OMITTED RAYS AND WEDGES OF FRACTIONAL CAUCHY TRANSFORMS

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Abstract

For $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the set of functions which can be expressed

$$f(z) = \int_{|\zeta|=1} \frac{1}{(1-\zeta z)^\alpha} \, d\mu(\zeta) \quad \text{for } |z| < 1,$$

where $\mu$ is a complex-valued Borel measure on the unit circle. We show that if $f$ is an analytic function in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and there are two nonparallel rays in $\mathbb{C}\setminus f(\Delta)$ which do not meet, then $f \in \mathcal{F}_\alpha$ where $\alpha \pi$ denotes the largest of the two angles determined by the rays. Also if the range of a function analytic in $\Delta$ is contained in an angular wedge of opening $\alpha \pi$ and $1 < \alpha < 2$, then $f \in \mathcal{F}_\alpha$.

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1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\Lambda = \{z \in \mathbb{C} : |z| = 1\}$, and let $\mathcal{M}$ denote the set of complex-valued Borel measures on $\Lambda$. For $\alpha > 0$, let $\mathcal{F}_\alpha$ denote the set of functions $f : \Delta \to \mathbb{C}$ for which there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_{\Lambda} \frac{1}{(1-\zeta z)^\alpha} \, d\mu(\zeta) \quad \text{for } |z| < 1.$$

The power function in (1) denotes the principal branch. Each function given by (1) is analytic in $\Delta$. For $f \in \mathcal{F}_\alpha$, let $\|f\|_{\mathcal{F}_\alpha} = \inf \|\mu\|$, where $\|\mu\|$ denotes the total variation of $\mu$, and $\mu$ varies over all measures in $\mathcal{M}$ for which (1) holds. This defines a norm on $\mathcal{F}_\alpha$, and $\mathcal{F}_\alpha$ is a Banach space with respect to this norm. The family $\mathcal{F}_1$...
was first studied by Havin [2]. The general class $\mathcal{F}_\alpha$, where $\alpha > 0$, was introduced in [5] and has been studied extensively. In [6] a survey is given about these so-called fractional Cauchy transforms.

Several conditions on an analytic function $f$ are known to be sufficient to imply $f \in \mathcal{F}_\alpha$. Most of these conditions are analytic. Here we are concerned with geometric conditions. The Riesz-Herglotz formula provides information of this type. It implies that if $f : \Delta \to \mathbb{C}$ is analytic and $f(\Delta)$ is contained in a half-plane, then $f \in \mathcal{F}_1$. Another result of this kind was obtained by Bourdon and Cima in [1], namely, if $f : \Delta \to \mathbb{C}$ is analytic and there are two oppositely directed rays in $\mathbb{C} \setminus f(\Delta)$, then $f \in \mathcal{F}_1$.

This paper contains generalizations of the two results described above. We show that if $f(\Delta)$ is contained in an angular wedge with opening $\alpha \pi$ and $1 < \alpha < 2$, then $f \in \mathcal{F}_\alpha$. Also if there are two nonparallel rays in $\mathbb{C} \setminus f(\Delta)$ which do not meet and the angles at infinity between these two rays are $\alpha \pi$ and $\beta \pi$, then $f \in \mathcal{F}_\gamma$, where $\gamma = \max \{\alpha, \beta\}$.

If $f(\Delta)$ is contained in an angular wedge of opening less than $\pi$, then $f \in \mathcal{F}_1$, but $f$ need not belong to $\mathcal{F}_\alpha$ for any $\alpha, 0 < \alpha < 1$. This holds more generally for bounded analytic functions. To see this, let

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1.$$ 

The function $f$ is analytic and bounded in $\Delta$. However $f \notin \mathcal{F}_\alpha$ when $0 < \alpha < 1$. This is because the Taylor coefficients of $f$ do not satisfy the condition $a_n = O\left(n^{\alpha-1}\right)$, which is necessary for membership in $\mathcal{F}_\alpha$.

Finally we mention that if $f : \Delta \to \mathbb{C}$ is analytic and $\mathbb{C} \setminus f(\Delta)$ contains a ray, then $f \in \mathcal{F}_2$ [5, Theorem 5].

2. Preliminaries

This section contains lemmas which will be used to prove the main results. The first two lemmas are in [5, Lemma 1]. Lemma 2.3 is in [3, Theorem 2]. Lemma 2.4 is known but we give a proof.

**Lemma 2.1.** For every $\alpha > 0$, $f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$.

**Lemma 2.2.** If $0 < \alpha < \beta$, then $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$.

**Lemma 2.3.** If $\alpha \geq 1$, $f \in \mathcal{F}_\alpha$ and the function $\varphi : \Delta \to \Delta$ is analytic, then the composition $f \circ \varphi \in \mathcal{F}_\alpha$. 
LEMMA 2.4. Suppose that $f : \Delta \to \mathbb{C}$ is analytic and $f'$ belongs to Hardy space $H^1$. Then $f \in \mathcal{F}_\alpha$ for every $\alpha > 0$.

PROOF. Suppose that $f' \in H^1$ and let $g = f'$. Let $\zeta = e^{i\theta}$. Then

$$G(\zeta) = \lim_{r \to 1^-} g(r\zeta)$$

exists for almost all $\theta$ in $[-\pi, \pi]$ and $G(e^{i\theta}) \in L^1([-\pi, \pi])$. Also $g$ is represented by the Cauchy formula

$$g(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{G(\zeta)}{\zeta - z} d\zeta, \quad |z| < 1.$$  \hspace{1cm} (2)

Equation (2) yields (1), where $d\mu(\zeta) = (G(\zeta)/2\pi i\zeta) d\zeta$ and hence $g = f' \in \mathcal{F}_1$. By Lemma 2.2, $f' \in \mathcal{F}_\alpha$ for $\alpha > 1$. Lemma 2.1 implies that $f \in \mathcal{F}_\alpha$ for all $\alpha > 0$. \hfill $\Box$

LEMMA 2.5. Suppose that the function $g$ is analytic in a neighborhood of $\Delta$. Let $N$ be a positive integer and suppose that $|\zeta_n| = 1$, $\alpha_n > 0$ for $n = 1, 2, \ldots, N$, and $\zeta_n \neq \zeta_m$ for $n \neq m$. Let

$$f(z) = \frac{g(z)}{\prod_{n=1}^{N} (z - \zeta_n)^{\alpha_n}}, \quad |z| < 1.$$  \hspace{1cm} (3)

Then $f \in \mathcal{F}_\alpha$, where $\alpha = \max\{\alpha_n : 1 \leq n \leq N\}$.

PROOF. We give the proof for the case $N = 2$. A similar argument can be given for other values of $N$.

Suppose that $|\zeta| = |\sigma| = 1$, $\zeta \neq \sigma$, $\beta > 0$, and $\gamma > 0$. Suppose that the function $g$ is analytic in a neighborhood of $\Delta$ and let

$$f(z) = \frac{g(z)}{(z - \zeta)^\beta (z - \sigma)^\gamma}, \quad |z| < 1.$$  \hspace{1cm} (4)

We shall show that $f \in \mathcal{F}_\alpha$, where $\alpha = \max\{\beta, \gamma\}$.

The function $z \mapsto g(z)/(z - \sigma)^\gamma$ is analytic at $z = \zeta$, and hence

$$\frac{g(z)}{(z - \sigma)^\gamma} = \sum_{m=0}^{\infty} a_m (z - \zeta)^m,$$

for $z$ in some neighborhood of $\zeta$. Let $p$ be the least integer such that $p \geq \beta$ and let $s = p - \beta$. Then

$$f(z) = \sum_{m=0}^{p-1} \frac{a_m}{(z - \zeta)^{p-m}} + (z - \zeta)^s h(z),$$  \hspace{1cm} (5)
where the function $h$ is analytic in some neighbourhood of $\zeta$. Suppose that $\beta$ is not an integer. Then
\[
\frac{d}{dz}[(z - \zeta)^{\beta}h(z)] = (z - \zeta)^{\beta}h'(z) + s(z - \zeta)^{s-1}h(z).
\]
Since $(z - \zeta)^{\beta}$ is bounded in $\overline{\Delta} \setminus \{\zeta\}$, this implies that there is a positive constant $A$ such that
\[
|\frac{d}{dz}[(z - \zeta)^{\beta}h(z)]| \leq A|z - \zeta|^{\beta-1},
\]
for $z \in \overline{\Delta}$, $z$ near $\zeta$, and $z \neq \zeta$. Likewise if $\gamma$ is not an integer, $q$ is the least integer such that $q \geq \gamma$ and $t = q - \gamma$, then
\[
f(z) = \sum_{m=0}^{q-1} \frac{b_m}{(z - \sigma)^{\gamma-m}} + (z - \sigma)^t k(z),
\]
where $k$ is a function analytic in some neighbourhood of $\sigma$ and $b_m (m = 0, 1, \ldots, q-1)$ are suitable constants. Thus
\[
|\frac{d}{dz}[(z - \sigma)^{t}k(z)]| \leq B|z - \sigma|^{t-1},
\]
for $z \in \overline{\Delta}$, $z$ near $\sigma$, and $z \neq \sigma$, where $B$ is a positive constant.

For $z \in \overline{\Delta} \setminus \{\zeta, \sigma\}$, let
\[
r(z) = f(z) - \sum_{m=0}^{p-1} \frac{a_m}{(z - \zeta)^{\beta-m}} - \sum_{m=0}^{q-1} \frac{b_m}{(z - \sigma)^{\gamma-m}}.
\]
The relations (5), (6) and (9) imply that there is a constant $C$ such that
\[
|r'(z)| \leq C|z - \zeta|^{\beta-1},
\]
for $z \in \overline{\Delta}$, $z$ near $\zeta$, and $z \neq \zeta$. Likewise (7)--(9) imply that
\[
|r'(z)| \leq D|z - \sigma|^{t-1},
\]
for $z \in \overline{\Delta}$, $z$ near $\sigma$, and $z \neq \sigma$, where $D$ is a positive constant.

The function $z \mapsto (z - \tau)^u$ belongs to $H^1$ when $|\tau| = 1$ and $u > -1$. Hence the inequalities (10) and (11) and the fact that $r'$ is analytic in $\overline{\Delta} \setminus \{\zeta, \sigma\}$ imply that $r' \in H^1$. This proves that $r' \in H^1$ when $\beta$ and $\gamma$ are not integers. A similar argument shows that $r' \in H^1$ when only one of the numbers $\beta$ and $\gamma$ is not an integer. If both
\[ f = f_1 + f_2 + r, \]

where

\[ f_1(z) = \sum_{m=0}^{p-1} \frac{a_m}{(z - \zeta)^{\delta - m}}, \quad \text{and} \quad f_2(z) = \sum_{m=0}^{q-1} \frac{b_m}{(z - \sigma)^{\gamma - m}}. \]

The function \( z \mapsto 1/(z - \zeta)^\delta \) belongs to \( \mathcal{F}_\delta \) when \( 0 < \delta \leq \beta \) and hence \( f_1 \in \mathcal{F}_\delta \). Likewise \( f_1 \in \mathcal{F}_\gamma \). Lemma 2.2 yields \( f_1 \in \mathcal{F}_\alpha \) and \( f_2 \in \mathcal{F}_\alpha \). Since \( r \in \mathcal{F}_\alpha \), (12) implies that \( f \in \mathcal{F}_\alpha \). \( \square \)

Lemma 2.5 contrasts with the following result obtained in [5, Lemma 1].

**THEOREM 2.6.** \( f \in \mathcal{F}_\alpha \) and \( g \in \mathcal{F}_\beta \) then \( f \cdot g \in \mathcal{F}_{\alpha + \beta} \).

Since the function \( g \) in Lemma 2.5 is analytic in \( \Delta \), \( g \) is a multiplier of \( \mathcal{F}_\delta \) for every \( \delta > 0 \) [4, Theorem 3.5]. This fact and Theorem 2.6 imply that the function \( f \) in (3) belongs to \( \mathcal{F}_{\alpha'} \), where \( \alpha' = \sum_{n=1}^{N} \alpha_n \). Lemma 2.5 is clearly an improvement of this result. The assumption that \( \zeta_n \neq \zeta_m \) for \( n \neq m \) is critical in Lemma 2.5. To see how this is reflected in our argument, suppose that the numbers \( \zeta_n (n = 1, 2, \ldots, N) \) are distinct, \( \zeta_2 \to \zeta_1 \), and the numbers \( \alpha_n \) are fixed. Suppose that \( \alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_N\} \). Then the norm \( \|f\|_{\mathcal{F}_\alpha} \) of the corresponding function in (3) goes to infinity as \( \zeta_2 \to \zeta_1 \).

3. The main results

Let \( f \) be analytic in \( \Delta \). In this section we give two geometric conditions on \( f(\Delta) \) sufficient to imply that \( f \in \mathcal{F}_\alpha \).

**THEOREM 3.1.** Suppose that the function \( f : \Delta \to \mathbb{C} \) is analytic and let \( \Phi = \mathbb{C} \setminus f(\Delta) \).

(a) Suppose that \( \Phi \) contains two nonparallel rays. Let \( \alpha \pi \) and \( \beta \pi \) denote the angles at \( \infty \) between these two rays, where \( \alpha \geq \beta \). If \( \alpha < 2 \), then \( f \in \mathcal{F}_\alpha \).

(b) If \( \Phi \) contains a ray then \( f \in \mathcal{F}_2 \).

**PROOF.** First assume that \( \Phi \) contains two nonparallel rays. Since \( \alpha + \beta = 2 \), the assumptions imply that \( 1 < \alpha < 2 \). We may assume that the rays do not intersect.
Let $F$ denote a conformal mapping of $\Delta$ onto the complement of the two rays. The Schwarz-Christoffel formula gives

$$
F(z) = b \int_0^z \frac{(w - \zeta_1)(w - \zeta_2)}{(w - \zeta_3)^{a+1}(w - \zeta_4)^{3-a}} \, dw + c,
$$

where $\zeta_1$, $\zeta_2$, $\zeta_3$, and $\zeta_4$ are distinct points on $\Lambda$, and $b$ and $c$ are suitable complex numbers. Hence

$$
F'(z) = \frac{g(z)}{(w - \zeta_3)^{a+1}(w - \zeta_4)^{3-a}},
$$

where $g$ is a quadratic polynomial. Since $3 - \alpha < \alpha + 1$, Lemma 2.5 yields $F' \in \mathcal{F}_{a+1}$. Hence Lemma 2.1 implies $F \in \mathcal{F}_\alpha$.

Since $f(\Delta) \subseteq F(\Delta)$ and $F$ is univalent, the function $\varphi = F^{-1} \circ f$ is analytic in $\Delta$ and maps $\Delta$ into $\Delta$. Since $F \in \mathcal{F}_\alpha$ and $\alpha > 1$, Lemma 2.3 yields $f = F \circ \varphi \in \mathcal{F}_\alpha$. This proves the first assertion.

The second assertion can be proved in a similar way. The conformal mapping of $\Delta$ onto the complement of a ray has the form $F(z) = P(z)/(z - \xi)^2$, where $P$ is a quadratic polynomial and $|\xi| = 1$. This yields $F \in \mathcal{F}_2$ and hence Lemma 2.3 yields $f \in \mathcal{F}_2$.

**Theorem 3.2.** Suppose that $f$ is analytic in $\Delta$. If $f(\Delta)$ is contained in an angular wedge of opening $\alpha \pi$ and $1 < \alpha < 2$, then $f \in \mathcal{F}_\alpha$.

**Proof.** The function $z \mapsto [(1 + z)/(1 - z)]^\alpha$ maps $\Delta$ one-to-one onto the wedge $\{w : |\arg w| < \alpha \pi/2\}$. Hence there are complex numbers $b$ and $c$ such that the function defined by $F(z) = b[(1 + z)/(1 - z)]^\alpha + c$ maps $\Delta$ one-to-one onto the angular wedge containing $f(\Delta)$. The function $z \mapsto 1/(1 - z)^\alpha$ belongs to $\mathcal{F}_\alpha$. Let $h(z) = (1 + z)^\alpha$. Since $\alpha > 1$, $h'$ is bounded. Thus $h' \in H^1$ and it follows that $h$ is a multiplier of $\mathcal{F}_\delta$ for every $\delta > 0$ [4, Theorem 3.5]. Therefore $F \in \mathcal{F}_\alpha$. Since $f(\Delta) \subseteq F(\Delta)$ and $F$ is univalent, we have $f = F \circ \varphi$, where the function $\varphi : \Delta \rightarrow \Delta$ is analytic. Since $F \in \mathcal{F}_\alpha$ and $\alpha > 1$, Lemma 2.3 gives $f \in \mathcal{F}_\alpha$.

**References**


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