2 u_2(x, t) \left[ 1 - b_2 u_1(x, t - \tau_3) - a_2 u_2(x, t - \tau_4) \right].
\end{align*} \quad (1.4)$$

Here \( u_i(t, x), i = 1, 2, \) are the densities of the populations of two species at location \( x \) and time \( t \). With a diffusion rate \( d_i, i = 1, 2, \) individuals move from their current
location $x$ and instantaneously arrive at some new location $y$. This process is represented by integro-differential equations (1.3) and (1.4): the second terms on the right-hand sides, $-d_i u_i$, $i = 1, 2$, represent individuals leaving location $x$, while the first terms, the convolutions $d_i J_i \ast u_i$, $i = 1, 2$, represent the total number of individuals arriving at $x$ from all possible locations $y$. Here $r_i$, $i = 1, 2$, are net birth rates, $1/b_i$, $i = 1, 2$, are carrying capacities, $a_i$, $i = 1, 2$, are competition coefficients and the delays $\tau_i$, $i = 1, \ldots, 4$, are nonnegative constants. In particular, letting $J_1(x) = J_2(x) = \delta(x) + \delta''(x)$, (1.3) and (1.4) reduce to the corresponding Laplacian diffusion competition systems which are investigated in [10, 18, 24]. In addition, if $\tau_i = 0$, $i = 1, \ldots, 4$, (1.3) and (1.4) reduce to competition systems used to describe the local interaction between the externally introduced gray squirrel and the indigenous red squirrel in Britain [8, 14] (see also [20, 21]).

In order to focus on the mathematical ideas, we will discuss the following general convolution diffusion systems with delays:

\[
\begin{align*}
\frac{\partial}{\partial t} u_1(x, t) &= d_1 [(J_1 \ast u_1)(x, t) - u_1(x, t)] + f_1(u_{1,t}(x), u_{2,t}(x)), \\
\frac{\partial}{\partial t} u_2(x, t) &= d_2 [(J_2 \ast u_2)(x, t) - u_2(x, t)] + f_2(u_{1,t}(x), u_{2,t}(x)),
\end{align*}
\tag{1.5}
\]

where $d_i > 0$, $u_{i,t}(x)(\theta) = u_i(x, t + \theta)$, $-\tau \leq \theta \leq 0$, $\tau$ denotes the maximal time delay [24]. $f_i : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $J_i : \mathbb{R} \to \mathbb{R}$ are nonnegative functions with $\int_{\mathbb{R}} J_i(y) dy = 1$, for $i = 1, 2$. Applying the method developed in [10], we prove the existence of travelling wave solutions of (1.5) with nonlocal dispersal. If we take $J_1(x) = J_2(x) = \delta(x) + \delta''(x)$ in (1.5), then (1.5) reduces to the Laplacian reaction diffusion equations with weak quasimonotonicity or weak exponential quasimonotonicity, which were investigated in [10]. Thus, our results contain those of [10].

This paper is organized as follows. Section 2 is devoted to preliminary and abstract results. More precisely, we investigate the existence of travelling wave solutions of a class of nonlocal convolution diffusion systems with weak quasimonotonicity (WQM) or weak exponential quasimonotonicity (WQM*) by applying the method developed in [10]. In Section 3, we apply our main results to the convolution diffusion-competition systems (1.3) and (1.4) and prove the existence of travelling wave solutions by constructing a pair of suitable upper–lower solutions. Because of the introduction of nonlocal diffusion, it is more difficult to construct and verify upper–lower solutions. Thus we choose a pair of suitable upper–lower solutions which make a slight difference to our calculations from those in [10] (see Remark 1).

## 2. Preliminary and abstract results

We first introduce the usual notation for the standard ordering in $\mathbb{R}^2$. That is, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we denote $u \leq v$ if $u_i \leq v_i$, $i = 1, 2$, and $u < v$ if $u \leq v$ but $u \neq v$. In particular, we denote $u \ll v$ if $u \leq v$ but $u_i \neq v_i$, $i = 1, 2$. 


If \( u \leq v \), we also denote \((u, v] = \{w \in \mathbb{R}^2, u < w \leq v\}\), \([u, v) = \{w \in \mathbb{R}^2, u \leq w < v\}\) and \([u, v) = \{w \in \mathbb{R}^2, u \leq w \leq v\}\). Let \(| \cdot |\) denote the Euclidean norm in \(\mathbb{R}^2\) and \(\| \cdot \|\) denote the supremum norm in \(C([−\tau, 0], \mathbb{R}^2)\). A travelling wave solution of (1.5) is a special translation invariant solution of the form \(u_1(x, t) = \phi(x + ct)\), \(u_2(x, t) = \psi(x + ct)\), where \((\phi, \psi) \in C^1(\mathbb{R}, \mathbb{R}^2)\) are the profiles of the wave that propagates through the one-dimensional spatial domain at a constant velocity \(c > 0\). Substituting \(u_1(x, t) = \phi(x + ct)\), \(u_2(x, t) = \psi(x + ct)\) into (1.5), then \((\phi, \psi)\) satisfies

\[
\begin{align*}
\phi'(t) &= d_1 (J_1 * \phi)(t) - d_1 \phi(t) + f_1^c(\phi_1, \psi_1), \\
\psi'(t) &= d_2 (J_2 * \psi)(t) - d_2 \psi(t) + f_2^c(\phi_1, \psi_1),
\end{align*}
\]

(2.1)

where \(f_i^c(\phi, \psi): C([−\tau, 0], \mathbb{R}^2) \to \mathbb{R}\) is defined by \(f_i(\phi^c, \psi^c)\), \(i = 1, 2\), and \(\phi^c(s) = \phi(cs)\) and \(\psi^c(s) = \psi(cs)\), for \(s \in [−\tau, 0]\).

Motivated by the background of travelling wave solutions, we also require that \((\phi, \psi)\) satisfies asymptotic boundary conditions

\[
\lim_{t \to -\infty} (\phi(t), \psi(t)) = (0, 0), \quad \lim_{t \to \infty} (\phi(t), \psi(t)) = (k_1, k_2),
\]

(2.2)

where \((0, 0)\) and \((k_1, k_2)\) are two equilibria of (2.1).

In this paper we are interested in travelling wave solutions of (2.1) and (2.2) when the reaction terms satisfy the following WQM or WQM* conditions.

**Definition 2.1.** Suppose that for any \(\phi_1(s), \phi_2(s), \psi_1(s), \psi_2(s) \in C([−\tau, 0], \mathbb{R})\) such that

\[
0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, \quad 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_2 \quad \text{for } s \in [−\tau, 0],
\]

(2.3)

there exist two positive numbers \(\beta_1\) and \(\beta_2\) such that

\[
\begin{align*}
f_1(\phi_1(s), \psi_1(s)) - f_1(\phi_2(s), \psi_1(s)) + (\beta_1 - d_1)(\phi_1(0) - \phi_2(0)) & \geq 0, \\
f_1(\phi_1(s), \psi_1(s)) - f_1(\phi_1(s), \psi_2(s)) & \leq 0, \\
f_2(\phi_1(s), \psi_1(s)) - f_2(\phi_1(s), \psi_2(s)) + (\beta_2 - d_2)(\psi_1(0) - \psi_2(0)) & \geq 0, \\
f_2(\phi_1(s), \psi_1(s)) - f_2(\phi_2(s), \psi_1(s)) & \leq 0.
\end{align*}
\]

(2.4)

Then the reaction terms have the property of WQM.

Suppose that for any \(\phi_1(s), \phi_2(s), \psi_1(s), \psi_2(s) \in C([−\tau, 0], \mathbb{R})\) satisfying (2.3) there exist two positive numbers \(\beta_1\) and \(\beta_2\) such that (2.4) holds and \(e^{\beta_1 s} [\phi_1(s) - \phi_2(s)]\) and \(e^{\beta_2 s} [\psi_1(s) - \psi_2(s)]\) are nondecreasing in \(s \in [−\tau, 0]\). In this case, the reaction terms have the property of WQM*.

For \(0 = (0, 0)\) and \(M = (M_1, M_2)\), let

\[C_{[0, M]}(\mathbb{R}, \mathbb{R}^2) = \{ (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) \mid 0 \leq \phi(s) \leq M_1, 0 \leq \psi(s) \leq M_2, s \in \mathbb{R} \}. \]

Define the operator \(H = (H_1, H_2): C_{[0, M]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)\) by

\[
\begin{align*}
H_1(\phi, \psi)(t) &= f_1^c(\phi_1, \psi_1) + d_1 (J_1 * \phi)(t) + (\beta_1 - d_1) \phi(t), \\
H_2(\phi, \psi)(t) &= f_2^c(\phi_1, \psi_1) + d_2 (J_2 * \psi)(t) + (\beta_2 - d_2) \psi(t),
\end{align*}
\]

(2.5)
and the operator $\mathbf{F} = (F_1, F_2) : C_{[0,M]}([\mathbb{R}, \mathbb{R}^2]) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$
\begin{align*}
F_1(\phi, \psi)(t) &= \frac{1}{c} e^{-\beta_1 t/c} \int_{-\infty}^{t} e^{\beta_1 s/c} H_1(\phi(s), \psi(s)) \, ds, \\
F_2(\phi, \psi)(t) &= \frac{1}{c} e^{-\beta_2 t/c} \int_{-\infty}^{t} e^{\beta_2 s/c} H_2(\phi(s), \psi(s)) \, ds.
\end{align*}
$$

(2.6)

Then the problem is changed into investigating whether the fixed point of $\mathbf{F}$, which is a travelling wave solution of (1.5) connecting $0 = (0, 0)$ and $K = (k_1, k_2)$, satisfies (2.2).

In the following we introduce the exponential decay norm. For $0 < \mu < \min \{\beta_1/c, \beta_2/c\}$, define

$$
B_\mu(\mathbb{R}, \mathbb{R}^2) = \{ \Phi | \Phi(t) \in C(\mathbb{R}, \mathbb{R}^2) \text{ and } \sup_{t \in \mathbb{R}} |\Phi(t)| e^{-\mu |t|} < \infty \}.
$$

It is easy to check that $B_\mu(\mathbb{R}, \mathbb{R}^2)$ is a Banach space equipped with the norm $|\cdot|_\mu$ defined by $|\Phi|_\mu = \sup_{t \in \mathbb{R}} |\Phi(t)| e^{-\mu |t|}$ for $\Phi \in B_\mu(\mathbb{R}, \mathbb{R}^2)$.

We make the following assumptions concerning (1.5) throughout this paper:

(A1) $f_i(0, 0) = f_i(k_1, k_2) = 0$ for $i = 1, 2$;

(A2) there exist two positive constants $L_1$ and $L_2$ such that

$$
|f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_2)| \leq L_1 \|\Phi - \Psi\|, \quad i = 1, 2,
$$

for $\Phi = (\phi_1, \psi_1), \Psi = (\phi_2, \psi_2) \in C([-\tau, 0], \mathbb{R}^2)$ where $0 \leq \phi_i(s), \psi_i(s) \leq M_i$, $s \in [-\tau, 0]$ and $M_i > k_i$ is positive constant, $i = 1, 2$;

(A3) $J_i : [\mathbb{R}] \to [\mathbb{R}]$ is a nonnegative function with $\int_{\mathbb{R}} J_i(y) \, dy = 1$, $i = 1, 2$;

(A4) $\int_{\mathbb{R}} J_i(y)e^{\lambda y} \, dy < \infty$, $i = 1, 2$ for any $\lambda \in \mathbb{R}$.

2.1. The WQM case In this subsection, we consider the existence of travelling wave solutions of (2.1) when the delayed reaction terms $f_1$ and $f_2$ are WQM.

**Definition 2.2.** The two functions $\Phi = (\phi, \psi)$ and $\Phi = (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2)$ are called the upper and lower solutions of (2.1), respectively, if there exists $\mathbb{T} = \{T_i | i = 1, \ldots, m\}$ such that $\Phi$ and $\Phi$ are once continuously differentiable in $\mathbb{R} \setminus \mathbb{T}$ and satisfy

$$
\begin{align*}
c \phi' \geq d_1(J_1 * \phi)(t) - d_1 \phi(t) + f_1^c(\phi, \psi) \quad &\text{in } \mathbb{R} \setminus \mathbb{T} \\
c \psi' \geq d_2(J_2 * \psi)(t) - d_2 \psi(t) + f_2^c(\phi, \psi) \quad &\text{in } \mathbb{R} \setminus \mathbb{T}
\end{align*}
$$

(2.7)

and

$$
\begin{align*}
c \phi' \leq d_1(J_1 * \phi)(t) - d_1 \phi(t) + f_1^c(\phi, \psi) \quad &\text{in } \mathbb{R} \setminus \mathbb{T} \\
c \psi' \leq d_2(J_2 * \psi)(t) - d_2 \psi(t) + f_2^c(\phi, \psi) \quad &\text{in } \mathbb{R} \setminus \mathbb{T}.
\end{align*}
$$

(2.8)

We assume that (2.1) has an upper solution $\Phi$ and a lower solution $\Phi$ satisfying the following hypotheses:

(P1) $(0, 0) \leq (\phi(t), \psi(t)) \leq (\Phi(t), \Psi(t)) \leq (M_1, M_2)$;
Assume that

$$\lim_{t \to -\infty} (\bar{\phi}(t), \bar{\psi}(t)) = (0, 0),$$
$$\lim_{t \to +\infty} (\underline{\phi}(t), \underline{\psi}(t)) = \lim_{t \to +\infty} (\bar{\phi}(t), \bar{\psi}(t)) = (k_1, k_2).$$

Now we formulate our main result as follows.

**Theorem 2.1.** Assume that (A1)–(A4) and WQM hold. If (2.1) has an upper solution $(\bar{\phi}(t), \bar{\psi}(t)) \in C_{[0, M]}(\mathbb{R}, \mathbb{R}^2)$ and a lower solution $(\underline{\phi}(t), \underline{\psi}(t)) \in C_{[0, M]}(\mathbb{R}, \mathbb{R}^2)$ such that (P1) and (P2) hold, then (2.1) has a travelling wave solution satisfying (2.2).

**Proof.** Assume that $\overline{\phi}$ and $\Phi \in C(\mathbb{R}, \mathbb{R}^2)$ are a pair of upper–lower solutions of (2.1) satisfying (P1) and (P2). Define the wave profile set $\Gamma$ by

$$\Gamma = \{ (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) \mid \underline{\phi}(t) \leq \phi(t) \leq \bar{\phi}(t), \underline{\psi}(t) \leq \psi(t) \leq \bar{\psi}(t), t \in \mathbb{R} \}.$$

Obviously, $\Gamma((\phi, \psi), [\bar{\phi}, \bar{\psi}])$ is a nonempty, closed and bounded convex set. Similarly to the proof in [10, 16, 17], it is easily seen that $F = (F_1, F_2) : C_{[0, M]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ is continuous with respect to the norm $| \cdot |_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R}^2)$ and

$$\underline{\phi} \leq F_1(\bar{\phi}, \bar{\psi}) \leq F_1(\overline{\phi}, \overline{\psi}) \leq \bar{\phi}, \quad \underline{\psi} \leq F_2(\overline{\phi}, \overline{\psi}) \leq F_2(\bar{\phi}, \bar{\psi}) \leq \bar{\psi}.$$

Moreover, $F(\Gamma((\phi, \psi), [\bar{\phi}, \bar{\psi}])) \subset \Gamma((\phi, \psi), [\bar{\phi}, \bar{\psi}])$ and the map $F$ is compact with respect to the decay norm $| \cdot |_{\mu}$. By Schauder’s fixed point theorem, there exists a fixed point $(\phi^*, \psi^*) \in \Gamma((\phi, \psi), [\bar{\phi}, \bar{\psi}])$, so it is a solution of (2.1).

Next, we need to verify the asymptotic boundary condition (2.2). By (P2) and

$$0 \leq (\underline{\phi}(t), \underline{\psi}(t)) \leq (\phi^*(t), \psi^*(t)) \leq (\bar{\phi}(t), \bar{\psi}(t)) \leq (M_1, M_2),$$

we see that $\lim_{t \to -\infty} (\phi^*(t), \psi^*(t)) = (0, 0)$ and $\lim_{t \to +\infty} (\phi^*(t), \psi^*(t)) = (k_1, k_2)$. This completes the proof. \hfill \Box

**2.2. The WQM* case** We assume that (2.1) has an upper solution $\overline{\phi} = (\bar{\phi}, \bar{\psi})$ and a lower solution $\Phi = (\underline{\phi}, \underline{\psi})$ not only satisfying the hypotheses (P1) and (P2) but also:

(P3) $e^{\beta_1 t} [\bar{\phi}(t) - \phi(t)]$ and $e^{\beta_2 t} [\bar{\psi}(t) - \psi(t)]$ are nondecreasing for $t \in \mathbb{R}$.

Define the set $\Gamma^*$ consisting of wave profiles by

$$\Gamma^*(\overline{\phi}, \overline{\psi}) = \left\{ (\phi, \psi) \in C_{[0, M]}(\mathbb{R}, \mathbb{R}^2) \mid \begin{array}{l}
(i) \quad \phi(t) \leq \phi(t) \leq \bar{\phi}(t), \quad \psi(t) \leq \psi(t) \leq \bar{\psi}(t), \quad t \in \mathbb{R}, \\
(ii) \quad e^{\beta_1 t} [\phi(t) - \Phi(t)], \quad e^{\beta_1 t} [\bar{\phi}(t) - \phi(t)], \\
e^{\beta_2 t} [\psi(t) - \psi(t)], \quad e^{\beta_2 t} [\bar{\psi}(t) - \psi(t)] \quad \text{are nondecreasing for} \; t \in \mathbb{R} \end{array} \right\}.$$
Obviously, $\Gamma^*([\underline{\phi}, \underline{\psi}], [\underline{\phi}, \underline{\psi}])$ is nonempty. In fact, by (P3), $(\phi, \psi), (\phi_t, \psi_t)$ satisfy (i) and (ii) of $\Gamma^*([\phi, \psi], [\phi_t, \psi_t])$. Moreover, $\Gamma^*([\phi, \psi], [\phi_t, \psi_t])$ is a closed and bounded convex set. Thus, we have the following result.

**Theorem 2.2.** Assume that (A1)–(A4) and WQM* hold. Suppose that (2.1) has an upper solution $(\bar{\phi}(t), \bar{\psi}(t)) \in C_{[0,M]}([\mathbb{R}, \mathbb{R}^2)$ and a lower solution $(\underline{\phi}(t), \underline{\psi}(t)) \in C_{[0,M]}([\mathbb{R}, \mathbb{R}^2)$ such that (P1), (P2) and (P3) hold. Then, for any $c \geq 1$, (2.1) has a travelling wave solution in $\Gamma^*([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$ satisfying (2.2).

**Proof.** Letting $(\phi, \psi) \in \Gamma^*([\underline{\phi}, \underline{\psi}], [\underline{\phi}, \underline{\psi}])$, it is easy to see that

$$
\phi \leq F_1(\phi, \psi) \leq F_1(\phi, \psi) \leq F_1(\bar{\phi}, \underline{\psi}) \quad \text{and} \quad \underline{\psi} \leq F_2(\bar{\phi}, \underline{\psi}) \leq F_2(\phi, \psi) \leq F_2(\bar{\phi}, \underline{\psi}) \leq \bar{\psi}.
$$

This implies that $F(\phi, \psi) = (F_1(\phi, \psi), F_2(\phi, \psi))$ satisfies statement (i) of $\Gamma^*([\phi, \psi], [\bar{\phi}, \bar{\psi}])$.

Now we need to prove that $(F_1(\phi, \psi), F_2(\phi, \psi))$ satisfies statement (ii) of $\Gamma^*([\phi, \psi], [\bar{\phi}, \bar{\psi}])$. Letting $F_1(\phi, \psi) = \phi_1$ for $(\phi, \psi) \in \Gamma^*([\phi, \psi], [\bar{\phi}, \bar{\psi}])$, we then have

$$
e^\beta_1 t [\bar{\phi}(t) - \phi_1(t)] = \frac{1}{c} e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^t e^{\beta_1 s/c} \left\{ [c\phi_1'(s) + \beta_1 \phi(s)] - [c\phi_1'(s) + \beta_1 \phi_1(s)] \right\} ds
$$

$$
= \frac{1}{c} e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^t e^{\beta_1 s/c} \left\{ [c\phi_1'(s) + \beta_1 \phi(s) - H_1(\phi, \psi)(s)] - [c\phi_1'(t) + \beta_1 \phi_1(t) - H_1(\phi, \psi)(s)] \right\} ds
$$

$$
= \frac{1}{c} e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^t e^{\beta_1 s/c} \left\{ [c\phi_1'(s) + \beta_1 \phi(s) - H_1(\phi, \psi)(s)] \right\} ds.
$$

Hence, for $c \geq 1$, we have, for $t \in \mathbb{R}$,

$$
\frac{d}{dt} \left\{ e^{\beta_1 t} [\bar{\phi}(t) - \phi_1(t)] \right\}
$$

$$
= \frac{1}{c} \left( \beta_1 - \frac{\beta_1}{c} \right) e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^t e^{\beta_1 s/c} \left\{ [c\phi_1'(s) + \beta_1 \phi(s) - H_1(\phi, \psi)(s)] \right\} ds
$$

$$
+ \frac{1}{c} e^{\beta_1 t} [c\phi_1'(t) + \beta_1 \phi_1(t) - H_1(\phi, \psi)(t)]
$$

$$
\geq \frac{1}{c} \left( \beta_1 - \frac{\beta_1}{c} \right) e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^t e^{\beta_1 s/c} \left\{ [c\phi_1'(s) + \beta_1 \phi(s) - H_1(\phi, \psi)(s)] \right\} ds
$$

$$
+ \frac{1}{c} e^{\beta_1 t} [c\phi_1'(t) + \beta_1 \phi_1(t) - H_1(\phi, \psi)(t)] \geq 0.
$$

Thus, $e^{\beta_1 t} [\bar{\phi}(t) - F_1(\phi, \psi)(t)]$ is nondecreasing for $t \in \mathbb{R}$. 


Similarly, we can prove that $e^{i2t}[\psi(t) - F_2(\phi, \psi)(t)]$, $e^{i2t}[F_2(\phi, \psi)(t) - \psi(t)]$ and $e^{i2t}[F_1(\phi, \psi)(t) - \phi(t)]$ are nondecreasing in $t \in \mathbb{R}$. Thus

$$F\left(\Gamma^*([\phi, \psi], [\overline{\phi}, \overline{\psi}])\right) \subset \Gamma^*([\phi, \psi], [\overline{\phi}, \overline{\psi}])$$

and, similar to the proof in [10, 16, 17], $F$ is compact with respect to the decay norm $|\cdot|_\mu$. Therefore, there exists $(\phi^*(t), \psi^*(t))$ satisfying the asymptotic boundary condition (2.2). This completes the proof. \hfill \Box

3. Convolution diffusion-competition models

In this section, we apply Theorems 2.1 and 2.2 to establish the existence of travelling wave solutions for Systems (1.3) and (1.4).

3.1. Model (1.3) We consider the existence of travelling wave solutions for the nonlocal diffusion-competition system (1.3), where $d_i > 0$, $a_i > 0$, $b_i > 0$, $\tau_i \geq 0$, $i = 1, 2$ are constants. Suppose that $J_1$ and $J_2$ are even functions which satisfy (A3) and (A4).

Let $c > 0$. A travelling wave solution $(\phi(x + ct), \psi(x + ct))$ of (1.3) must satisfy

$$\begin{cases}
c\phi'(t) = d_1(J_1 * \phi)(t) - d_1\phi(t) + r_1\phi(t)(1 - a_1\phi(t) - b_1\psi(t - ct_1)), \\
c\psi'(t) = d_2(J_1 * \psi)(t) - d_2\psi(t) + r_2\psi(t)(1 - b_2\phi(t) - c\tau_2) - a_2\psi(t)).
\end{cases} \tag{3.1}$$

As with the theory established in Section 2, we are interested in the solution of (3.1) with asymptotic boundary conditions

$$\lim_{t \to -\infty} (\phi(t), \psi(t)) = (0, 0), \quad \lim_{t \to +\infty} (\phi(t), \psi(t)) = (k_1, k_2),$$

where

$$k_1 = \frac{a_2 - b_1}{a_1a_2 - b_1b_2} > 0, \quad k_2 = \frac{a_1 - b_2}{a_1a_2 - b_1b_2} > 0,$$

provided that

$$a_1 > b_2, \quad a_2 > b_1. \tag{3.2}$$

For $\phi, \psi \in C([-\tau, 0], \mathbb{R})$ where $\tau = \max\{\tau_1, \tau_2\}$, denote

$$f_1(\phi, \psi) = r_1\phi(0)[1 - a_1\phi(0) - b_1\psi(-\tau_1)],$$

$$f_2(\phi, \psi) = r_2\psi(0)[1 - b_2\phi(-\tau_2) - a_2\psi(0)].$$

Obviously, (A1) and (A2) hold. We now verify that $F = (f_1, f_2)$ is WQM.

Lemma 3.1. The function $F = (f_1, f_2)$ is WQM.

Proof. Take $(\phi_1(s), \phi_2(s)), (\psi_1(s), \psi_2(s)) \in C([-\tau, 0], \mathbb{R}^2)$ where

$$0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, \quad 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_2 \quad \text{for } s \in [-\tau, 0].$$

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 26 May 2019 at 09:42:30, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446181109000406
According to $M_1 > k_1$ and $M_2 > k_2$, we have $\beta_1 := r_1(2a_1M_1 + b_1M_2 - 1) + d_1 > 0$ and 
\[2a_1M_1 + b_1M_2 - 1 > 2a_1k_1 + b_1k_2 - 1 = a_1k_1 > 0.
\]
Then, defining $\Upsilon = [\phi_1(0) - \phi_2(0)]$, we have
\[
f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) \\
= r_1 \phi_1(0)[1 - a_1 \phi_1(0) - b_1 \psi_1(-\tau_1)] - r_1 \phi_2(0)[1 - a_1 \phi_2(0) - b_1 \psi_1(-\tau_1)] \\
= r_1 \Upsilon - r_1 a_1(\phi_1^2(0) - \phi_2^2(0)) - r_1 b_1 \psi_1(-\tau_1) \Upsilon \\
\geq r_1 \Upsilon(1 - 2a_1M_1 - b_1M_2) \\
= -r_1(2a_1M_1 + b_1M_2 - 1) \Upsilon - d_1 \Upsilon + d_1 \Upsilon = -\beta_1 \Upsilon + d_1 \Upsilon.
\]
So, $f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) + (\beta_1 - d_1)[\phi_1(0) - \phi_2(0)] \geq 0$ and
\[
f_1(\phi_1, \psi_1) - f_1(\phi_1, \psi_2) \\
= r_1 \phi_1(0)[1 - a_1 \phi_1(0) - b_1 \psi_1(-\tau_1)] - r_1 \phi_1(0)[1 - a_1 \phi_1(0) - b_1 \psi_2(-\tau_1)] \\
= -r_1 b_1 \phi_1(0)[\psi_1(-\tau_1) - \psi_2(-\tau_1)] \leq 0.
\]
In a similar way, we can prove that $f_2$ is WQM. This completes the proof. □

Define
\[
\Delta_1(\lambda, c) = d_1J_1 * e^{\lambda \cdot} - d_1 - c\lambda + r_1, \\
\Delta_2(\lambda, c) = d_2J_2 * e^{\lambda \cdot} - d_2 - c\lambda + r_2.
\]

We can easily obtain the following conclusions.

**Lemma 3.2.**

(i) There exists a $c^*_1 > 0$ such that (3.3) has two distinct positive roots, $\lambda_1(c)$ and $\lambda_2(c)$, with $\lambda_1(c) < \lambda_2(c)$ for any $c > c^*_1$.
(ii) There exists a $c^*_2 > 0$ such that (3.4) has two distinct positive roots, $\lambda_3(c)$ and $\lambda_4(c)$, with $\lambda_3(c) < \lambda_4(c)$ for any $c > c^*_2$.
(iii) There exists a $c' = \min\{c^*_1, c^*_2\}$ such that (3.3) and (3.4) have no real root if $c < c'$.

For fixed
\[
\eta \in \left(1, \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}, \frac{\lambda_4(c)}{\lambda_3(c)}, \frac{\lambda_1(c) + \lambda_3(c)}{\lambda_1(c)}, \frac{\lambda_2(c) + \lambda_4(c)}{\lambda_2(c)}\right\}\right),
\]
where $c > c^* = \max\{c^*_1, c^*_2\}$ and we have a large constant $q > 0$, we consider the functions $l_1(t) = e^{\lambda_1(c)t} - q e^{\eta \lambda_1(c)t}$ and $l_3(t) = e^{\lambda_3(c)t} - q e^{\eta \lambda_3(c)t}$. It is easy to see that $l_1(t)$ and $l_3(t)$ have global maxima $m_1 > 0$ and $m_3 > 0$, respectively. Then there exist $t_1$ and $t_3$ such that $l_1(t_1) = m_1$ and $l_3(t_3) = m_3$, where $t_i = [1/(\eta \lambda_i - \lambda_i)] \ln(1/(q \eta)) < 0$, $i = 1, 3$, and $t_i$ ($i = 1, 3$) is large and negative enough for large enough $q$. Furthermore, $l_1'(t) \geq 0$ for $t \leq t_1$ and $l_1'(t) \leq 0$ for $t \geq t_1$ ($i = 1, 3$).
Then, for any given \( \lambda > 0 \), there exist \( \varepsilon_2 > 0 \) and \( \varepsilon_4 > 0 \) such that
\[
k_1 - \varepsilon_2 e^{-\lambda t_1} = l_1(t_1) = m_1 \quad \text{and} \quad k_2 - \varepsilon_4 e^{-\lambda t_3} = l_3(t_3) = m_3.
\]
According to (3.2), there exist \( \varepsilon_0, \varepsilon_1 \) and \( \varepsilon_3 \) such that
\[
\begin{cases}
  a_1 \varepsilon_1 - b_1 \varepsilon_4 > \varepsilon_0, & a_2 \varepsilon_3 - b_2 \varepsilon_2 > \varepsilon_0, \\
  a_1 \varepsilon_2 - b_1 \varepsilon_3 > \varepsilon_0, & a_2 \varepsilon_4 - b_2 \varepsilon_1 > \varepsilon_0.
\end{cases}
\]

For the above constants and for suitable constants \( t_2 \) and \( t_4 \), we can define continuous functions as follows:
\[
\overline{\phi}(t) = \begin{cases}
  e^{\lambda_1(c) t}, & t \leq t_2, \\
  k_1 + \varepsilon_1 e^{-\lambda t}, & t \geq t_2,
\end{cases} \quad \overline{\psi}(t) = \begin{cases}
  e^{\lambda_3(c) t}, & t \leq t_4, \\
  k_2 + \varepsilon_3 e^{-\lambda t}, & t \geq t_4,
\end{cases}
\]
and
\[
\underline{\phi}(t) = \begin{cases}
  e^{\lambda_1(c) t} - q e^{\eta \lambda_1(c) t}, & t \leq t_1, \\
  k_1 - \varepsilon_2 e^{-\lambda t}, & t \geq t_1,
\end{cases} \quad \underline{\psi}(t) = \begin{cases}
  e^{\lambda_3(c) t} - q e^{\eta \lambda_1(c) t}, & t \leq t_3, \\
  k_2 - \varepsilon_4 e^{-\lambda t}, & t \geq t_3,
\end{cases}
\]
where \( q > 0 \) is large enough and \( \lambda > 0 \) is small enough. It is easy to see that \( M_1 = \sup_{t \in \mathbb{R}} \overline{\phi}(t) > k_1, \ M_2 = \sup_{t \in \mathbb{R}} \overline{\psi}(t) > k_2, \ \overline{\phi}(t), \ \overline{\psi}(t), \ \underline{\phi}(t) \) and \( \underline{\psi}(t) \) satisfy (P1) and (P2), since \( t_i = \lfloor 1/(\eta \lambda_1 - \lambda_i) \rfloor \ln(1/\eta q) < 0 \), \( i = 1, 3, \) are small enough and
\[
\min\{t_2, t_4\} - c \max\{\tau_1, \tau_2\} \geq \max\{t_1, t_3\}
\]
for sufficiently large \( q > 0 \) and sufficiently small \( \lambda > 0 \). We now prove that the continuous functions \((\overline{\phi}(t), \overline{\psi}(t))\) and \((\underline{\phi}(t), \underline{\psi}(t))\) are an upper solution and a lower solution of (3.1), respectively.

**Remark 1.** The \( t_i, i = 1, 3, \) used here are different from \( t_i = \max\{t \mid l_i(t) = m_i/2\}, \ i = 1, 3, \) in [10]. In the current paper \( l_i(t) \ i = 1, 3 \) are nondecreasing for \( t \leq t_i, \ i = 1, 3, \) and \( l_i(t) \ i = 1, 3, \) are nonincreasing for \( t \geq t_i, \ i = 1, 3, \) where \( t_i, i = 1, 3, \) are negative and \( -t_i, i = 1, 3, \) are large enough. In the current paper the lower solutions are increasing functions but they are not monotone in [10].

**Lemma 3.3.** Assume that (A3), (A4) and (3.2) hold and that \( J_1 \) and \( J_2 \) are even functions. Then \((\overline{\phi}(t), \overline{\psi}(t))\) is an upper solution of (3.1).

**Proof.** For \((\overline{\phi}(t), \overline{\psi}(t)) \in C(\mathbb{R}, \mathbb{R}^2), \) if \( t \leq t_2, \) then \( \overline{\psi}(t - c \tau_1) \geq 0, \overline{\phi}(t) = e^{\lambda_1(c) t} \) and
\[
d_1(J_1 * \overline{\phi})(t) - d_1 \overline{\phi}(t) - \overline{c \phi}'(t) + r_1 \overline{\phi}(t)[1 - a_1 \overline{\phi}(t) - b_1 \overline{\psi}(t - c \tau_1)]
\]
\[
= d_1 \left( \int_{-\infty}^{t_2} J_1(t - s) \overline{\phi}(s) \, ds + \int_{t_2}^{\infty} J_1(t - s) \overline{\phi}(s) \, ds \right) - d_1 \overline{\phi}(t) - c \overline{\phi}'(t)
\]
\[
+ r_1 \overline{\phi}(t)[1 - a_1 \overline{\phi}(t) - b_1 \overline{\psi}(t - c \tau_1)]
\]
\[
= d_1(J_1 * (e^{\lambda_1(c) t}))(t) + d_1 \int_{t_2}^{\infty} J_1(t - s)[k_1 + \varepsilon_1 e^{-\lambda s} - e^{\lambda_1 s}] \, ds
\]
\[
- d_1 \overline{\phi}(t) - c \overline{\phi}'(t) + r_1 \overline{\phi}(t)[1 - a_1 \overline{\phi}(t) - b_1 \overline{\psi}(t - c \tau_1)].
\]

(3.7)
The expression (3.7) is less than or equal to
\[
d_1(J_1 \ast (e^{\lambda_1(t)}))(t) + d_1 \int_{t_2}^{\infty} J_1(t - s)[k_1 + \varepsilon_1 e^{-\lambda t} - e^{\lambda_1 t}] \, ds
\]
\[
- d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)[1 - a_1\bar{\phi}(t) - b_1\psi(t - c\tau_1)]
\]
\[
\leq d_1(J_1 \ast (e^{\lambda_1(t)}))(t) - d_1 e^{\lambda_1(t)} - c(e^{\lambda_1(t)})' + r_1 e^{\lambda_1(t)}
\]
\[
= [d_1(J_1 \ast (e^{\lambda_1(t)})) - d_1 - c\lambda_1(t) + r_1] e^{\lambda_1(t)} = 0.
\]
Similarly,
\[
d_2(J_2 \ast \psi)(t) - d_2\psi(t) - c\psi'(t) + r_2\psi(t)[1 - b_2\bar{\phi}(t - c\tau_2) - a_2\bar{\psi}(t)] \leq 0.
\]
If \( t \geq t_2, \) then \( t - c\tau_1 \geq t_2 - c\tau_1 \geq t_3 \) and \( \psi(t - c\tau_1) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_1)}. \) For any \( t \in \mathbb{R}, \) we have that \( 0 < \bar{\psi}(t) < k_1 + \varepsilon_1, \) and it follows that
\[
d_1(J_1 \ast \bar{\phi})(t) - d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)[1 - a_1\bar{\phi}(t) - b_1\psi(t - c\tau_1)]
\]
\[
\leq d_1(J_1 \ast (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) - c(k_1 + \varepsilon_1 e^{-\lambda t})'
\]
\[
+ r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_1)})]
\]
\[
= d_1(J_1 \ast (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t}
\]
\[
+ r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_1)})] =: I(\lambda).
\]
Obviously,
\[
I(0) = d_1 J_1 \ast (k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1)
\]
\[
+ r_1(k_1 + \varepsilon_1)(1 - a_1 k_1 - b_1 k_2 - a_1 \varepsilon_1 + b_1 \varepsilon_4)
\]
\[
= d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1) + r_1(k_1 + \varepsilon_1)(b_1 \varepsilon_4 - a_1 \varepsilon_1)
\]
\[
= r_1(k_1 + \varepsilon_1)(b_1 \varepsilon_4 - a_1 \varepsilon_1).
\]
Since \( a_1 \varepsilon_1 - b_1 \varepsilon_4 > \varepsilon_0, \) we have \( I(0) < -r_1(k_1 + \varepsilon_1)\varepsilon_0 < 0, \) and there exists a \( \lambda_1^* > 0 \) such that \( I(\lambda) < 0 \) for \( \lambda \in (0, \lambda_1^*). \) Thus,
\[
d_1(J_1 \ast \bar{\phi})(t) - d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)[1 - a_1\bar{\phi}(t) - b_1\psi(t - c\tau_1)] \leq 0.
\]
Similarly, there exists a \( \lambda_2^* > 0 \) such that, for \( \lambda \in (0, \lambda_2^*), \)
\[
d_2(J_2 \ast \psi)(t) - d_2\psi(t) - c\psi'(t) + r_2\psi(t)[1 - b_2\bar{\phi}(t - c\tau_2) - a_2\bar{\psi}(t)] \leq 0.
\]
This completes the proof. \( \square \)

**Lemma 3.4.** Assume that (A3), (A4) and (3.2) hold and, in addition, \( J_1, J_2 \) are even functions. Then \( \bar{\phi}, \psi \) is a lower solution of (3.1).
PROOF. Equation (3.5) implies that $\lambda_1(c) < \eta \lambda_1(c) < \lambda_2(c)$, $2 \lambda_1(c) > \eta \lambda_1(c)$, $\lambda_3(c) + \lambda_1(c) > \eta \lambda_1(c)$ and $\Delta_1(\eta \lambda_1(c), c) < 0$.

For $t \leq t_1$,

$$d_1(J_1 * \phi(t) - d_1\phi(t) - c\phi'(t) + r_1\phi(t) (1 - a_1\phi(t) - b_1 \bar{\psi}(t - c \tau_1))$$

$$= d_1 \left( \int_{t_1}^{\infty} J_1(t - s) \phi(s) \, ds + \int_{-\infty}^{t_1} J_1(t - s) \phi(s) \, ds \right) - d_1\phi(t) - c\phi'(t)$$

$$+ r_1\phi(t) (1 - a_1\phi(t) - b_1 \bar{\psi}(t - c \tau_1))$$

$$= d_1 \left\{ J_1[e^{\lambda_1(c)} - qe^{\eta \lambda_1(c)}] \right\}(t)$$

$$+ \int_{t_1}^{\infty} J_1(t - s)[k_1 - \varepsilon_2 e^{-\lambda s} - (e^{\lambda_1(c)s} - qe^{\eta \lambda_1(c)s})] \, ds$$

$$- d_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}] - c[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}']$$

$$+ r_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t} \right\} \left\{ 1 - a_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}] - b_1 e^{\lambda_3(c)(t-c \tau_1)} \right\}$$

$$\geq d_1 \left\{ J_1[e^{\lambda_1(c)}] \right\}(t) - d_1[e^{\lambda_1(c)t}] - c\lambda_1(c)e^{\lambda_1(c)t} + r_1 e^{\lambda_1(c)t}$$

$$- \left\{ d_1 J_1 * [qe^{\eta \lambda_1(c)}] \right\}(t) - d_1 qe^{\eta \lambda_1(c)t} - c q \lambda_1(c) e^{\eta \lambda_1(c)t} + r_1 qe^{\eta \lambda_1(c)t}$$

$$+ d_1 \int_{t_1}^{\infty} J_1(t - s)[k_1 - \varepsilon_2 e^{-\lambda t} - (e^{\lambda_1(c)t_1} - qe^{\eta \lambda_1(c)t_1})] \, ds$$

$$- r_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}] \left\{ a_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}] + b_1 e^{\lambda_3(c)(t-c \tau_1)} \right\}$$

$$= \Delta_1(\lambda_1(c), c) e^{\lambda_1(c)t} - \Delta_1(\eta \lambda_1(c), c) q e^{\eta \lambda_1(c)t}$$

$$- r_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}] \left\{ a_1[e^{\lambda_1(c)t} - qe^{\eta \lambda_1(c)t}] + b_1 e^{\lambda_3(c)(t-c \tau_1)} \right\}$$

$$\geq -\Delta_1(\eta \lambda_1(c), c) q e^{\eta \lambda_1(c)t} - r_1 e^{\lambda_1(c)t} [a_1 e^{\lambda_1(c)t} + b_1 e^{\lambda_3(c)t}]$$

$$= -q e^{\eta \lambda_1(c)t} \left\{ \frac{\Delta_1(\eta \lambda_1(c), c) + r_1}{q} [a_1 e^{(2\lambda_1(c)-\eta \lambda_1(c)t) \eta_1} + b_1 e^{(\lambda_3(c)+\lambda_1(c)-\eta \lambda_1(c)\lambda_1)}] \right\}$$

$$\geq -q e^{\eta \lambda_1(c)t} \left\{ \frac{\Delta_1(\eta \lambda_1(c), c) + r_1}{q} [a_1 e^{(2\lambda_1(c)-\eta \lambda_1(c)t) \eta_1} + b_1 e^{(\lambda_3(c)+\lambda_1(c)-\eta \lambda_1(c)\lambda_1)}] \right\}$$

$$\geq -q e^{\eta \lambda_1(c)t} \left\{ \frac{\Delta_1(\eta \lambda_1(c), c) + r_1}{q} [a_1 e^{(2\lambda_1(c)-\eta \lambda_1(c)t) \eta_1} + b_1 e^{(\lambda_3(c)+\lambda_1(c)-\eta \lambda_1(c)\lambda_1)}] \right\}$$

$$\geq 0$$ for large enough $q$,

where $\alpha = (\ln(1/\eta q))/(\lambda_1(c)(\eta - 1))$.

By a similar argument,

$$d_2(J_2 * \psi(t) - d_2\psi(t) - c\psi'(t) + r_2\psi(t) \left( 1 - b_2\bar{\psi}(t - c \tau_2) - a_2 \bar{\psi}(t) \right) \geq 0.$$
If \( t > t_1 \) and \(-t_1\) is large enough, then for any \( \varepsilon'(0 < \varepsilon' < r_1 \varepsilon_0/d_1) \) there exists a large enough \( T(q) > 0 \) such that \( t_1 < -T(q) \) and

\[
\int_{-t_1}^{t_1} J_1(s) \, ds > \int_{-\infty}^{\infty} J_1(s) \, ds - \varepsilon' = 1 - \varepsilon'.
\]

By the definition of \( \overline{\psi}(t) \), it is easy to verify that \( 0 < \overline{\psi}(t) \leq k_2 + \varepsilon_3 \) for any \( t \in \mathbb{R} \).

Then,

\[
d_1(J_1 \ast \phi)(t) - d_1 \phi(t) - c \phi'(t) + r_1 \phi(t)(1 - a_1 \phi(t) - b_1 \overline{\psi}(t - c \tau_1)) \\
\geq d_1(J_1 \ast \phi)(t) - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + c \lambda e^{-\lambda t} \\
+ r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_2 + \varepsilon_3)] \\
= d_1 \int_{-\infty}^{t_1} J_1(t-s)(e^{\lambda_1 c s} - q e^{\lambda_2 c s}) \, ds + d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2 e^{-\lambda s}) \, ds \\
- d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + c \lambda e^{-\lambda t} \\
+ r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_2 + \varepsilon_3)] \\
\geq d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2 e^{-\lambda s}) \, ds - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + c \lambda e^{-\lambda t} \\
+ r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_2 + \varepsilon_3)] =: \tilde{I}(\lambda).
\]

Thus,

\[
\tilde{I}(0) = d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2) \, ds - d_1(k_1 - \varepsilon_2)(a_1 \varepsilon_2 - b_1 \varepsilon_3) \\
\geq d_1(k_1 - \varepsilon_2)(1 - \varepsilon') - d_1(k_1 - \varepsilon_2) + r_1(k_1 - \varepsilon_2) \varepsilon_0 \\
= -d_1(k_1 - \varepsilon_2)\varepsilon' + r_1(k_1 - \varepsilon_2) \varepsilon_0 > 0,
\]

which implies that there exists a \( \lambda_3^* > 0 \) such that \( \tilde{I}(\lambda) > 0 \) for \( \lambda \in (0, \lambda_3^*) \). Thus,

\[
d_1(J_1 \ast \phi)(t) - d_1 \phi(t) - c \phi'(t) + r_1 \phi(t)(1 - a_1 \phi(t) - b_1 \overline{\psi}(t - c \tau_1)) \geq 0.
\]

Similarly, there exists a \( \lambda_4^* > 0 \) such that, for \( \lambda \in (0, \lambda_4^*) \),

\[
d_2(J_2 \ast \psi)(t) - d_2 \psi(t) - c \psi'(t) + r_2 \psi(t)(1 - b_2 \overline{\phi}(t - c \tau_2) - a_2 \overline{\psi}(t)) \geq 0.
\]

This completes the proof. \( \square \)

By Theorem 2.1 and Lemmas 3.3 and 3.4, the following result holds.

**Theorem 3.5.** Assume that (A3), (A4) and (3.2) hold and, in addition, \( J_1, J_2 \) are even functions. Then, for every \( c > c^* \), Model (3.1) has a travelling wave solution \( (\phi(x + ct), \psi(x + ct)) \), which connects \((0, 0)\) and \((k_1, k_2)\). Furthermore, \( \lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1 k_1 \xi} = \lim_{\xi \to -\infty} \psi(\xi)e^{-\lambda_4 k_2 \xi} = 1 \).

**3.2. Model (1.4)** Next we consider the existence of travelling wave solutions for the nonlocal diffusion-competition model (1.4) with delays, where all constants and
assumptions are the same as the above arguments in Model (1.3) and \( \tau_i \geq 0, i = 1, 2, 3, 4 \).

Letting \( c > 0 \), the corresponding wave profile equations of (1.4) are
\[
\begin{align*}
    c \phi'(t) &= d_1(J_1 \ast \phi)(t) - d_1 \phi(t) + r_1 \phi(t)(1 - a_1 \phi(t - c \tau_1) - b_1 \psi(t - c \tau_2)), \\
    c \psi'(t) &= d_2(J_2 \ast \psi)(t) - d_2 \psi(t) + r_2 \psi(t)(1 - b_2 \phi(t - c \tau_3) - a_2 \psi(t - c \tau_4)).
\end{align*}
\]

For \( \phi, \psi \in C([-\tau, 0], \mathbb{R}) \), where \( \tau = \max(\tau_1, \tau_2, \tau_3, \tau_4) \), denote
\[
\begin{align*}
    f_1(\phi, \psi) &= r_1 \phi(0)(1 - a_1 \phi(-\tau_1) - b_1 \psi(-\tau_2)), \\
    f_2(\phi, \psi) &= r_2 \psi(0)(1 - b_2 \phi(-\tau_3) - a_2 \psi(-\tau_4)).
\end{align*}
\]

It is obvious that (A1) and (A2) hold.

**Lemma 3.6.** For \( \tau_1 \) and \( \tau_4 \) small enough, the function \( F = (f_1, f_2) \) is WQM*.

**Proof.** Take \((\phi_1(s), \phi_2(s)), (\psi_1(s), \psi_2(s)) \in C([-\tau, 0], \mathbb{R}^2)\) where:
- (i) \( 0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, \ 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_2 \), for \( s \in [-\tau, 0] \); and
- (ii) \( e^{\beta_1 s}[\phi_1(s) - \phi_2(s)] \) and \( e^{\beta_1 s}[\psi_1(s) - \psi_2(s)] \) are nondecreasing in \( s \in [-\tau, 0] \).

If \( \tau_1 \) is small enough, then we can choose \( \beta_1 > 0 \) such that
\[
r_1(a_1 M_1 + b_1 M_2 + a_1 M_1 e^{\beta_1 \tau_1} - 1) + d_1 < \beta_1.
\]

Thus, we have
\[
\begin{align*}
    f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) &= r_1 \phi_1(0)(1 - a_1 \phi_1(-\tau_1) - b_1 \psi_1(-\tau_2)) \\
                               &= r_1 \phi_2(0)[1 - a_1 \phi_2(-\tau_1) - b_1 \psi_1(-\tau_2)] \\
                               &= r_1 \gamma_{1} - r_1 a_1 \phi_1(0)[\phi_1(-\tau_1) - \phi_2(-\tau_1)] \\
                               &
\end{align*}
\]
\[
\begin{align*}
    f_2(\phi_1, \psi_1) - f_2(\phi_2, \psi_1) &= r_2 \psi_1(0)(1 - a_2 \psi_1(-\tau_2) - b_2 \phi_1(-\tau_2)) \\
                               &= r_2 \phi_2(0)[1 - a_2 \phi_2(-\tau_2) - b_2 \psi_2(-\tau_2)] \\
                               &= r_2 \gamma_{1} - r_2 a_2 \psi_1(0)[\psi_1(-\tau_2) - \psi_2(-\tau_2)] \\
                               &
\end{align*}
\]

So, \( f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) \geq 0 \) and
\[
\begin{align*}
    f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_2) &= r_1 \phi_1(0)[1 - a_1 \phi_1(-\tau_1) - b_1 \psi_1(-\tau_2)] \\
                               &= r_1 \phi_2(0)[1 - a_1 \phi_2(-\tau_1) - b_1 \psi_2(-\tau_2)] \\
                               &= -r_1 b_1 \phi_1(0)[\psi_1(-\tau_2) - \psi_2(-\tau_2)] \leq 0.
\end{align*}
\]

Similarly, \( f_2 \) is WQM* if \( \tau_4 \) is small enough. This completes the proof. \( \Box \)

Now we define \( \bar{\phi}(t), \bar{\psi}(t), \bar{\phi}(t) \) and \( \bar{\phi}(t) \) as in Model (1.3), where \( c > c^* = \max\{c_1^*, c_2^*, 1\} \) and
\[
\min\{\tau_2, \tau_4\} - c \max\{\tau_1, \tau_2, \tau_3, \tau_4\} \geq \max\{\tau_1, \tau_3\}.
\]

It is easy to see that \( \bar{\phi}(t), \bar{\psi}(t), \bar{\phi}(t) \) and \( \bar{\phi}(t) \) satisfy (P1)–(P3). In fact, (P3) holds for \( \lambda \) small enough and \( \beta > \lambda \).
**Lemma 3.7.** Assume that (A3), (A4) and (3.2) hold and, in addition, $J_1$ and $J_2$ are even functions. If $\tau_1$ and $\tau_4$ are small enough, then $(\bar{\phi}(t), \bar{\psi}(t))$ is an upper solution and $(\underline{\phi}(t), \underline{\psi}(t))$ is a lower solution of (3.8).

**Proof.** First, we prove that $(\bar{\phi}(t), \bar{\psi}(t)) \in C(\mathbb{R}, \mathbb{R}^2)$ is an upper solution. For $t \geq t_2 + c\tau_1$ or $t \leq t_2$, the proof is similar to that of Lemma 3.3. If $t \leq t_2$, it follows that

$$d_1(J_1 \ast \bar{\phi})(t) - d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)[1 - a_1\bar{\phi}(t - c\tau_1) - b_1\bar{\psi}(t - c\tau_2)]$$

$$\leq d_1(J_1 \ast \bar{\phi})(t) - d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)$$

$$\leq d_1(J_1 \ast (e^{\lambda_1(c)}))(t) - d_1c\lambda_1(c)\psi + r_1c\lambda_1(c)t$$

$$= [d_1(J_1 \ast (e^{\lambda_1(c)})) - d_1c\lambda_1(c) + r_1]c\lambda_1(c)t = 0.$$

Similarly, we can obtain

$$d_2(J_2 \ast \bar{\psi})(t) - d_2\bar{\psi}(t) - c\bar{\psi}'(t) + r_2\bar{\psi}(t)(1 - b_2\bar{\phi}(t - c\tau_3) - a_2\bar{\psi}(t - c\tau_4)) \leq 0.$$

For $t \geq t_2 + c\tau_1$, then $t - c\tau_2 \geq t_3$ and $\bar{\psi}(t - c\tau_2) = k_2 - e_4e^{-\lambda(t-c\tau_2)}$. Since $0 < \bar{\phi}(t) \leq k_1 + \varepsilon_1$ for any $t \in \mathbb{R}$, then

$$d_1(J_1 \ast \bar{\phi})(t) - d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)(1 - a_1\bar{\phi}(t - c\tau_1) - b_1\bar{\psi}(t - c\tau_2))$$

$$\leq d_1(J_1 \ast (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1e^{-\lambda t}) - c(k_1 + \varepsilon_1e^{-\lambda t})'$$

$$+ r_1(k_1 + \varepsilon_1e^{-\lambda t})(1 - a_1(k_1 + \varepsilon_1e^{-\lambda(t-c\tau_1)}) - b_1(k_2 - e_4e^{-\lambda(t-c\tau_2)})$$

$$= d_1(J_1 \ast (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t}$$

$$+ r_1(k_1 + \varepsilon_1e^{-\lambda t})(1 - a_1(k_1 + \varepsilon_1e^{-\lambda(t-c\tau_1)}) - b_1(k_2 - e_4e^{-\lambda(t-c\tau_2)})$$

$$=: I_1(\lambda).$$

Obviously,

$$I_1(0) = d_1(J_1 \ast (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1)$$

$$+ r_1(k_1 + \varepsilon_1)[1 - a_1(k_1 - b_1k_2 - a_1\varepsilon_1 + b_1\varepsilon_4]$$

$$= d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1) + r_1(k_1 + \varepsilon_1)(b_1\varepsilon_4 - a_1\varepsilon_1)$$

$$= r_1(k_1 + \varepsilon_1)(b_1\varepsilon_4 - a_1\varepsilon_1).$$

Since $a_1\varepsilon_1 - b_1\varepsilon_4 > \varepsilon_0$, we have $I_1(0) < -r_1(k_1 + \varepsilon_1)\varepsilon_0 < 0$, and there exists a $\lambda_1 > 0$ such that $I_1(\lambda) < 0$ for $\lambda \in (0, \lambda_1)$. Thus,

$$d_1(J_1 \ast \bar{\phi})(t) - d_1\bar{\phi}(t) - c\bar{\phi}'(t) + r_1\bar{\phi}(t)(1 - a_1\bar{\phi}(t - c\tau_1) - b_1\bar{\psi}(t - c\tau_2)) \leq 0.$$

Similarly, there exists a $\lambda_2 > 0$ such that, for $\lambda \in (0, \lambda_2)$,

$$d_2(J_2 \ast \bar{\psi})(t) - d_2\bar{\psi}(t) - c\bar{\psi}'(t) + r_2\bar{\psi}(t)(1 - b_2\bar{\phi}(t - c\tau_3) - a_2\bar{\psi}(t - c\tau_4)) \leq 0.$$

For $t_2 + c\tau_1 > t > t_2$, then $\bar{\phi}(t) = k_1 + \varepsilon_1e^{-\lambda t}$, $\bar{\phi}'(t) = -\varepsilon_1\lambda e^{-\lambda t}$ and $\bar{\phi}(t - c\tau_1) = e^{\lambda_1(t-c\tau_1)}$. 


Since $0 < \tilde{\phi}(t) \leq k_1 + \varepsilon_1$ for any $t \in \mathbb{R}$, then
\begin{align*}
d_1(J_1 * \tilde{\phi})(t) - d_1\tilde{\phi}(t) - c\tilde{\phi}'(t) + r_1\tilde{\phi}(t)(1 - a_1\tilde{\phi}(t - c\tau_1) - b_1\psi(t - c\tau_2))
\leq d_1(J_1 * (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1)e^{-\lambda t} + c\varepsilon_1\lambda e^{-\lambda t} + r_1(k_1 + \varepsilon_1)e^{-\lambda t}[1 - a_1e^{\lambda_1(t - c\tau_1)} - b_1(k_2 - \varepsilon_4e^{-\lambda(t - c\tau_2)})]
\leq d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1)e^{-\lambda t} + c\varepsilon_1\lambda e^{-\lambda t} + r_1(k_1 + \varepsilon_1)e^{-\lambda t}[1 - a_1e^{\lambda_1(t - c\tau_1)} - b_1(k_2 - \varepsilon_4e^{-\lambda(t - c\tau_2)})]
= d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1)e^{-\lambda t} + c\varepsilon_1\lambda e^{-\lambda t} + r_1(k_1 + \varepsilon_1)e^{-\lambda t}[1 - a_1e^{\lambda_1(t - c\tau_1)}(k_1 + \varepsilon_1)e^{-\lambda t} - b_1(k_2 - \varepsilon_4e^{-\lambda(t - c\tau_2)})]
=: I_2(\lambda).
\end{align*}

For enough small $\tau_1$, there exists an $\varepsilon^* > 0$ such that $e^{-\lambda_1(t - c\tau_1)} > 1 - \varepsilon^*$. Thus,
\begin{align*}
I_2(0) = r_1(k_1 + \varepsilon_1)[1 - a_1e^{\lambda_1(t - c\tau_1)}(k_1 + \varepsilon_1) - b_1k_2 + b_1\varepsilon_4]
= r_1(k_1 + \varepsilon_1)[a_1k_1 - a_1e^{\lambda_1(t - c\tau_1)}(k_1 + \varepsilon_1) + b_1\varepsilon_4]
\leq r_1(k_1 + \varepsilon_1)[a_1k_1 - a_1(1 - \varepsilon^*)(k_1 + \varepsilon_1) + b_1\varepsilon_4]
= r_1(k_1 + \varepsilon_1)[b_1\varepsilon_4 - a_1\varepsilon_4 + a_1(k_1 + \varepsilon_1)\varepsilon^*]
< r_1(k_1 + \varepsilon_1)[-\varepsilon_0 + a_1(k_1 + \varepsilon_1)\varepsilon^*] < 0.
\end{align*}

Thus, there exists a $\tilde{\lambda}_3 > 0$ such that for $\lambda \in (0, \tilde{\lambda}_3)$,
\begin{align*}
d_1(J_1 * \tilde{\phi})(t) - d_1\tilde{\phi}(t) - c\tilde{\phi}'(t) + r_1\tilde{\phi}(t)(1 - a_1\tilde{\phi}(t - c\tau_1) - b_1\psi(t - c\tau_2)) \leq 0.
\end{align*}

Similarly, there exists a $\tilde{\lambda}_4 > 0$ such that, for $\lambda \in (0, \tilde{\lambda}_4)$,
\begin{align*}
d_2(J_2 * \psi)(t) - d_2\psi(t) - c\psi'(t) + r_2\psi(t)(1 - b_2\psi(t - c\tau_3) - a_2\psi(t - c\tau_4)) \leq 0.
\end{align*}

Next, we prove that $(\phi(t), \psi(t)) \in C(\mathbb{R}, \mathbb{R}^2)$ is a lower solution of (3.8).

For $t \leq t_1$ and $t \geq t_1 + c\tau_1$, the proof of a lower solution $(\phi(t), \psi(t))$ of (3.8) is similar to that of Lemma 3.4.

Now, for $t_1 < t < t_1 + c\tau_1$, it is easy to see that $\phi(t) = k_1 - \varepsilon_2e^{-\lambda t}, \phi(t - c\tau_1) = e^{\lambda_1(t - c\tau_1)} - qe^{\eta\lambda_1(t - c\tau_1)}$ and $\psi(t - c\tau_2) = e^{\lambda_3(t - c\tau_2)}$. Then
\begin{align*}
d_1(J_1 * \phi)(t) - d_1\phi(t) - c\phi'(t) + r_1\phi(t)(1 - a_1\phi(t - c\tau_1) - b_1\psi(t - c\tau_2))
= d_1(J_1 * \phi)(t) - d_1(k_1 - \varepsilon_2e^{-\lambda t}) + \lambda\varepsilon_2e^{-\lambda t}
+ r_1(k_1 - \varepsilon_2e^{-\lambda t}) \left[1 - a_1[e^{\lambda_1(t - c\tau_1)} - qe^{\eta\lambda_1(t - c\tau_1)}] - b_1e^{\lambda_3(t - c\tau_2)} \right]
\geq d_1 \int_{t_1}^\infty J_1(t - s)(k_1 - \varepsilon_2e^{-\lambda s}) ds - d_1(k_1 - \varepsilon_2e^{-\lambda t}) + \lambda\varepsilon_2e^{-\lambda t}
+ r_1(k_1 - \varepsilon_2e^{-\lambda t}) \left[1 - a_1[e^{\lambda_1t_1} - qe^{\eta\lambda_1t_1}] - b_1e^{\lambda_3t_1} \right].
\end{align*}
\[ I_3(0) = d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2 e^{-\lambda t}) \, ds - d_1(k_1 - \varepsilon_2) \]
\[ + r_1(k_1 - \varepsilon_2)[1 - a_1(k_1 - \varepsilon_2) - b_1(k_2 + \varepsilon_3)] \]
\[ \geq d_1(k_1 - \varepsilon_2)(1 - \varepsilon') - d_1(k_1 - \varepsilon_2) + r_1(k_1 - \varepsilon_2)(a_1 \varepsilon_2 - b_1 \varepsilon_3) \]
\[ \geq -d_1(k_1 - \varepsilon_2)\varepsilon' + r_1(k_1 - \varepsilon_2)\varepsilon_0 > 0. \]

Hence, there exists a $\lambda^*_S > 0$ such that, for $\lambda \in (0, \lambda^*_S)$,
\[ d_1(J_1 \star \phi)(t) - d_1 \phi(t) - c\phi'(t) + r_1 \phi(t)(1 - a_1 \phi(t) - b_1 \overline{\psi}(t - c\tau_2)) \geq 0. \]

In a similar way, there exists a $\lambda^*_S > 0$ such that for $\lambda \in (0, \lambda^*_S)$,
\[ d_2(J_2 \star \overline{\psi})(t) - d_2 \overline{\psi}(t) - c\overline{\psi}'(t) + r_2 \overline{\psi}(t)(1 - b_2 \overline{\psi}(t - c\tau_3) - a_2 \overline{\psi}(t - c\tau_4)) \geq 0. \]

This completes the proof. \hfill \Box

By Theorem 2.2 and Lemma 3.7, we have the following result.

**Theorem 3.8.** Assume that (A3), (A4) and (3.2) hold and, in addition, $J_1, J_2$ are even functions. If $\tau_1$ and $\tau_2$ are small enough, then, for every $c > c^*$, Model (3.8) has a travelling wave solution $(\phi(x + ct), \psi(x + ct))$, which connects $(0, 0)$ and $(k_1, k_2)$. Furthermore, \[ \lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1 \xi} = \lim_{\xi \to -\infty} \psi(\xi)e^{-\lambda_3 \xi} = 1. \]

**Acknowledgements**

We would like to thank an anonymous referee and Professor Wan-Tong Li for their carefully reading and helpful suggestions which led to an improvement of our original manuscript. This research was supported by the National Natural Science Foundation of China and RFDP.

**References**


