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RESEARCH ARTICLE

Localization in the random XXZ quantum spin chain

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Abstract
We study the many-body localization (MBL) properties of the Heisenberg XXZ spin-$\frac{1}{2}$ chain in a random magnetic field. We prove that the system exhibits localization in any given energy interval at the bottom of the spectrum in a nontrivial region of the parameter space. This region, which includes weak interaction and strong disorder regimes, is independent of the size of the system and depends only on the energy interval. Our approach is based on the reformulation of the localization problem as an expression of quasi-locality for functions of the random many-body XXZ Hamiltonian. This allows us to extend the fractional moment method for proving localization, previously derived in a single-particle localization context, to the many-body setting.

Contents

1. Introduction

The last two decades have seen an explosion of physics research on the behavior of isolated quantum systems in which both disorder and interactions are present. The appearance of these two features has been linked to the existence of materials that fail to thermalize and consequently cannot be described using equilibrium statistical mechanics. These materials are presumed to remain insulators at nonzero temperature, a phenomenon called many-body localization (MBL). We refer the reader to the physics reviews [1,9,38] for the general description of this phenomenon. MBL-type behavior has been observed in cold atoms experiments [31, 41]. The stability of the MBL phase for infinite systems and all times remains a topic of intense debate [25, 35, 43, 44, 45].

In this paper, we consider the random spin-$\frac{1}{2}$ Heisenberg XXZ chain in the Ising phase, a one-dimensional random quantum spin system. This is the most studied model in the context of MBL both in the physics and mathematics literature (going back to [39, 47]). It can be mapped by the Jordan-Wigner transformation into an interacting spinless fermionic model closely related to the disordered

Fermi-Hubbard Hamiltonian, a paradigmatic model in condensed matter physics that provides crucial insights into the electronic and magnetic properties of materials. One interesting feature of the random *one-dimensional* XXZ quantum spin system is the emergence of a many-body localization-delocalization transition. (In contrast, prototypical non-interacting one-dimensional random Schrödinger operators do not exhibit a phase transition and are completely localized.) Numerical evidence for this transition in the disordered XXZ model has been provided in a number of simulations (e.g., [3, 10, 11, 30, 39]), but remains contested on theoretical grounds (e.g., [14]).

Until quite recently, mathematical results related to the proposed MBL characteristics, including zero-velocity Lieb-Robinson bounds, exponential clustering, quasi-locality, slow spreading of information and area laws, have been confined to quasi-free systems. The latter are models whose study can effectively be reduced to one of a (disordered) one-particle Hamiltonian. Examples of such systems include the XY spin chain in a random transversal field (going back to [29]; see [2] for a review on this topic), the disordered Tonks-Girardeau gas [42] and systems of quantum harmonic oscillators [36]. Another direction of research considers the effect of many-body interaction on a single-particle localization (rather than MBL) within the framework of the effective field theories. This allows to consider a realistic Hilbert space for a single particle, such as $\ell^2(\mathbb{Z}^d)$, rather than finite dimensional ones that are typically used in the MBL context. In particular, the persistence of the dynamical localization in the Hartree-Fock approximation for the disordered Hubbard model has been established in [16, 34].

In the last few years, there has been some (modest) progress in understanding genuine many-body systems, all of which is concerned with the XXZ model, either in the quasi-periodic setting (where the exponential clustering property for the ground state of the André-Aubry model has been established [32, 33]) or in the droplet spectrum regime in the random case [12, 19]. In the latter case, several MBL manifestations have been established, including some that have never been previously discussed in the physics literature [18].

While not exactly solvable, the XXZ spin chain does have a symmetry; namely, it preserves the particle number. This enables a reduction to an infinite system of discrete N-body Schrödinger operators on the fermionic subspaces of $\mathbb{Z}^N$ [21, 37]. For the XXZ spin chain in the Ising phase, in the absence of a magnetic field, the low energy eigenstates above the ground state are characterized by a *droplet regime*. In this regime, spins form a droplet (i.e., a single cluster of down spins (particles) in a sea of up spins). This reduction has been effectively exploited inside the droplet spectrum (the interval I_1 in (2.14) below) using methods that resemble the fractional moment method for random Schrödinger operators, yielding the small number of rigorous results [12, 19]. However, these methods seem to be inadequate above this energy interval (i.e, inside the multi-cluster spectrum), and a new set of ideas that do not rely on a reduction to Schrödinger operators are required to tackle this case.

In this paper, we extend the energy interval for which MBL holds well beyond the droplet spectrum, deep inside the multi-cluster spectrum. We develop a suitable method, formulated and proved in terms of spin systems concepts. In particular, our method does not rely on the reduction of the XXZ Hamiltonian to a direct sum of Schrodinger operators (and the subsequent analysis that uses single-particle tools).

Localization phenomenon in condensed matter physics is usually associated with non-spreading of wave packets in a disordered medium. Experimentally, it is observed in semiconductors whose properties are predominantly caused by crystal defects or impurities, as well as in the variety of other systems. This phenomenon is by now well understood for quantum single particle models. A prototypical system studied in this context is the Anderson Hamiltonian H_A, which is a self-adjoint operator acting on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ of the form $H_A = -\Delta + \lambda V_\omega$. Here, Δ is the (discrete) Laplacian describing the kinetic hopping, V_ω is a randomly generated multiplication operator (ω is the random parameter) describing the electric potential, and λ is a parameter measuring the strength of the disorder.

Let us denote by $\delta_x \in \mathcal{H}$ the indicator of $x \in \mathbb{Z}^d$, and fix the random parameter ω. An important feature of H_A as a map on $\ell^2(\mathbb{Z}^d)$ is its locality, meaning $\langle \delta_x, H_A \delta_y \rangle = 0$ if $|x - y| > 1$. As a consequence, the resolvent $(H_A - z)^{-1}$ retain a measure of locality, which we will call quasi-locality, given by the Combes-Thomas estimate

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$$\left|\langle\delta_x,(H_A-z)^{-1}\delta_y\rangle\right|\le C_z\mathrm{e}^{-m_z|x-y|}, \tag{1.1}$$

where C_z and m_z are constants independent of ω such that $C_z < \infty$ and $m_z > 0$ if $z \in \mathbb{C}$ is outside the spectrum of H_A. Maps given by smooth functions of H_A also express a measure of quasi-locality – namely,

$$\left|\langle\delta_x,f(H_A)\delta_y\rangle\right|\le C_{f,n}(1+|x-y|)^{-n}, \tag{1.2}$$

where $C_{f,n} < \infty$ for all $n \in \mathbb{N}$ and infinitely differentiable functions f. Moreover, these quasi-locality estimate hold with the same constants for the restriction H_A^Λ of H_A to a finite volume $\Lambda \subset \mathbb{Z}^d$. (See, for example, [26, 24, 40].)

The two mainstream approaches for proving localization in the single particle setting, the multi-scale analysis (MSA) and the fractional moment method (FMM), going back to [15, 23, 22] and [4, 6], respectively – establish localization for the (random) Anderson model H_A by proving quasi-locality estimates for the finite volume resolvent inside the spectrum of H_A. In particular, the fractional moment method shows that, fixing $s \in (0, 1)$, for large disorder λ, we have

$$\mathbb{E}\left\{\left|\langle\delta_x,(H_A^\Lambda-E)^{-1}\delta_y\rangle\right|^s\right\}\le C\mathrm{e}^{-m|x-y|}, \tag{1.3}$$

for all finite $\Lambda \subset \mathbb{Z}^d$, $x, y \in \Lambda$, and energies $E \in \mathbb{R}$, where the constants $C < \infty$ and $m > 0$ are independent of Λ. Moreover, one also gets a quasi-locality estimate for Borel functions of H^A (dynamical localization),

$$\mathbb{E}\left\{\sup_f\left|\langle\delta_x,f(H_A^\Lambda)\delta_y\rangle\right|\right\}\le C\mathrm{e}^{-m|x-y|}, \tag{1.4}$$

where the supremum is taken over all Borel functions on $\mathbb{R}$ bounded by one. Various manifestations of one-particle localization, such as non-spreading of wave packets, vanishing of conductivity in response to electric field, and statistics of the spacing between nearby energy levels, can be derived from these quasi-locality estimates. (See, for example, [8].) On the mathematical level, the quasi-locality estimates provides an effective description of single particle localization.

The MSA and the FMM prove localization for random Schrödinger operators, both in the discrete and continuum settings. We refer the reader to the lecture notes [26, 27] and the monograph [8] for an introduction to the multi-scale analysis and the fractional moment method, respectively.

Both methods have been extended to quantum system consisting of an arbitrary, but fixed, number of interacting particles, showing that many characteristics of single-particle localization remain valid in this case (e.g., [7, 13, 28]). But truly many-body systems (where the number of particles is proportional to the system's size) present new challenges. A major difficulty lies in the fact that the concepts of MBL proposed in the physics literature are not easily tractable on the mathematical level, and it is not clear what could be chosen as the fundamental description of the theory from which other properties can be derived, as in a single particle case. For example, the available concept of quasi-locality in the many-body systems looks very different from the one for single particle quantum systems.

To introduce a simple many-body system Hamiltonian, we consider a finite graph $\mathbf{\Gamma} = (\mathcal{V}, \mathcal{E})$ (where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges) and a family $\{\mathcal{H}_i\}_{i\in\mathcal{V}}$ of Hilbert spaces. The Hilbert space of the subsystem associated with a set $X \subset \mathcal{V}$ is given by $\mathcal{H}_X = \bigotimes_{i\in X}\mathcal{H}_i$, and the full Hilbert space (we ignore particles' statistics) is $\mathcal{H}_\mathcal{V}$. For each $X \subset \mathcal{V}$, one introduces the *algebra of observables* $\mathcal{A}_X$ measurable in this subsystem, which is the collection $\mathcal{B}(\mathcal{H}_X)$ of bounded linear operators on the Hilbert space $\mathcal{H}_X$. An observable $\mathcal{O} \in \mathcal{A}_\mathcal{V}$ is said to be supported by $X \subset \mathcal{V}$ if $\mathcal{O} = \mathcal{O}_X \otimes \mathbb{1}_{\mathcal{H}_{\mathcal{V}\setminus X}}$, where $\mathcal{O}_X \in \mathcal{A}_X$ (i.e., if $\mathcal{O}$ acts trivially on $\mathcal{H}_{\mathcal{V}\setminus X}$). Slightly abusing the notation, we will usually identify $\mathcal{O}$ with $\mathcal{O}_X$ and call X a support for $\mathcal{O}$. Since we are primarily interested here in understanding the way particles interact, the structure of a single particle Hilbert space $\mathcal{H}_i$ will be only of marginal importance

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for us. So we will be considering the simplest possible realization of such system, where each $\mathcal{H}_i$ is the two dimensional vector space $\mathbb{C}^2$ describing a spin-$\frac{1}{2}$ particle.

We next describe the interactions between our spins. We again are going to consider the simplest possible arrangement, where only nearest neighboring spins are allowed to interact. Explicitly, for each pair of vertices $(i,j) \in \mathcal{V}$ that share an edge (i.e., $\{i,j\} \in \mathcal{E}$), we pick an observable (called an *interaction*) $h_{i,j} \in \mathcal{A}_{\{i,j\}}$ such that $h_{i,j} = h_{i,j}^*$, an observable (called a *local transverse field*) $v_i = v_i^* \in \mathcal{A}_{\{i\}}$, and associate a Hamiltonian $H^{\mathcal{V}} = \sum_{\{i,j\}\in\mathcal{E}} h_{i,j} + \sum_{i\in\mathcal{V}} v_i$ with our spin system. In particular, $H^{\mathcal{V}}$ is the sum of local observables and is consequently referred to as a *local Hamiltonian*. Locality is manifested by $[[H^{\mathcal{V}}, \mathcal{O}], \mathcal{O}'] = 0$ for any pair of observables $\mathcal{O} \in \mathcal{A}_X$, $\mathcal{O}' \in \mathcal{A}_Y$, with $\operatorname{dist}(X,Y) > 1$. (To compare it with the concept of (single particle) locality for the map H_A, we need to define a local observable for the space $\ell^2(\mathbb{Z}^d)$. We will say that an observable $\mathcal{O} \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ has support $X \subset \mathbb{Z}^d$ if $\mathcal{O} = \mathcal{O}_X \oplus 0_{\ell^2(\mathbb{Z}^d\setminus X)}$ with $\mathcal{O}_X \in \mathcal{L}(\ell^2(X))$. With this definition, locality of the map H_A (i.e., the property $\langle \delta_x, H_A \delta_y\rangle = 0$ whenever $|x-y| > 1$) is equivalent to the statement that $[[H_A, \mathcal{O}], \mathcal{O}'] = 0$ for any pair of observables $\mathcal{O}, \mathcal{O}'$ with $\operatorname{dist}(\operatorname{supp}(\mathcal{O}), \operatorname{supp}(\mathcal{O}')) > 1$.)

The XXZ spin chain is defined as above on finite subgraphs Λ of the graph $\mathbb{Z}$ (see Section 2.1). Consider $\Lambda \subset \mathbb{Z}$ connected, and let $|\Lambda|$ be its cardinality. We say we have a particle at the site $i \in \Lambda$ if we have spin down in the copy $\mathcal{H}_i$ of $\mathbb{C}^2$. Let $\mathcal{N}_i$ be the orthogonal projection onto configurations with a particle at the site i, and set $[i]_p^\Lambda = \{j \in \Lambda, |j-i| \le p\}$ for $p = 0, 1, \ldots$. Given $B \subset \Lambda$, let P_+^B be the orthogonal projection onto configurations with no particles in B. In the Ising phase, H^Λ is a 2-local, gapped, frustration-free system, and P_+^Λ describes the projection onto the ground state of H^Λ (see Remark 2.3).

We can now informally state our main results. We first prove that the resolvent $R_z^\Lambda = (H^\Lambda - z)^{-1}$ exhibits quasi-locality in the form (see Lemma 3.1 and Remark 3.2)

$$\left\| \mathcal{N}_i R_z^\Lambda P_+^{[i]_p^\Lambda} \right\| \le C_z \mathrm{e}^{-m_z p}, \tag{1.5}$$

where C_z and m_z are constants, independent of Λ and of the transverse field, such that $C_z < \infty$ and $m_z > 0$ if $z \in \mathbb{C}$ is outside the spectrum of H^Λ. We also establish the many-body analogue of (1.2):

$$\left\| \mathcal{N}_i f(H^\Lambda) P_+^{[i]_p^\Lambda} \right\| \le C_{f,n} (1+p)^{-n}, \tag{1.6}$$

where $C_{f,n} < \infty$ for all $n \in \mathbb{N}$ and infinitely differentiable functions f on $\mathbb{R}$ with compact support. (See Appendix B.)

We next consider the random XXZ spin chain (see Definition 2.2). The relations (1.5)–(1.6) suggest, by analogy with random Schrödinger operators, that localization should be manifested as quasi-locality inside the spectrum of H^Λ. This is indeed what we prove in Theorem 2.4. We introduce increasing energy intervals $I_{\le k}$, $k = 0, 1, 2, \ldots$, in (2.14) and prove that quasi-locality of the form given in (1.5) holds for the resolvent for energies in $I_{\le k}$ for any fixed k. In particular, given $s \in (0, \frac{1}{3})$, we prove, in the appropriate (k dependent) parameter region, that

$$\mathbb{E}\left\{ \left\| \mathcal{N}_i R_E^\Lambda P_+^{[i]_p^\Lambda} \right\|^s \right\} \le C_k |\Lambda|^{\xi_k} \mathrm{e}^{-m_k p} \quad \text{for all} \quad E \in I_{\le k}, \tag{1.7}$$

where the constants $C_k < \infty$, $\xi_k > 0$, $m_k > 0$ do not depend on Λ. As a consequence, we derive a quasi-locality estimate for Borel functions of H^Λ (Corollary 2.6):

$$\mathbb{E}\left(\sup_f \left\| \mathcal{N}_i f(H^\Lambda) P_+^{[i]_p^\Lambda} \right\| \right) \le C_k |\Lambda|^{\xi_k} \mathrm{e}^{-m_k p}, \tag{1.8}$$

where the supremum is taken over all Borel functions on $\mathbb{R}$ that are equal to zero outside the interval $I_{\le k}$ and bounded by one.

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While the estimates (1.7) and (1.8) are very natural from the mathematical perspective, it is far from obvious whether they yield any of the MBL-type features proposed by physicists. Nevertheless, in a sequel to this paper [17], we derive slow propagation of information, a putative MBL manifestation, from Theorem 2.4 and Corollary 2.6, for any $k \in \mathbb{N}$.

In the droplet spectrum, [19, Theorem 2.1] imply Corollary 2.6 (with $k = 1$), and a converse can established using [17, Remark 3.3]. While [19] and the follow-up paper [18] contain several MBL-type properties such as the (dynamical) exponential clustering property, (properly defined) zero-velocity Lieb-Robinson bounds, and slow propagation (non-spreading) of information, they are all derived using [19, Theorem 2.1] as the starting point. We stress that [19, Theorem 2.1], by its very nature, can only hold in the droplet regime, so while it provides us with very strong consequences in the $k = 1$ case, we do not expect the methods of [19, 18] to be of any use in the multi-cluster case – that is, for $k \geq 2$.

Although the methods derived in this work are not universal (which is typical for many-body results), they are sufficiently powerful for investigation of MBL phenomena in this context, as shown in [17]. We have to admit, however, that in the physics literature, MBL is usually associated with energies that are not fixed (as we assumed in this work) but are comparable with the system size $|\Lambda|$. We do not expect that our techniques will be sufficient to probe such energies. To be able to do so would require non-perturbative techniques similar to the ones used in the investigations of one dimensional random Schrödinger operators.

The model description and main results (Theorem 2.4 and Corollary 2.6) are presented in Section 2. In Section 3, we outline the main ideas used in the proof of Theorem 2.4, which is completed in Section 4. Corollary 2.6 is proven in Section 5. Appendix A contains some useful identities. Appendix B contains the proof of of the many-body quasi-locality estimate (1.6).

Throughout the paper, we will use generic constants C, c, m, etc., whose values will be allowed to change from line to line, even in a displayed equation. These constants will not depend on subsets of $\mathbb{Z}$, but they will, in general, depend on the parameters of the model introduced in Section 2.1 (such as μ, k, Δ_0, λ_0 and s). When necessary, we will indicate the dependence of a constant on k explicitly by writing it as C_k, m_k, etc. These constants can always be estimated from the arguments, but we will not track the changes to avoid complicating the arguments.

2. Model description and main results

2.1. Model description

The random XXZ quantum spin-$\frac{1}{2}$ chain on an finite subset Λ of $\mathbb{Z}$ is given by a self-adjoint Hamiltonian H^{Λ} acting on the finite dimensional Hilbert space $\mathcal{H}_{\Lambda} = \otimes_{i\in\Lambda}\mathcal{H}_i$, where $\mathcal{H}_i = \mathbb{C}^2$ for each $i \in \Lambda$. For a vector $\phi \in \mathbb{C}^2$, we let ϕ_i denote the vector as an element of $\mathcal{H}_i$; for an operator (2×2 matrix) A on $\mathbb{C}^2$, we let A_i denote the operator acting on $\mathcal{H}_i$.

We consider only finite subsets of $\mathbb{Z}$, so by a subset of $\mathbb{Z}$ we will always mean a finite subset. If $S \subset T \subset \mathbb{Z}$, and A_S is an operator on $\mathcal{H}_S$, we consider A_S as operator on $\mathcal{H}_T$ by identifying it with $A_S \otimes \mathbb{1}_{T\setminus S}$, where $\mathbb{1}_R$ denotes the identity operator on $\mathcal{H}_R$. We thus identify $\mathcal{A}_S$ with a subset of $\mathcal{A}_T$, where $\mathcal{A}_R$ denotes the algebra of bounded operators on $\mathcal{H}_R$.

We now fix $\Lambda \subset \mathbb{Z}$, and consider Λ as a subgraph of $\mathbb{Z}$. We denote by dist_{Λ} the graph distance in Λ, which can be infinite if Λ is not a connected subset of $\mathbb{Z}$. We write $K^c = \Lambda \setminus K$ for $K \subset \Lambda$. To define H^{Λ}, we introduce some notation and definitons.

1. By $\sigma^{x,y,z}$ and $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ we will denote the standard Pauli matrices and ladder operators, respectively.
2. By $\uparrow\rangle := \begin{pmatrix}1\\0\end{pmatrix}$ and $\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ we will denote the elements of the canonical basis of $\mathbb{C}^2$, called spin-up and spin-down, respectively. Letting $\mathcal{N} = \frac{1}{2}(\mathbb{1} - \sigma^z)$, we note that $\mathcal{N}\uparrow\rangle = 0$ and $\mathcal{N}\downarrow\rangle =\downarrow\rangle$, and interpret $\downarrow\rangle$ as a particle.

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3. $\mathcal{N}_i$, the matrix $\mathcal{N}$ acting on $\mathcal{H}_i$, is the projection onto the spin-down state (also called the local number operator) at site i. Given $S \subset \Lambda$, $\mathcal{N}_S = \sum_{i \in S} \mathcal{N}_i$ is the total (spin-down) number operator in S.
4. The total number operator $\mathcal{N}_\Lambda$ has eigenvalues $0, 1, 2, \ldots, |\Lambda|$. ($|S|$ denotes the cardinality of $S \subset \mathbb{Z}$.) We set $\mathcal{H}_\Lambda^{(N)} = \operatorname{Ran}(\chi_N(\mathcal{N}_\Lambda))$, obtaining the Hilbert space decomposition $\mathcal{H}_\Lambda = \bigoplus_{N=0}^{|\Lambda|} \mathcal{H}_\Lambda^{(N)}$. We will use the notation $\chi_N^\Lambda = \chi_{\{N\}}(\mathcal{N}_\Lambda)$.
5. The canonical (orthonormal) basis Φ_Λ for $\mathcal{H}_\Lambda$ is constructed as follows: Let $\Omega_\Lambda = \phi_\emptyset = \otimes_{i \in \Lambda} \uparrow\rangle_i$ be the vacuum state. Then

$$\Phi_\Lambda = \left\{ \phi_A = \left(\prod_{i \in A} \sigma_i^- \right) \Omega_\Lambda : \ A \subset \Lambda \right\} = \bigcup_{N=0}^{|\Lambda|} \Phi_\Lambda^{(N)}, \tag{2.1}$$

where $\Phi_\Lambda^{(N)} = \{\phi_A : A \subset \Lambda, \ |A| = N\}$. Note that $\Phi_\Lambda^{(0)} = \{\Omega_\Lambda\}$.

We now define the free XXZ quantum spin-$\frac{1}{2}$ Hamiltonian on $\Lambda \subset \mathbb{Z}$ by

$$H_0^\Lambda = H_0^\Lambda(\Delta) = -\tfrac{1}{2\Delta} \mathbf{\Delta}^\Lambda + \mathcal{W}^\Lambda \quad \text{on} \quad \mathcal{H}_\Lambda, \tag{2.2}$$

where

$$\mathbf{\Delta}^\Lambda = \sum_{\{i,i+1\} \subset \Lambda} \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ \right), \tag{2.3}$$

$$\mathcal{W}^\Lambda = \mathcal{N}_\Lambda - \sum_{\{i,i+1\} \subset \Lambda} \mathcal{N}_i \mathcal{N}_{i+1}, \tag{2.4}$$

and $\Delta > 1$ is the anisotropy parameter, specifying the Ising phase ($\Delta = 1$ selects the Heisenberg chain and $\Delta = \infty$ corresponds to the the Ising chain).

We will consider the XXZ model in the presence of a transversal field λV_ω^Λ, given by $V_\omega^\Lambda = \sum_{i \in \Lambda} \omega_i \mathcal{N}_i$, where $\omega_i \geq 0$, and the parameter $\lambda > 0$ is used to modulate the strength of the field. The full Hamiltonian is then

$$H^\Lambda = H_\omega^\Lambda = H_\omega^\Lambda(\Delta, \lambda) = H_0^\Lambda(\Delta) + \lambda V_\omega^\Lambda = -\tfrac{1}{2\Delta} \mathbf{\Delta}^\Lambda + \mathcal{W}^\Lambda + \lambda V_\omega^\Lambda. \tag{2.5}$$

Remark 2.1.

1. The operator $\mathbf{\Delta}^\Lambda$ can be viewed as the analog of the Laplacian operator on $\mathcal{H}_\Lambda$.
2. $\mathcal{N}_i$ is diagonalized by the canonical basis for all $i \in \Lambda$: $\mathcal{N}_i \phi_A = \phi_A$ if $i \in A$ and 0 otherwise. It follows that the total number operator $\mathcal{N}_\Lambda$ is also diagonalized by the canonical basis: $\mathcal{N}_\Lambda \phi_A = |A| \phi_A$.
3. $\mathcal{W}^\Lambda$, the number of clusters operator, is diagonalized by the canonical basis: $\mathcal{W}^\Lambda \phi_A = W_A^\Lambda \phi_A$, where $W_A^\Lambda \in [0, |A|] \cap \mathbb{Z}$ is the number of clusters of A in Λ (i.e., the number of connected components of A in Λ (considered as a subgraph of $\mathbb{Z}$)).
4. V_ω^Λ is diagonalized by the canonical basis: $V_\omega^\Lambda \phi_A = \omega^{(A)} \phi_A$, where $\omega^{(A)} = \sum_{i \in A} \omega_i$.
5. The operators $\mathcal{N}_\Lambda$, $\mathcal{W}^\Lambda$, and V_ω^Λ commute.
6. The XXZ Hamiltonian H^Λ preserves the total particle number,

$$[H^\Lambda, \mathcal{N}_\Lambda] = -\tfrac{1}{2\Delta} [\mathbf{\Delta}^\Lambda, \mathcal{N}_\Lambda] = 0. \tag{2.6}$$

We will consider the XXZ model in the presence of a random transversal field; that is, $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ is a family of random variables. More precisely, we make the following definition.

Definition 2.2. The random XXZ spin Hamiltonian on $\Lambda \subset \mathbb{Z}$ is the operator $H^\Lambda = H_\omega^\Lambda(\Delta, \lambda)$ given in (2.5), where $\Delta > 1$, $\lambda > 0$, and $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ is a family of independent identically distributed random

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variables, whose common probability distribution μ satisfies

$$\{0,1\} \subset \operatorname{supp}\mu \subset [0,1] \tag{2.7}$$

and is assumed to be absolutely continuous with a bounded density.

From now on, H^Λ always denotes the random XXZ spin Hamiltonian on Λ. The corresponding resolvent is given by $R_E^\Lambda = \left(H^\Lambda - E\right)^{-1}$, which is well-defined for almost every energy $E \in \mathbb{R}$. We set $\omega_S = \{\omega_i\}_{i\in S}$ for $S \subset \mathbb{Z}$ and denote the corresponding expectation and probability by $\mathbb{E}_S$ and $\mathbb{P}_S$.

It is convenient to introduce the local interaction terms

$$h_{i,i+1} = -\mathcal{N}_i\mathcal{N}_{i+1} - \tfrac{1}{2\Delta}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right), \tag{2.8}$$

which allows us to rewrite

$$H_0^\Lambda = \sum_{\{i,i+1\}\subset\Lambda} h_{i,i+1} + \mathcal{N}_\Lambda. \tag{2.9}$$

It can be verified that on $\mathcal{H}_{\{i,i+1\}} = \mathcal{H}_i^2 \otimes \mathcal{H}_{i+1}^2$, we have

$$\tfrac{1}{2}(\mathcal{N}_i + \mathcal{N}_{i+1}) - \mathcal{N}_i\mathcal{N}_{i+1} \mp \tfrac{1}{2}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right) \geq 0, \tag{2.10}$$

which implies that $\mathcal{W}_\Lambda \pm \frac{1}{2}\mathbf{\Delta}_\Lambda \geq 0$; that is,

$$-2\mathcal{W}_\Lambda \leq -\mathbf{\Delta}_\Lambda \leq 2\mathcal{W}_\Lambda. \tag{2.11}$$

It follows that

$$\left(1 - \tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda \leq H_0^\Lambda \leq \left(1 + \tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda, \quad \text{so} \quad \left(1 - \tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda \leq H^\Lambda. \tag{2.12}$$

We conclude that the spectrum of H^Λ is of the form

$$\sigma(H^\Lambda) = \{0\} \cup \left(\left[1 - \tfrac{1}{\Delta}, \infty\right) \cap \sigma(H^\Lambda)\right). \tag{2.13}$$

The lower bound in (2.12) suggests the introduction of the energy thresholds $k\left(1 - \frac{1}{\Delta}\right)$, $k = 0,1,2\ldots$. We define the energy intervals

$$\begin{aligned} \widehat{I}_{\leq k} &= \left(-\infty, (k+1)\left(1 - \tfrac{1}{\Delta}\right)\right), \quad & \widehat{I}_k &= \left[1 - \tfrac{1}{\Delta}, (k+1)\left(1 - \tfrac{1}{\Delta}\right)\right), \\ I_{\leq k} &= \left(-\infty, (k + \tfrac{3}{4})\left(1 - \tfrac{1}{\Delta}\right)\right), \quad & I_k &= \left[1 - \tfrac{1}{\Delta}, (k + \tfrac{3}{4})\left(1 - \tfrac{1}{\Delta}\right)\right). \end{aligned} \tag{2.14}$$

We call $\widehat{I}_k$ the k-cluster spectrum.

Given $\emptyset \neq S \subset \Lambda$, we define the orthogonal projections $P_\pm^S$ on $\mathcal{H}_\Lambda$ by

$$P_+^S = \bigotimes_{i\in S}\left(\mathbb{1}_{\mathcal{H}_i} - \mathcal{N}_i\right) = \chi_{\{0\}}(\mathcal{N}_S) \quad \text{and} \quad P_-^S = \mathbb{1}_{\mathcal{H}_\Lambda} - P_+^S = \chi_{\mathbb{N}}(\mathcal{N}_S). \tag{2.15}$$

P_+^S is the orthogonal projection onto states with no particles in the set S; P_-^S is the orthogonal projection onto states with at least one particle in S. We also set

$$P_+^\emptyset = \mathbb{1}_{\mathcal{H}_\Lambda} \quad \text{and} \quad P_-^\emptyset = 0. \tag{2.16}$$

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Remark 2.3. In the Ising phase (i.e., $\Delta > 1$), we have (2.12) and (2.13) for all $\Lambda \subset \mathbb{Z}$. It follows that the XXZ chain Hamiltonian H^{Λ} has ground state Ω_{Λ} and the ground state energy is 0 ($H_{\Lambda}\Omega_{\Lambda}$=0), and, moreover, the ground state energy is gapped. This makes H^{Λ} a 2-local, gapped, frustration-free system. These features, plus the preservation of the total particle number, make the XXZ model especially amenable to analysis. In particular, the number of eigenstates of H^{Λ} in the intervals $I_{\le k}$ grows only polynomially in the volume of Λ (not exponentially as the dimension of $\mathcal{H}_{\Lambda}$) as shown in Lemma 3.5 below.

2.2. Main results

Our main result establishes quasi-locality for the resolvent of the random XXZ chain inside the spectrum of H^{Λ}.

Theorem 2.4 (Quasi-locality for resolvents). *Fix $\Delta_0 > 1$, $\lambda_0 > 0$, and let $s \in (0, \frac{1}{3})$. Then for all $k \in \mathbb{N}^0$, there exist constants $D_k, F_k, \xi_k, \theta_k > 0$ (depending on k, Δ_0, λ_0 and s) such that, for all $\Delta \ge \Delta_0$ and $\lambda \ge \lambda_0$ with $\lambda\Delta^2 \ge D_k$, $\Lambda \subset \mathbb{Z}$ finite, and energy $E \in I_{\le k}$, we have*

$$\mathbb{E}\left\{\left\|P_-^A R_E^\Lambda P_+^B\right\|^s\right\} \le F_k |\Lambda|^{\xi_k} \mathrm{e}^{-\theta_k \operatorname{dist}_\Lambda(A,B^c)}, \tag{2.17}$$

for $A \subset B \subset \Lambda$ with A connected in Λ.

The theorem is proven in Section 4.

Remark 2.5. If A is not connected in Λ, the theorem still holds with (2.17) replaced by

$$\mathbb{E}\left\{\left\|P_-^A R_E^\Lambda P_+^B\right\|^s\right\} \le F_k \Upsilon_A^\Lambda |\Lambda|^{\xi_k} \mathrm{e}^{-\theta_k \operatorname{dist}_\Lambda(A,B^c)}, \tag{2.18}$$

where Υ_A^Λ denotes the number of connected components of A in Λ. This follows from (2.17) and

$$P_-^A = \sum_{j=1}^{\Upsilon_A^\Lambda} P_+^{\bigcup_{i=i}^{j-1} A_i} P_-^{A_j}, \tag{2.19}$$

where A_j, $j = 1, 2, \ldots, \Upsilon_A^\Lambda$, are the connected components of A in Λ .

As a consequence of Theorem 2.4, we prove the following quasi-locality estimate for Borel functions of H^{Λ}. By $B(I_{\le k})$ we denote the collection of Borel functions on $\mathbb{R}$ that are equal to zero outside the interval $I_{\le k}$.

Corollary 2.6 (Quasi-locality for Borel functions). *Assume the hypotheses and conclusions of Theorem 2.4, Then for all $k \in \mathbb{N}^0$, there exist constants $\widetilde{F}_k, \widetilde{\xi}_k, \widetilde{\theta}_k > 0$ (depending on k, Δ_0, λ_0 and s) such that, for all $\Delta \ge \Delta_0$ and $\lambda \ge \lambda_0$ with $\lambda\Delta^2 \ge D_k$, and $\Lambda \subset \mathbb{Z}$ finite, we have*

$$\mathbb{E}_\Lambda\left(\sup_{\substack{f \in B(I_{\le k}):\\ \|f\|_\infty \le 1}} \left\|P_-^A f(H^\Lambda) P_+^B\right\|\right) \le \widetilde{F}_k |\Lambda|^{\widetilde{\xi}_k} \mathrm{e}^{-\widetilde{\theta}_k \operatorname{dist}_\Lambda(A,B^c)}, \tag{2.20}$$

for all $A \subset B \subset \Lambda$, A connected in Λ.

The proof of the Corollary is given in Section 5.

3. Key ingredients for the proofs

In this section, we collect a number of definitions, statements and lemmas that will facilitate the proof of Theorem 2.4.

Λ will always denote a finite subset of $\mathbb{Z}$, and $A \subset \Lambda$ will always denote a nonempty subset connected in Λ. ($B \subset \Lambda$, $S \subset \Lambda$, etc., may not be connected in Λ.)

3.1. Some definitions

- Given $M \subset \Lambda$ and $q \in \mathbb{Z}$, we define enlarged (for $q \geq 0$) and trimmed (for $q < 0$) set $[M]_q^\Lambda$ by

$$[M]_q^\Lambda := \begin{cases} \{x \in \Lambda : \operatorname{dist}_\Lambda(x, M) \leq q\} & \text{if } q \in \mathbb{N}^0 = \{0\} \cup \mathbb{N} \\ \{x \in \Lambda : \operatorname{dist}_\Lambda(x, M^c) \geq 1 - q\} = M \setminus [M^c]_{-q}^\Lambda & \text{if } q \in -\mathbb{N} \\ \{x \in \Lambda : \operatorname{dist}_\Lambda(x, M) < \infty\} = \bigcup_{p \in \mathbb{N}^0} [M]_p^\Lambda & \text{if } q = \infty \end{cases}. \tag{3.1}$$

Note that $[M]_{-|M|}^\Lambda = \emptyset$. Moreover, $[M]_\infty^\Lambda = [M]_{|\Lambda|-1}^\Lambda$ is the connected component of Λ containing M, and we have

$$[H^\Lambda, P_\pm^{[M]_\infty^\Lambda}] = 0. \tag{3.2}$$

We define $\partial_{ex}^\Lambda M$ (the external boundary of M in Λ), $\partial_{in}^\Lambda M$ (the inner boundary of M in Λ), and $\partial^\Lambda M$ (the boundary of M in Λ), by

$$\begin{aligned} \partial_{ex}^\Lambda M &:= \{x \in \Lambda : \ \operatorname{dist}_\Lambda(x, M) = 1\} = [M]_1^\Lambda \setminus M, \\ \partial_{in}^\Lambda M &:= \{x \in \Lambda : \ \operatorname{dist}_\Lambda(x, M^c) = 1\} = M \setminus [M]_{-1}^\Lambda, \\ \partial^\Lambda M &:= \partial_{in}^\Lambda M \cup \partial_{ex}^\Lambda M. \end{aligned} \tag{3.3}$$

It follows that

$$]M[_q^\Lambda := [M]_{q+1}^\Lambda \setminus [M]_q^\Lambda = \begin{cases} \partial_{ex}^\Lambda [M]_q^\Lambda, & q \in \mathbb{N}^0 \\ \partial_{in}^\Lambda [M]_{q+1}^\Lambda & q \in -\mathbb{N} \end{cases}, \tag{3.4}$$

and we have

$$]M[_p^\Lambda =]M^c[_{-p-1}^\Lambda \quad \text{for} \quad p \in \mathbb{Z}. \tag{3.5}$$

If $M = \{j\}$, we write $[j]_q^\Lambda = [\{j\}]_q^\Lambda$.

- Given $A \subset B \subset \Lambda$, we let $\rho^\Lambda(A, B)$ be the largest $q \in \mathbb{N}^0 \cup \{\infty\}$ such that $[A]_q^\Lambda \subset B$; that is,

$$\rho^\Lambda(A, B) = \sup\left\{q \in \mathbb{N}^0 : [A]_q^\Lambda \subset B\right\} = \operatorname{dist}_\Lambda(A, B^c) - 1. \tag{3.6}$$

It will be more convenient to use $\rho^\Lambda(A, B)$ instead of $\operatorname{dist}_\Lambda(A, B^c)$ in the proofs.Note that

$$\rho^\Lambda(A, B) = \infty \iff \operatorname{dist}_\Lambda(A, B^c) = \infty \iff [A]_\infty^\Lambda \subset B. \tag{3.7}$$

- It follows from (3.2) and (3.7) that

$$P_-^A R_E^\Lambda P_+^B = 0 \quad \text{if} \quad A \subset B \subset \Lambda \quad \text{and} \quad \rho^\Lambda(A, B) = \infty, \tag{3.8}$$

so it suffices to prove Theorem 2.4 for $\rho^\Lambda(A, B) < \infty$. Moreover, since $A \subset B$, we have $[A]_{\rho^\Lambda(A,B)}^\Lambda \subset B$, and hence,

$$\left\|P_-^A R_E^\Lambda P_+^B\right\| \leq \left\|P_-^A R_E^\Lambda P_+^{[A]_{\rho^\Lambda(A,B)}^\Lambda}\right\|, \tag{3.9}$$

so without loss of generality, it suffices to prove (2.17) for $B = [A]_\rho^\Lambda$ with $\rho \in \mathbb{N}^0$.

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∘ Given $K \subset \Lambda$, we consider the operator $H^K = H^K \otimes \mathbb{1}_{\mathcal{H}_{K^c}}$ acting on $\mathcal{H}_\Lambda$. We also consider the operators on $\mathcal{H}_\Lambda$ given by

$$H^{K,K^c} = H^K + H^{K^c}, \quad R_E^{K,K^c} = \left(H^{K,K^c} - E\right)^{-1}, \quad \Gamma^K = H^\Lambda - H^{K,K^c}. \tag{3.10}$$

3.2. *Quasi-locality for resolvents*

The following lemma and remark yields (deterministic) quasi-locality for the resolvent of the XXZ chain outside the spectrum of H^Λ.

Lemma 3.1. *Let $\Theta \subset \Lambda$, and consider the Hilbert space $\mathcal{H}_\Lambda$. Let the operator $T \in \mathcal{A}_\Lambda$ be of the form*

$$T = T^\Theta + T^{\Theta^c}; \quad \textit{where} \quad T^\Theta \in \mathcal{A}_\Theta \quad \textit{and} \quad T^{\Theta^c} \in \mathcal{A}_{\Theta^c}, \tag{3.11}$$

and let $\mathcal{X} \in \mathcal{A}_\Lambda$ be a projection such that $[\mathcal{X}, T] = 0$ and $[\mathcal{X}, P_\pm^K] = 0$ for all $K \subset \Theta$.

Suppose

1. *For all $K \subset \Theta$, we have $[P_-^K, T] P_+^{[K]_1^\Theta} = 0$.*
2. *For all $K \subset \Theta$, with K connected in Θ, we have $\left\| [P_-^K, T] \right\| \le \gamma$.*
3. *$T_\mathcal{X}$, the restriction of the operator T to* Ran $\mathcal{X}$, *is invertible with $\|T_\mathcal{X}^{-1}\|_{\operatorname{Ran} \mathcal{X}} \le \eta^{-1}$, where $\eta > 0$.*

Then for all $A \subset B \subset \Theta$, with A connected in Θ, we have

$$\left\| P_-^A \, T_\mathcal{X}^{-1} \, P_+^B \right\|_{\operatorname{Ran} \mathcal{X}} \le \eta^{-1} \mathrm{e}^{-m\rho^\Theta(A,B)}, \quad \textit{with} \quad m = \ln\left(\gamma^{-1}\eta\right). \tag{3.12}$$

Proof. We consider first the case $\mathcal{X} = \mathbb{1}_{\mathcal{H}_\Lambda}$. Let $A \subset B \subset \Theta$, with A connected in Θ. Let $1 \le t \le \rho^\Theta(A, B)$, so $[A]_t^\Theta \subset B$. We have

$$P_-^A \, T^{-1} \, P_+^B = T^{-1} [T, P_-^A] T^{-1} \, P_+^B = T^{-1} [T, P_-^A] P_-^{[A]_1^\Theta} T^{-1} \, P_+^B, \tag{3.13}$$

using condition (i) of the Lemma. Proceeding recursively, we get

$$P_-^A \, T^{-1} \, P_+^B = \left(\prod_{p=0}^{t-1} T^{-1} [T, P_-^{[A]_p^\Theta}] \right) P_-^{[A]_t^\Theta} T^{-1} \, P_+^B. \tag{3.14}$$

Since A is connected in Θ, $[A]_r^\Theta$, $r = 1, 2, \ldots, t$, are also connected in Θ. Using assumptions (ii) and (iii), we get

$$\left\| P_-^A \, T^{-1} \, P_+^B \right\| \le \left(\gamma \eta^{-1}\right)^t \eta^{-1}. \tag{3.15}$$

Since (3.15) holds for all $1 \le t \le \rho^\Theta(A, B)$, we get

$$\left\| P_-^A \, T^{-1} \, P_+^B \right\| \le \eta^{-1} \mathrm{e}^{-m\rho^\Theta(A,B)}, \quad \text{with} \quad m = \ln\left(\gamma^{-1}\eta\right). \tag{3.16}$$

If condition (iii) holds with a projection $\mathcal{X} \in \mathcal{A}_\Lambda$ such that $[\mathcal{X}, T] = 0$ and $[\mathcal{X}, P_\pm^K] = 0$ for all $K \subset \Theta$, then $\widetilde{T} = T\mathcal{X} + \eta(1 - \mathcal{X})$ satisfies conditions (i), (ii), and condition (iii) with $\mathcal{X} = \mathbb{1}_{\mathcal{H}_\Lambda}$, and the estimate (3.16) for $\widetilde{T}$ implies (3.12). □

Remark 3.2. Lemma 3.1 yields quasi-locality for the resolvent of the operator H^Λ. The operator $H^\Lambda - z$ satisfies the hypotheses of Lemma 3.1 for $z \notin \sigma(H^\Lambda)$, with $\Theta = \Lambda$, $\gamma = \frac{1}{\Delta}$ (use (A.6)), $\mathcal{X} = \mathbb{1}_{\mathcal{H}_\Lambda}$, and

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$\eta = \operatorname{dist}(z, \sigma(H^\Lambda))$. It follows that, with $R_z^\Lambda = (H^\Lambda - z)^{-1}$, for all $A \subset B \subset \Lambda$, we have

$$\left\| P_-^A R_z^\Lambda P_+^B \right\| \le \left(\operatorname{dist}(z, \sigma(H^\Lambda)) \right)^{-1} \mathrm{e}^{-m\rho^\Theta(A,B)}, \text{ with } m = \ln\left(\Delta \operatorname{dist}(z, \sigma(H^\Lambda))\right). \tag{3.17}$$

From now on, we fix $\Delta_0 > 5$, $\lambda_0 > 0$, and assume $\Delta \ge \Delta_0$ and $\lambda \ge \lambda_0$. The constants will depend on Δ_0 and λ_0.

Given $m \in \mathbb{N}^0$, we set $Q_m^\Lambda = \chi_{\{m\}}(\mathcal{W}^\Lambda)$, the orthogonal projection onto configurations with exactly m clusters, and let $Q_B^\Lambda = \chi_B(\mathcal{W}^\Lambda) = \sum_{m\in B} Q_m^\Lambda$ for $B \subset \mathbb{N}^0$. Note that $Q_0^\Lambda = P_+^\Lambda$ and $Q_{\mathbb{N}}^\Lambda = \chi_{\mathbb{N}}(\mathcal{N}^\Lambda)$. For $k \in \mathbb{N}$, we set

$$Q_{\le k}^\Lambda = Q_{\{1,2,\ldots,k\}}^\Lambda = \sum_{m=1}^{k} Q_m^\Lambda \quad \text{and} \quad \widehat{Q}_{\le k}^\Lambda = Q_{\le k}^\Lambda + \tfrac{k+1}{k} Q_0^\Lambda. \tag{3.18}$$

We also set

$$\begin{aligned} \widehat{H}_0^\Lambda &= H^\Lambda + \left(1 - \tfrac{1}{\Delta}\right) Q_0^\Lambda, \\ \widehat{H}_k^\Lambda &= H^\Lambda + k\left(1 - \tfrac{1}{\Delta}\right) \widehat{Q}_{\le k}^\Lambda \quad \text{for} \quad k \in \mathbb{N}. \end{aligned} \tag{3.19}$$

We use the notation

$$\widehat{R}_{k,E}^\Lambda = \left(\widehat{H}_k^\Lambda - E\right)^{-1} \text{ for } E \notin \sigma(\widehat{H}_k^\Lambda),\ k \in \mathbb{N}^0. \tag{3.20}$$

It follows from (2.12) and (2.14) that for $k \in \mathbb{N}^0$, we have

$$\widehat{H}_k^\Lambda \ge (k+1)\left(1 - \tfrac{1}{\Delta}\right) \quad \text{and} \quad \left(\widehat{H}_k^\Lambda - E\right) \ge \tfrac{1}{4}\left(1 - \tfrac{1}{\Delta}\right) \text{ for } E \in I_{\le k}. \tag{3.21}$$

For $k \in \mathbb{N}^0$ and $E \in I_{\le k}$, the operator $T = \widehat{H}_k^\Lambda - E$ satisfies the assumptions of Lemma 3.1 with $\Theta = \Lambda$, $\gamma = \frac{1}{\Delta}$, $\mathcal{X} = \mathbb{1}_{\mathcal{H}_\Lambda}$ and $\eta = \frac{1}{4}\left(1 - \frac{1}{\Delta}\right)$ (see (3.21)). In this case, $m = \ln \frac{\Delta-1}{4}$, and hence, for $A \subset B \subset \Lambda$, (3.12) yields

$$\left\| P_-^A \widehat{R}_{k,E}^\Lambda P_+^B \right\| \le \tfrac{4}{1-\frac{1}{\Delta}} \mathrm{e}^{-\left(\ln \frac{\Delta-1}{4}\right)\rho^\Lambda(A,B)}. \tag{3.22}$$

To have decay in (3.22), we need $\frac{\Delta-1}{4} > 1$; that is, $\Delta > 5$. In the proof of Theorem 2.4, we will fix $\Delta_0 > 5$ and $\lambda_0 > 0$ and require $\Delta \ge \Delta_0$ and $\lambda \ge \lambda_0$. In this case, we have $\frac{4}{1-\frac{1}{\Delta}} \le \frac{4}{1-\frac{1}{\Delta_0}}$ and $\ln \frac{\Delta-1}{4} \ge \ln \frac{\Delta_0-1}{4}$, so we have

$$\left\| P_-^A \widehat{R}_{k,E}^\Lambda P_+^B \right\| \le C_0 \mathrm{e}^{-m_0\rho^\Lambda(A,B)}, \text{ with } C_0 = \tfrac{4}{1-\frac{1}{\Delta_0}},\ m_0 = \ln \tfrac{\Delta_0-1}{4} > 0. \tag{3.23}$$

It follows from (3.2), which also holds for the operator $\widehat{H}_k^\Lambda$, that

$$P_-^M R_E^\Lambda P_+^{[M]_\infty^\Lambda} = 0 \quad \text{and} \quad P_-^M \widehat{R}_{k,E}^\Lambda P_+^{[M]_\infty^\Lambda} = 0 \quad \text{for} \quad M \subset \Lambda. \tag{3.24}$$

Remark 3.3. We will prove Theorem 2.4 with $\Delta_0 > 5$ to simplify our analysis. The proof can be extended to arbitrary $\Delta_0 > 1$ with minor modifications. Specifically, for $1 < \Delta_0 \le 5$, we need to improve the decay rate in (3.22), which is derived from the lower bound in (3.21). To do so, we would

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replace $\widehat{H}_k^\Lambda$ in the proof by $\widehat{H}_{k+r}^\Lambda$, where $r \in \mathbb{N}$, so (3.21) yields $\widehat{H}_{k+r}^\Lambda - E \geq (r+\frac{1}{4})\left(1-\frac{1}{\Delta}\right)$ for $E \in I_{\leq k}$, leading to $m_0 = \ln\left((r+\frac{1}{4})(\Delta_0 - 1)\right) > 0$ for an appropriate choice of r.

3.3. An a priori estimate

The first step toward the proof of Theorem 2.4 is to understand why the expression on the left-hand side of (2.17) is actually finite. A useful technical device for this purpose is the following bound, where $\|T\|_{HS}$ denotes the Hilbert-Schmidt norm of the operator T.

Lemma 3.4 (A priori estimate). *Let $i, j \in \Lambda$ ($i = j$ is allowed), and let T_1, T_2 be a pair of Hilbert-Schmidt operators on $\mathcal{H}_\Lambda$ that are $\omega_{\{i,j\}}$-independent. Then we have*

$$\mathbb{E}_{\{i,j\}}\left(\left\|T_1 \mathcal{N}_i R_E^\Lambda \mathcal{N}_j T_2\right\|_{HS}^s\right) \leq C\lambda^{-s}\|T_1\|_{HS}^s \|T_2\|_{HS}^s \text{ for all } E \in \mathbb{R} \text{ and } s \in (0,1). \tag{3.25}$$

The lemma follows from [5, Proposition 3.2], used with $U_1 = \mathcal{N}_j$, $U_2 = \mathcal{N}_k$ there, and the layer-cake representation for a non-negative random variable X_ω: $\mathbb{E}(X_\omega^s) = \int_0^\infty \mathbb{P}(X_\omega > t^{1/s})\, dt$ for $s \in (0,1)$.

The Hilbert-Schmidt operators for Lemma 3.4 are provided by the following result.

Lemma 3.5. *Let $k \in \mathbb{N}$. Then*

$$\left\|Q_{\leq k}^\Lambda\right\|_{HS} \leq \sqrt{k}|\Lambda|^k, \tag{3.26}$$

$$\operatorname{tr}\chi_{\widehat{I}_{\leq k}}(H^\Lambda) \leq k|\Lambda|^{2k} + 1. \tag{3.27}$$

Proof. For $m \geq 1$ and $N \geq 1$, we have the rough estimate

$$\operatorname{tr}\chi_N^\Lambda Q_m^\Lambda \leq |\Lambda|^m N^{m-1}. \tag{3.28}$$

Thus,

$$\operatorname{tr}\chi_N^\Lambda Q_{\leq k}^\Lambda \leq \sum_{m=1}^k |\Lambda|^m N^{m-1} = \frac{1}{N}\frac{(|\Lambda|N)^{k+1} - (|\Lambda|N)}{(|\Lambda|N)-1} \leq k|\Lambda|^k N^{k-1}. \tag{3.29}$$

It follows that

$$\operatorname{tr} Q_{\leq k}^\Lambda \leq k|\Lambda|^k \sum_{N=1}^{|\Lambda|} N^{k-1} \leq k|\Lambda|^{2k}. \tag{3.30}$$

To prove (3.27), let $\widehat{H}_k^\Lambda$ be as in (3.19), and note that (3.21) implies $\operatorname{tr}\chi_{\widehat{I}_{\leq k}}(\widehat{H}_k^\Lambda) = 0$. Since the spectral shift is bounded by the rank of the perturbation, it follows from (3.19) that

$$\operatorname{tr}\chi_{\widehat{I}_{\leq k}}(H^\Lambda) \leq \operatorname{tr}\chi_{\widehat{I}_{\leq k}}(\widehat{H}_k^\Lambda) + \operatorname{Rank}\left(k\left(1-\frac{1}{\Delta}\right)\widehat{Q}_{\leq k}^\Lambda\right) = \operatorname{tr}\widehat{Q}_{\leq k}^\Lambda = \operatorname{tr} Q_{\leq k}^\Lambda + 1. \tag{3.31}$$

□

Lemmas 3.4 and 3.5 yield the a priori estimate

$$\mathbb{E}_{\{i,j\}}\left\|Q_{\leq k}^\Lambda \mathcal{N}_i R_E^\Lambda \mathcal{N}_j Q_{\leq k}^\Lambda\right\|_{HS}^s \leq C\lambda^{-s}k^s|\Lambda|^{2sk} \text{ for all } i,j \in \Lambda \text{ and } s \in (0,1). \tag{3.32}$$

More generally, we have

$$\mathbb{E}_{\{A\cup B\}}\left\|Q_{\leq k}^\Lambda P_-^A R_E^\Lambda P_-^B Q_{\leq k}^\Lambda\right\|_{HS}^s \leq C\lambda^{-s}k^s|\Lambda|^{2sk}|A||B| \quad \text{for} \quad \emptyset \neq A, B \subset \Lambda. \tag{3.33}$$

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Those a priori estimates are only useful if we can 'dress' the resolvent with factors of $Q^{\Lambda}_{\leq k}$ on both sides. To be able to do so, we will decorate R^{Λ}_{E} with resolvents of positive operators that satisfy the quasi-locality property.

3.4. Dressing resolvents with Hilbert-Schmidt operators

For $k = 1, 2, \ldots$, and $E \in I_{\leq k}$, we use the resolvent identity

$$R^{\Lambda}_{E} = \widehat{R}^{\Lambda}_{k,E} + k\left(1 - \tfrac{1}{\Delta}\right) R^{\Lambda}_{E} \widehat{Q}^{\Lambda}_{\leq k} \widehat{R}^{\Lambda}_{k,E} = \widehat{R}^{\Lambda}_{k,E} + k\left(1 - \tfrac{1}{\Delta}\right) \widehat{R}^{\Lambda}_{k,E} \widehat{Q}^{\Lambda}_{\leq k} R^{\Lambda}_{E}. \tag{3.34}$$

Using it twice, we get

$$R^{\Lambda}_{E} = \widehat{R}^{\Lambda}_{k,E} + k\left(1 - \tfrac{1}{\Delta}\right) \widehat{R}^{\Lambda}_{k,E} \widehat{Q}^{\Lambda}_{\leq k} \widehat{R}^{\Lambda}_{k,E} + k^2\left(1 - \tfrac{1}{\Delta}\right)^2 \widehat{R}^{\Lambda}_{k,E} \widehat{Q}^{\Lambda}_{\leq k} R^{\Lambda}_{E} \widehat{Q}^{\Lambda}_{\leq k} \widehat{R}^{\Lambda}_{k,E}. \tag{3.35}$$

We use the notation $(p)_+ = \max(p, 0)$ for $p \in \mathbb{R}$.

Lemma 3.6. *Let $\mathcal{X}$ denote a spectral projection of $\mathcal{N}_\Lambda$ (say, $\mathcal{X} = \mathbb{1}_{\mathcal{H}_\Lambda}$ or $\mathcal{X} = \chi^{\Lambda}_{N}$). Let $A \subset B \subset \Lambda$, and $1 \leq t = \rho^{\Lambda}(A, B) < \infty$. Let $E \in I_{\leq k}$, and let m_0 be as in (3.23).*

1. *We have the following estimate on operator norms:*

$$\begin{aligned} \left\| \mathcal{X} P^{A}_{-} R^{\Lambda}_{E} P^{B}_{+} \right\| &\leq C_k \left(|\Lambda| \mathrm{e}^{-m_0 t} + \sum_{p=-|A|}^{|\Lambda|} \sum_{q=-|A|}^{|\Lambda|} \mathrm{e}^{-m_0 (p)_+} \mathrm{e}^{-m_0 (t-q-1)_+} \left\| \mathcal{X} F^{\Lambda}_{p,q}(E, A) \right\| \right), \\ \textit{where} \quad F^{\Lambda}_{p,q}(E, A) &= Q^{\Lambda}_{\leq k} P_{+}^{[A]^{\Lambda}_{p}} P_{-}^{]A[^{\Lambda}_{p}} R^{\Lambda}_{E} P_{+}^{[A]^{\Lambda}_{q}} P_{-}^{]A[^{\Lambda}_{q}} Q^{\Lambda}_{\leq k} \quad \textit{for} \quad p, q \in \mathbb{Z}. \end{aligned} \tag{3.36}$$

2. *We have the following estimates on Hilbert-Schmidt norms:*

$$\left\| \mathcal{X} P^{A}_{-} R^{\Lambda}_{E} P^{B}_{+} Q^{\Lambda}_{\leq k} \right\|_{HS} \leq C_k \left(|\Lambda|^k \mathrm{e}^{-m_0 t} + \sum_{q=-|A|}^{|\Lambda|} \mathrm{e}^{-m_0 (q)_+} \left\| \mathcal{X} Q^{\Lambda}_{\leq k} P_{-}^{]A[_q} R^{\Lambda}_{E} P^{B}_{+} Q^{\Lambda}_{\leq k} \right\|_{HS} \right). \tag{3.37}$$

Moreover, for $s \in (0, 1)$, we have

$$\mathbb{E}\left(\left\| \mathcal{X} P^{A}_{-} R^{\Lambda}_{E} P^{B}_{+} Q^{\Lambda}_{\leq k} \right\|^{s}_{HS} \right) \leq C_{k,s} |\Lambda|^{2sk+3}. \tag{3.38}$$

Proof. Let $A \subset B \subset \Lambda$, A connected in Λ. Since $\mathcal{X}$ commutes with all the relevants operators, we will just do the proof for $\mathcal{X} = I$.

Using (3.35), (3.18) and (3.23), we get

$$\left\| P^{A}_{-} R^{\Lambda}_{E} P^{B}_{+} \right\| \leq C_0 \mathrm{e}^{-m_0 t} + k \left\| P^{A}_{-} \widehat{R}^{\Lambda}_{k,E} Q^{\Lambda}_{\leq k} \widehat{R}^{\Lambda}_{k,E} P^{B}_{+} \right\| + k^2 \left\| P^{A}_{-} \widehat{R}^{\Lambda}_{k,E} Q^{\Lambda}_{\leq k} R^{\Lambda}_{E} Q^{\Lambda}_{\leq k} \widehat{R}^{\Lambda}_{k,E} P^{B}_{+} \right\|. \tag{3.39}$$

Using (3.24), (A.7) and the fact that $Q^{\Lambda}_{\leq k}$ commutes with $P_{\pm}$ operators, we get

$$\left\| P^{A}_{-} \widehat{R}^{\Lambda}_{k,E} Q^{\Lambda}_{\leq k+1} \widehat{R}^{\Lambda}_{k,E} P^{B}_{+} \right\| \leq \sum_{q=-|A|}^{|\Lambda|} \|D_q\| \|E_q\|, \tag{3.40}$$

where

$$D_q = P^{A}_{-} \widehat{R}^{\Lambda}_{k,E} P_{+}^{[A]_q} \text{ and } E_q = P_{-}^{]A[_q} \widehat{R}^{\Lambda}_{k,E} P^{B}_{+}. \tag{3.41}$$

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Using (3.21), (3.23) and $]A[_q \subset B$ for $q+1 \le t$, we get

$$\left\|D_q\right\| \le C_0 \mathrm{e}^{-m_0 (q)_+} \text{ and } \left\|E_q\right\| \le C_0 \mathrm{e}^{-m_0 (t-q-1)_+} \text{ for all } q \in \mathbb{Z}. \tag{3.42}$$

It follows that

$$\left\|P_-^A \widehat{R}_{k,E}^\Lambda Q_{\le k+1}^\Lambda \widehat{R}_{k,E}^\Lambda P_+^B\right\| \le C_0^2 \sum_{q=-|A|}^{|\Lambda|} \mathrm{e}^{-m_0 (q)_+} \mathrm{e}^{-m_0 (t-q-1)_+} \le C_0' |\Lambda| \mathrm{e}^{-m_0 t}. \tag{3.43}$$

This leaves us with the estimation of the last term in (3.39). To this end, we use (3.24), (A.7) and (3.42) to obtain

$$\begin{aligned} \left\|P_-^A \widehat{R}_{k,E}^\Lambda Q_{\le k+1}^\Lambda R_E^\Lambda Q_{\le k+1}^\Lambda \widehat{R}_{k,E}^\Lambda P_+^B\right\| &\le \sum_{p=-|A|}^{|\Lambda|} \sum_{q=-|A|}^{|\Lambda|} \|D_p\| \|F_{p,q}\| \|E_q\| \\ &\le C_0^2 \sum_{p=-|A|}^{|\Lambda|} \sum_{q=-|A|}^{|\Lambda|} \mathrm{e}^{-m_0 (p)_+} \mathrm{e}^{-m_0 (t-q-1)_+} \left\|F_{p,q}\right\|, \end{aligned} \tag{3.44}$$

where $F_{p,q} = F_{p,q}^\Lambda(E, A)$ is as in (3.36) for $p, q \in \mathbb{Z}$.

Combining (3.39), (3.43) and (3.44), we get (3.36).

To prove (3.37), we proceed as in (3.39) using (3.34), exploit $\|T_1 T_2\|_{HS} \le \|T_1\| \|T_2\|_{HS}$, and use (3.26), obtaining

$$\left\|P_-^A R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS} \le C_k \mathrm{e}^{-m_0 t} |\Lambda|^k + k \left\|P_-^A \widehat{R}_{k,E}^\Lambda Q_{\le k}^\Lambda R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS}. \tag{3.45}$$

We then use (3.24), (A.7) and (3.42) to get

$$\begin{aligned} \left\|P_-^A \widehat{R}_{k,E}^\Lambda Q_{\le k}^\Lambda R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS} &\le \sum_{q=-|A|}^{|\Lambda|} \left\|D_q\right\| \left\|Q_{\le k}^\Lambda P_-^{]A[_q} R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS} \\ &\le \sum_{q=-|A|}^{|\Lambda|} C_0 \mathrm{e}^{-m_0 (q)_+} \left\|Q_{\le k}^\Lambda P_-^{]A[_q} R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS}. \end{aligned} \tag{3.46}$$

Given $s \in (0,1)$, it follows from (3.37) and (3.33) that

$$\mathbb{E}\left(\left\|\mathcal{X} P_-^A R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS}^s\right) \le C_{k,s} |\Lambda|^{2sk+3}. \tag{3.47}$$

□

3.5. Large deviation estimate

Using a large deviation argument, we get the following refinement of (3.33). Recall we may assume $\rho^\Lambda(A,B) < \infty$ in view of (3.8).

Lemma 3.7. *Let $k \in \mathbb{N}$. Let $A \subset B \subset \Lambda$, with $\rho^\Lambda(A,B) < \infty$. Given $s \in (0, \frac{1}{2})$, there exist constants $C_{k,s}, c_\mu > 0$ such that for all $E \in I_{\le k}$, we have*

$$\mathbb{E}\left(\left\|\chi_N^\Lambda Q_{\le k}^\Lambda P_-^A R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|_{HS}^s\right) \le C_{k,s} |\Lambda|^{2(sk+1)} \left(\mathrm{e}^{-c_\mu N} + \mathrm{e}^{-m_0 \rho^\Lambda(A,B)}\right). \tag{3.48}$$

In particular,

$$\mathbb{E}\left\|\chi_N^\Lambda Q_{\le k}^\Lambda P_-^A R_E^\Lambda P_+^B Q_{\le k}^\Lambda\right\|^s \le C_{k,s} |\Lambda|^{2(sk+1)} \mathrm{e}^{-m_{0,\mu} \rho^\Lambda(A,B)} \quad \text{if } 8kN \ge \rho^\Lambda(A,B), \tag{3.49}$$

where $m_{0,\mu} > 0$.

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Proof. Recall $\mathcal{H}_\Lambda^{(N)} = \operatorname{Ran}\chi_N^\Lambda$, and let $\mathcal{H}_\Lambda^{(N,k)} = \operatorname{Ran}\chi_N^\Lambda Q_{\le k}^\Lambda$. Recall also that the restriction of V_ω^Λ to $\mathcal{H}_\Lambda^{(N)}$ is diagonalized by the canonical basis $\Phi_\Lambda^{(N)}$ as in Remark 2.1(iii).

Let us first assume that N is such that $N\lambda\bar{\mu} \ge 2k\left(1-\frac{1}{\Delta}\right)$, where $\bar{\mu}$ denotes the mean of the probability distribution μ (see Definition 2.2). The standard large deviation estimate (Cramer's Theorem) gives

$$\mathbb{P}\left\{\lambda\omega^{(M)} < k\left(1-\tfrac{1}{\Delta}\right)\right\} \le \mathbb{P}\left\{\omega^{(M)} < N\tfrac{\bar{\mu}}{2}\right\} \le \mathrm{e}^{-c_\mu N} \text{ for all } M \subset \Lambda \text{ with } |M| = N, \tag{3.50}$$

where c_μ is a constant depending only on the probability distribution μ. This implies that there exists $C_k > 0$ such that

$$\mathbb{P}\left\{\lambda\omega^{(M)} < k\left(1-\tfrac{1}{\Delta}\right)\right\} \le C_k\mathrm{e}^{-c_\mu N} \text{ for all } N \in \mathbb{N} \text{ and } \ M \subset \Lambda \text{ with } |M| = N. \tag{3.51}$$

It follows that for the event

$$\mathcal{B}_k^N = \left\{\exists M \subset \Lambda \text{ with } |M| = N,\ W_M^\Lambda = k \text{ and } \lambda\omega^{(M)} < k\left(1-\tfrac{1}{\Delta}\right)\right\}, \tag{3.52}$$

we have

$$\mathbb{P}_\Lambda\left(\mathcal{B}_k^N\right) \le C_k\mathrm{e}^{-c_\mu N}\operatorname{tr}Q_{\le k}^{\Lambda,N} \le C_k|\Lambda|^{2k}\mathrm{e}^{-c_\mu N} \quad \text{for} \quad N = 1,2\ldots,|\Lambda|, \tag{3.53}$$

where we also used Lemma 3.5. On the complementary event $\left(\mathcal{B}_k^N\right)^c$, we have

$$\lambda V_\omega\chi_N^\Lambda Q_{\le k}^\Lambda \ge k\left(1-\tfrac{1}{\Delta}\right)\chi_N^\Lambda Q_{\le k}^\Lambda. \tag{3.54}$$

If (3.54) holds, we conclude that

$$\begin{aligned} H^{\Lambda,N} &\ge \left(1-\tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda + \lambda V_\omega = \left(Q_{\le k}^{\Lambda,N} + Q_{\ge k+1}^{\Lambda,N}\right)\left(\left(1-\tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda + \lambda V_\omega\right) \\ &\ge \left(1-\tfrac{1}{\Delta}\right)Q_{\ge k+1}^{\Lambda,N}\mathcal{W}^\Lambda + Q_{\le k}^{\Lambda,N}\left(\left(1-\tfrac{1}{\Delta}\right)\mathcal{W}^\Lambda + \lambda V_\omega\right) \ge (k+1)\left(1-\tfrac{1}{\Delta}\right). \end{aligned} \tag{3.55}$$

We deduce that for $\omega \in \left(\mathcal{B}_k^N\right)^c$ and $E \in I_{\le k}$, we have

$$H^{\Lambda,N} - E \ge (k+1)\left(1-\tfrac{1}{\Delta}\right) - (k+\tfrac{3}{4})\left(1-\tfrac{1}{\Delta}\right) = \tfrac{1}{4}\left(1-\tfrac{1}{\Delta}\right). \tag{3.56}$$

Proceeding as in the derivation of (3.23), it follows from Lemma 3.1 and Remark 3.2 that for $\omega \in \left(\mathcal{B}_k^N\right)^c$, we have, for $A \subset B \subset \Lambda$ with A connected in Λ, that

$$\left\|\chi_N^\Lambda P_-^A R_E^\Lambda P_+^B\right\| \le C_0\mathrm{e}^{-m_0\rho^\Lambda(A,B)}. \tag{3.57}$$

Given $E \in I_{\le k}$, and letting $T = \chi_N Q_{\le k}^\Lambda P_-^A R_E^\Lambda P_+^B Q_{\le k}^\Lambda$, we obtain

$$\begin{aligned} \mathbb{E}(\|T\|_{HS}^s) &\le \mathbb{E}\left(\chi_{\mathcal{B}_k^N}\|T\|_{HS}^s\right) + \mathbb{E}\left(\chi_{(\mathcal{B}_k^N)^c}\|T\|_{HS}^s\right) \\ &\le \left(\mathbb{P}\left(\mathcal{B}_k^N\right)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\|T\|_{HS}^{2s}\right)\right)^{\frac{1}{2}} + C_0\mathrm{e}^{-m_0\rho^\Lambda(A,B)}\left\|\chi_N Q_{\le k}^\Lambda\right\|_{HS}^s \\ &\le C_{k,s}|\Lambda|^{2(sk+1)}\left(\mathrm{e}^{-\frac{1}{2}c_\mu N} + \mathrm{e}^{-m_0\rho^\Lambda(A,B)}\right), \end{aligned} \tag{3.58}$$

where we used (3.53), Lemma 3.5 and (3.33) with $2s$ instead of s. This estimate is (3.48), up to a redefinition of the constant c_μ.

The estimate (3.49) follows immediately from (3.48). □

3.6. Decoupling of resolvents

We now illustrate the basic idea that allows us to obtain the exponential decay of the left-hand side in (2.17), analogous to the decoupling argument in the single particle localization literature. For this purpose, we will consider a more convenient object than the one in (2.17). To do so, let $A \subset M \subset B \subset \Lambda$, and consider $P_+^{M^c} P_-^A R_E^\Lambda P_+^B$. Let $K \subset \mathbb{Z}$ be such that $M \subset [K]_{-1} \subset K \subset [K]_1 \subset B$. The resolvent identity yields (recall (3.10))

$$\begin{aligned} P_+^{M^c} P_-^A R_E^\Lambda P_+^B &= -P_+^{M^c} P_-^A R_E^{K,K^c} \Gamma^K R_E^\Lambda P_+^B = -P_+^{M^c} P_-^A R_E^{K,K^c} P_+^{K^c} \Gamma^K R_E^\Lambda P_+^B \\ &= -P_+^{M^c} P_-^A R_E^K P_+^{K^c} \Gamma^K R_E^\Lambda P_+^B, \end{aligned} \tag{3.59}$$

where we used that $P_-^A R_E^{K,K^c} P_+^K = 0$ by (3.2) since $[A]_\infty^K \subset K$, $P_+^{M^c} R_E^{K,K^c} = P_+^{M^c} R_E^{K,K^c} P_+^{K^c}$ by (3.2) since $K^c \subset M^c$, and $R_E^{K,K^c} P_+^{K^c} = R_E^K P_+^{K^c}$. Using the specific structure of the XXZ Hamiltonian – that is, (A.3)–(A.5) – we have $P_+^{K^c}\Gamma^K = P_+^{K^c} P_-^{\partial^\Lambda K}\Gamma^K P_-^{\partial^\Lambda K} = P_+^{K^c} P_-^{\partial_{in}^\Lambda K}\Gamma^K P_-^{\partial_{ex}^\Lambda K}$, so it follows from (3.59) that

$$P_+^{M^c} P_-^A R_E^\Lambda P_+^B = -P_+^{M^c} P_-^A R_E^K P_-^{\partial_{in}^\Lambda K} P_+^{K^c} \Gamma^K P_-^{\partial_{ex}^\Lambda K} R_E^\Lambda P_+^B. \tag{3.60}$$

We now use the resolvent identity for the operator $H^{[K]_1^\Lambda,([K]_1^\Lambda)^c}$ and (A.3), obtaining

$$P_-^{\partial_{ex}^\Lambda K} R_E^\Lambda P_+^B = -P_-^{\partial_{ex}^\Lambda K} R_E^\Lambda P_-^{\partial_{ex}^\Lambda K} \Gamma^{[K]_1} P_-^{\partial_{ex}^\Lambda [K]_1} P_+^{[K]_1} R_E^{[K]_1^c} P_+^B. \tag{3.61}$$

Combining (3.60)–(3.61), we obtain

$$\begin{aligned} &P_+^{M^c} P_-^A R_E^\Lambda P_+^B = \\ &(P_-^A P_+^{M^c \cap K} R_E^K P_-^{\partial_{in}^\Lambda K}) P_+^{K^c} \Gamma^K \left(P_-^{\partial_{ex}^\Lambda K} R_E^\Lambda P_-^{\partial_{ex}^\Lambda K}\right) \Gamma^{[K]_1} P_+^{[K]_1} \left(P_-^{\partial_{ex}^\Lambda [K]_1} R_E^{[K]_1^c} P_+^{B \cap [K]_1^c}\right). \end{aligned} \tag{3.62}$$

This is the basic decoupling formula, in a sense that the expressions in the first and last parentheses on the last line are statistically independent and of the same form as the left-hand side of (2.17). So, if we can perform the averaging over the random variables at sites $r \in \partial_{ex}^\Lambda K$ to get rid of the middle resolvent, we will effectively decouple the system into pieces supported by the disjoint subsets K and $[K]_1^c$. (Note that these pieces do not depend on the random variables at sites $r \in \partial_{ex}^\Lambda K$.) This decoupling will be performed using the a priori estimate (3.33), after we dress the corresponding resolvents with Hilbert-Schmidt operators on both sides as in Lemma 3.6. In broad strokes, we then will extract the (initial) exponential decay from the expression in the first parenthesis in (3.62) using reduction to lower energies and obtain the full exponential decay using a sub-harmonicity argument. We flesh out details of this process as we proceed with the proof.

3.7. Clusters classification

In preparation to initiate the FMM, we first inspect the structure of states in $\operatorname{Ran} Q_{\le k}^\Lambda$. Since $Q_{\le k}^\Lambda$ is a multiplication operator in the canonical basis $\left\{\Phi_\Lambda^{(N)}\right\}_{N=0}^{|\Lambda|}$ introduced in (2.1), we just need to consider the elements φ_M of this basis with M that belong to a set $\mathcal{S}_{N,k}^\Lambda := \left\{M \subset \Lambda : \ |M| = N, \ 1 \le W_M^\Lambda \le k\right\}$, $N \ge 1$. (Recall that W_M^Λ is the number of clusters of the configuration M – that is, the number of connected components of M in the graph Λ.) Denoting by π_φ the orthogonal projection onto $\mathbb{C}\varphi$, given

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$M \in \mathcal{S}^{\Lambda}_{N,k}$, we abuse the notation and write π_M for π_{φ_M}, so $\pi_M = \left(\prod_{j\in M} \mathcal{N}_j\right) P_+^{M^c}$, and note that $\chi^{\Lambda}_N Q^{\Lambda}_{\le k} = \sum_{M\in\mathcal{S}^{\Lambda}_{N,k}} \pi_M$.

Given $A \subset \Lambda$, we set $\mathcal{S}^{\Lambda,A}_{N,k} = \left\{M \in \mathcal{S}^{\Lambda}_{N,k} : M \cap A \neq \emptyset\right\}$, and note that $\chi^{\Lambda}_N Q^{\Lambda}_{\le k} P^A_- = \sum_{M\in\mathcal{S}^{\Lambda,A}_{N,k}} \pi_M$. We set

$$\gamma_A(M) = \max_{x\in M} \operatorname{dist}_{\Lambda}(x, A) \le \operatorname{diam}_{\Lambda}(M) = \max_{x,y\in M} \operatorname{dist}_{\Lambda}(x, y) \quad \text{for} \quad M \in \mathcal{S}^{\Lambda,A}_{N,k}. \tag{3.63}$$

Note that $\operatorname{diam}_{\Lambda}(M) = N - 1$ for $k = 1$ and $\operatorname{diam}_{\Lambda}(M) \ge N \ge 2$ for $k \ge 2$.

If $8kN < \rho^{\Lambda}(A, B)$, we will use the following lemma.

Lemma 3.8. *Fix $k \ge 2$. Let $A \subset B \subset \Lambda$ be such that $8kN < \rho^{\Lambda}(A, B) < \infty$, and let $M \in \mathcal{S}^{\Lambda,A}_{N,k}$.*

1. *Suppose $\gamma_A(M) < 4kN$. Then setting $Z = [A]^{\Lambda}_{6kN}$, we have*

$$A \cup M \subset [Z]_{-1} \subset Z \subset [Z]_1 \subset B;\ \rho^{\Lambda}(A \cup M, Z) \ge 2kN; \quad \rho^{\Lambda}(Z, B) \ge 2kN. \tag{3.64}$$

2. *Suppose $\rho^{\Lambda}(A, B) \le 2\gamma_A(M)$. Let $d_\rho := \left\lfloor \frac{\rho^{\Lambda}(A,B)}{6k} \right\rfloor$. Then there exists $a \in \{1, 2, \ldots, 3k - 1\}$, such that, letting $K = [A]^{\Lambda}_{ad_\rho}$, we have*

$$\rho^{\Lambda}\left(\partial^{\Lambda} K, \Lambda \setminus M\right) \ge d_\rho - 1. \tag{3.65}$$

   Moreover, letting $M_1 = M \cap K$ and $M_2 = M \cap K^c$, we have $K \subset B$ and $M_i \neq \emptyset$ for $i = 1, 2$.

3. *Suppose $8kN < 2\gamma_A(M) < \rho^{\Lambda}(A, B)$. Let $d_\gamma := \left\lfloor \frac{\gamma_A(M)}{3k} \right\rfloor$. Then there exists $a \in \{1, 2, \ldots, 3k - 1\}$, such that, letting*

$$K = [A]^{\Lambda}_{ad_\gamma} \cup \left([A]^{\Lambda}_{\gamma_A(M)+d_\gamma} \setminus [A]^{\Lambda}_{ad_\gamma+1}\right), \tag{3.66}$$

   we have

$$\rho^{\Lambda}\left(\partial^{\Lambda} K, \Lambda \setminus M\right) \ge d_\gamma - 1. \tag{3.67}$$

   Moreover, letting $M_1 = M \cap [A]^{\Lambda}_{jd_\gamma}$ and $M_2 = M \cap [A]^{\Lambda}_{\gamma_A(M)} \setminus [A]^{\Lambda}_{jd_\gamma+1}$, we have $M_1 \cup M_2 = M \subset K \subset B$ and $M_i \neq \emptyset$ for $i = 1, 2$.

Proof. Part (i) is obvious. To prove Parts (ii) and (iii), let $d = d_\rho$ in Part (ii), and $d = d_\gamma$ in Part (ii); note that $d \ge N$ in both cases. We set $Y_a = [A]^{\Lambda}_{ad} \setminus [A]^{\Lambda}_{(a-1)d} \subset B$ for $a = 1, 2, \ldots, 3k$; note $3kd \le \frac{\rho^{\Lambda}(A,B)}{2}$ in both cases.

The set M consists of s clusters where $2 \le s \le k$, so $N \ge 2$. Each cluster has length $\le N - 1$, so it can intersect at most two of the Y_a's (as $d \ge N$); hence, M can intersect at most $2k$ of the distinct Y_a's. Thus, there exists $a_* \in \{1, 2, \ldots, 3k - 1\}$ such that

$$M \cap \left(Y_{a_*} \cup Y_{a_*+1}\right) = \emptyset, \tag{3.68}$$

and $M_1 = M \cap [A]^{\Lambda}_{(a_*-1)d} \neq \emptyset$ since $A \cap M \neq \emptyset$.

To prove Part (ii) with $d = d_\rho$, set $K = [A]^{\Lambda}_{a_* d_\rho} \subset B$. Then $M_1 = M \cap K \neq \emptyset$ since $A \cap M \neq \emptyset$, and $M_2 = M \cap (\Lambda \setminus K) \neq \emptyset$ as $\rho^{\Lambda}(A, B) \le 2\gamma_A(M)$ by hypothesis. Moreover, (3.65) holds due to (3.68).

To prove Part (iii) with $d = d_\gamma$, let K be given in (3.66). Then $M_1 = M \cap K \neq \emptyset$ since $A \cap M \neq \emptyset$, and $M_2 = M \cap (\Lambda \setminus K) \neq \emptyset$ as $\rho^{\Lambda}(A, B) \le 2\gamma_A(M)$ by hypothesis. Moreover, (3.65) holds due to (3.68). □

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Motivated by Lemma 3.8, given $A \subset B \subset \Lambda$ with $8kN < \rho^{\Lambda}(A,B) < \infty$, we decompose $\mathcal{S}_{N,k}^{\Lambda,A}$ into three distinct groups:

1. *Small* $\gamma_A(M)$: $M \in \mathcal{G}_1^{\Lambda,N}(A,B)$ if $2\gamma_A(M) \le 8kN < \rho^{\Lambda}(A,B)$.
2. *Large* $\gamma_A(M)$: $M \in \mathcal{G}_2^{\Lambda,N}(A,B)$ if $8kN < \rho^{\Lambda}(A,B) \le 2\gamma_A(M)$.
3. *Intermediate* $\gamma_A(M)$: $M \in \mathcal{G}_3^{\Lambda,N}(A,B)$ if $8kN < 2\gamma_A(M) < \rho^{\Lambda}(A,B)$.

Note that for $8kN < \rho^{\Lambda}(A,B) < \infty$, we have

$$\chi_N^{\Lambda} Q_{\le k}^{\Lambda} P_-^A = \sum_{1=1}^{3} \pi_{\mathcal{G}_i^{\Lambda,N}(A,B)}, \quad \text{where} \quad \pi_{\mathcal{G}_i^{\Lambda,N}(A,B)}, = \sum_{M \in \mathcal{G}_i^{\Lambda,N}(A,B)} \pi_M. \tag{3.69}$$

3.8. Decoupling revisited

We will need to estimate $\chi_N^{\Lambda} Q_{\le k}^{\Lambda} P_-^A R_E^{\Lambda} P_+^B Q_{\le k}^{\Lambda}$. If $8kN \ge \rho^{\Lambda}(A,B)$, we use (3.49). If $8kN < \rho^{\Lambda}(A,B)$, we note that

$$\pi_M \chi_N^{\Lambda} Q_{\le k}^{\Lambda} P_-^A R_E^{\Lambda} P_+^B Q_{\le k}^{\Lambda} = \pi_M P_-^A R_E^{\Lambda} P_+^B Q_{\le k}^{\Lambda} \quad \text{for} \quad M \in S_{N,k}^{\Lambda,A}. \tag{3.70}$$

We will use different strategies for $M \in \mathcal{G}_i = \mathcal{G}_i^{\Lambda,N}(A,B)$, $i = 1,2,3$.

If $M \in \mathcal{G}_1$, we use the decoupling argument of Section 3.6, getting (3.62) with $K = [A]_{8kN}^{\Lambda}$. The estimation for the expression in the first parenthesis in (3.62) will be performed using directly the a priori estimate (3.48) and (3.64). (No energy reduction.) This yields exponential decay in $\gamma_A(M)$ for this type of contributions, and the sub-harmonicity argument concludes the analysis.

To handle $M \in \mathcal{G}_2$, we consider K, M_1, M_2 as in Lemma 3.8(ii), set $S = [\partial K]_{d_\gamma - 1}^{\Lambda}$, and note that

$$\pi_M P_-^A R_E^{\Lambda} P_+^B Q_{\le k}^{\Lambda} = \pi_M P_+^S P_-^K P_-^{K^c} P_-^A R_E^{\Lambda} P_+^B Q_{\le k}^{\Lambda}. \tag{3.71}$$

Using $M_1 \subset B$, we get

$$P_+^S P_-^K P_-^{K^c} R_E^{\Lambda} P_+^B = -\left(P_+^S P_-^K P_-^{K^c} R_E^{K,K^c} P_-^{\partial^\Lambda K} \right) \Gamma^{\mathcal{K}} P_-^{\partial^\Lambda K} R_E^{\Lambda} P_+^B. \tag{3.72}$$

The expression in parenthesis is estimated by reduction to lower energies $E' \in I_{\le k-1}$, allowing the use of the induction hypothesis (in k) together with the estimate (3.65) to obtain exponential decay in $\rho^{\Lambda}(A,B)$.

If $M \in \mathcal{G}_3$, we use a decoupling based on Lemma 3.8(iii), we get exponential decay in $\gamma_A(M)$ from the induction hypothesis (in k), and the sub-harmonicity argument concludes the analysis.

3.9. Reduction to lower energies

We first observe that $P_-^A R_E^{\Lambda} P_+^B = P_-^A \widehat{R}_{0,E}^{\Lambda} P_+^B$ decays exponentially in $\rho^{\Lambda}(A,B)$ for $E \le \frac{3}{4}\left(1 - \frac{1}{\Delta}\right)$ due to (3.23) with $k = 0$; that is, Theorem 2.4 holds for k=0. Suppose now that we already established (2.17) for all energies $E \in I_{\le k-1}$ and we want to push the allowable energies to the interval $I_{\le k}$. The principal idea here is to observe that if $\emptyset \ne K \subsetneq \Lambda$, then we have the nontrivial decoupling $H^{K,K^c} = H^K + H^{K^c}$, and R_E^{K,K^c} can be decomposed as

$$R_E^{K,K^c} = \sum_{\nu \in \sigma(H^{K^c})} R_{E-\nu}^K \otimes \pi_{\kappa_\nu}, \tag{3.73}$$

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where $\{\kappa_\nu\}_{\nu\in\sigma(H^{K^c})}$ is an orthonormal basis for $\mathcal{H}_{K^c}$ that diagonalizes H^{K^c}: $H^{K^c}\kappa_\nu = \nu\kappa_\nu$. In particular, if $K_1 \subset K$ and $K_2 \subset K^c$, we deduce that

$$P_-^{K_1}P_-^{K_2}R_E^{K,K^c} = \sum_{\nu\in\sigma(H^{K^c})\cap[1-\frac{1}{\Delta},\infty)} \left(P_-^{K_1}R_{E-\nu}^K\right)\otimes\left(P_-^{K_2}\pi_{\kappa_\nu}\right), \tag{3.74}$$

since $P_-^{K_2}\pi_{\kappa_0} = 0$, and we have $\min_{\nu\in\sigma(H^{K^c})\setminus\{0\}} \nu \geq 1-\frac{1}{\Delta}$. This is exactly the type of setup we have in (3.71)–(3.72). It means that the factor $P_-^{K_1}P_-^{K_2}$ allows us effectively to lower the energy $E \in I_{\leq k}$ to $E-\nu \in I_{\leq k-1}$ and therefore use the induction hypothesis to obtain exponential decay (we of course still need to control the summation over ν on the right-hand side of (3.74)).

4. Proof of the main theorem

In this section, we prove Theorem 2.4. We fix $\Delta_0 > 5$ and $\lambda_0 > 0$, and assume $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$. As discussed in Remark 3.3, the argument can be modified for $\Delta_0 > 1$.

The proof proceeds by induction on k. Theorem 2.4 holds for $k = 0$, since in this case, (2.17) follows from (3.23) with $F_0 = C_0$, $\xi_0 = 0$ and $\theta_0 = m_0$ as $P_-^A R_E^\Lambda = P_-^A R_{0,E}^\Lambda$. Given $k \in \mathbb{N}$, we assume the theorem holds for $k-1$, and we will prove the theorem holds for k.

We now fix $k \in \mathbb{N}$ and $\Lambda \subset \mathbb{Z}$, finite and nonempty. We also fix $A \subset B \subset \Lambda$, where A is a nonempty subset connected in Λ; it follows that $[A]_p^\Lambda$ is also connected in Λ and $\left|]A[_p^\Lambda\right| \leq 2$ for all $p \in \mathbb{Z}$.

To derive the bound (2.17) from Lemma 3.6(i), we will estimate $\mathbb{E}\left(\left\|F_{p,q}^\Lambda(E,A)\right\|_{HS}^s\right)$ for $p,q = -|A|, -|A|+1, \ldots, |\Lambda|$ for $E \in I_{\leq k}$, where $F_{p,q}^\Lambda(E,A)$ is given in (3.36). The estimate (3.33) gives the a priori bound ($F_{p,q} = F_{p,q}^\Lambda(E,A)$)

$$\mathbb{E}\left\|F_{p,q}\right\|_{HS}^s \leq C\lambda_0^{-s}k^s|\Lambda|^{2sk+2}. \tag{4.1}$$

Since $F_{p,q} = F_{q,p}^*$, we may assume $p \leq q$. If $p = q$, we use (4.1); if $p < q$, we note that

$$\left\|F_{p,q}\right\|_{HS} \leq \left\|Q_{\leq k}^\Lambda P_-^{]A[_p^\Lambda} R_E^\Lambda P_+^{[A]_q^\Lambda} Q_{\leq k}^\Lambda\right\|_{HS} \leq \sum_{j\in]A[_p^\Lambda} \left\|Q_{\leq k}^\Lambda \mathcal{N}_j R_E^\Lambda P_+^{[j]_{q-p-1}^\Lambda} Q_{\leq k}^\Lambda\right\|_{HS}, \tag{4.2}$$

where we used $[]A[_p^\Lambda]_{q-p-1}^\Lambda \subset [A]_q^\Lambda$ for $p < q$.

For $r \in \mathbb{N}^0$ and $E \in I_{\leq k}$, we set

$$f^\Lambda(k,E,r) = \max_{\Theta\subset\Lambda}\max_{j\in\Theta} \mathbb{E}\left(\left\|Q_{\leq k}^\Theta \mathcal{N}_j R_E^\Theta P_+^{[j]_r^\Theta} Q_{\leq k}^\Theta\right\|_{HS}^s\right) \tag{4.3}$$

and prove the following lemma.

Lemma 4.1. *Let $k \in \mathbb{N}$, $s \in (0,\frac{1}{3})$, and assume Theorem 2.4 holds for $k-1$. Then there exist constants $D_k, C_k, \zeta_k, m_k > 0$ (depending on k, Δ_0, λ_0 and s), such that such that, for all $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ with $\lambda\Delta^2 \geq D_k$, $\Lambda \subset \mathbb{Z}$ finite, energy $E \in I_{\leq k}$, and $r \in \mathbb{N}^0$, we have*

$$f^\Lambda(k,E,r) \leq C_k|\Lambda|^{\zeta_k}\mathrm{e}^{-m_k r}. \tag{4.4}$$

To finish the proof of the theorem, we assume that $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ with $\lambda\Delta^2 \geq D_k$ as in the lemma. Then, since $\mathbb{E}\left(\left\|F_{p,q}\right\|\right)_{HS}^s \leq 2f^\Lambda(k,E,|q-p|-1)$ for $|q-p| \geq 1$, and we have (4.1) for $q = p$, we obtain

$$\mathbb{E}\left(\sum_{p=-|A|}^{|\Lambda|}\sum_{q=-|A|}^{|\Lambda|} \mathrm{e}^{-m_0(p)_+}\mathrm{e}^{-m_0(t-q-1)_+}\left\|F_{p,q}\right\|\right)^s \leq C_k|\Lambda|^{\zeta_k}\mathrm{e}^{-sm_k t}. \tag{4.5}$$

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The estimate (2.17) now follows from (3.36) and (4.5) (recall (3.6)), so Theorem 2.4 holds for k.

To complete the proof of Theorem 2.4, we need to prove Lemma 4.1. To do so, we need the following lemma.

Lemma 4.2. *Let $k \in \mathbb{N}$, $s \in (0, \frac{1}{3})$, and assume Theorem 2.4 holds for $k-1$. Then there exist constants $C_k, \zeta_k, m_k > 0$ (depending on k, Δ_0, λ_0 and s), such that, for all $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$, $j \in \Lambda \subset \mathbb{Z}$ finite, energy $E \in I_{\leq k}$, $N \in \mathbb{N}$, and $r \in \mathbb{N}^0$ such that $8kN < r$, we have*

$$G_N^\Lambda(r) = \mathbb{E}\left(\left\|\chi_N^\Lambda Q_{\leq k}^\Lambda \mathcal{N}_j R_E^\Lambda P_+^{[j]_r^\Lambda} Q_{\leq k}^\Lambda\right\|_{HS}^s\right) \leq C_k\left(|\Lambda|^{\zeta_k} \mathrm{e}^{-m_k r} + \mathrm{e}^{-m_k N}\left(\lambda\Delta^2\right)^{-s} \sum_{p=0}^{r} \mathrm{e}^{-m_k(r-p)} f_N^\Lambda(p)\right). \tag{4.6}$$

Proof. Let $k \in \mathbb{N}$, $s \in (0, \frac{1}{3})$, and assume Theorem 2.4 holds for $k-1$. Let $j \in \Lambda \subset \mathbb{Z}$ finite and $E \in I_{\leq k}$, $N \in \mathbb{N}$, and $r \in \mathbb{N}^0$ such that $8kN < r$. Let $G_N^\Lambda(r)$ be as in (4.6). It follows from (3.69), setting $\mathcal{G}_i^N = \mathcal{G}_i^{\Lambda,N}(\{j\}, [j]_r^\Lambda)$, $i = 1, 2, 3$ (see Section 3.7), that

$$G_N^\Lambda(r) \leq \sum_{i=1}^{3} G_i(r), \text{ where } G_i(r) = G_i^{\Lambda,N}(r) = \mathbb{E}\left(\left\|\pi_{\mathcal{G}_i^N} \mathcal{N}_j R_E^\Lambda P_+^{[j]_r^\Lambda} Q_{\leq k}^\Lambda\right\|_{HS}^s\right). \tag{4.7}$$

To estimate $G_1(r)$, we use (3.62) with $M = [j]_{4kN}^\Lambda$ and $K = [j]_{6kN}^\Lambda$, (3.25) and (A.6), obtaining

$$\begin{aligned} &G_1(r) \leq C\left(\lambda\Delta^2\right)^{-s} \mathbb{E}_K\left(\|Y\|_{HS}^s\right) \mathbb{E}_{([K]_1^\Lambda)^c}\left(\|Z\|_{HS}^s\right);\\ &Y := \chi_N^K Q_{\leq k}^K P_+^{K\setminus M} R_E^K P_-^{\partial_{in}^\Lambda K}, \quad Z := P_-^{\partial_{ex}^\Lambda [K]_1} R_E^{([K]_1^\Lambda)^c} P_+^{[j]_r^\Lambda \cap ([K]_1^\Lambda)^c} Q_{\leq k}^{([K]_1^\Lambda)^c} \chi_N^{([K]_1^\Lambda)^c}. \end{aligned} \tag{4.8}$$

To estimate $\mathbb{E}_K\left(\|Y\|_{HS}^s\right)$, note that

$$\|Y\|_{HS} \leq \sum_{u \in \partial_{in}^\Lambda K} \|Y_u\|_{HS}, \text{ where } Y_u = \chi_N^K Q_{\leq k}^K P_+^{K\setminus M} R_E^K \mathcal{N}_u, \text{ and } \left|\partial_{in}^\Lambda K\right| \leq 2. \tag{4.9}$$

Using (3.37) and $\rho^K(\partial_{in}^\Lambda K, K \setminus M) \geq 2kN$, for $u \in \partial_{in}^\Lambda K$, we get

$$\begin{aligned} \mathbb{E}_K\left(\|Y_u\|_{HS}^s\right) &\leq C_k^s\left(|K|^{sk} \mathrm{e}^{-sm_0 2kN} + \sum_{q=-1}^{|K|} \mathrm{e}^{-sm_0(q)_+} \mathbb{E}\left(\left\|\chi_N^K Q_{\leq k}^K P_-^{]u[_q^K} R_E^K P_+^{K\setminus M} Q_{\leq k}^K{}_{HS}\right\|^s\right)\right)\\ &\leq C_{k,s}\left(|K|^{2sk+1} \mathrm{e}^{-sm_0 2kN} + 2\sum_{q=-1}^{2kN-1} \mathrm{e}^{-sm_0(q)_+} f_N^K(2kN - q - 1)\right)\\ &\leq C_{k,s} \mathrm{e}^{-m_{0,k}' kN}, \end{aligned} \tag{4.10}$$

where we used the a priori bounds (3.33) and (3.48).

Similarly,

$$\|Z\|_{HS} \leq \sum_{u \in \partial_{ex}^\Lambda [K]_1} \|Z_u\|_{HS}, \text{ where } Z_u = \mathcal{N}_u R_E^{([K]_1^\Lambda)^c} P_+^{[j]_r^\Lambda \cap ([K]_1^\Lambda)^c} Q_{\leq k}^{([K]_1^\Lambda)^c} \chi_N^{([K]_1^\Lambda)^c}, \tag{4.11}$$

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and $\left|\partial_{ex}^{\Lambda}[K]_1\right| \le 2$. Using (3.37), for $u \in \partial_{ex}[K]_1$, we get

$$\begin{aligned}
&\mathbb{E}_{([K]_1^\Lambda)^c}\left(\|Z_u\|_{HS}^s\right) \le C_k^s\Big(|\Lambda|^{sk}\mathrm{e}^{-sm_0(r-6kN-2)}+\\
&\quad\sum_{q=-1}^{|\Lambda|}\mathrm{e}^{-sm_0(q)_+}\mathbb{E}\Big(\Big\|\chi_N^{([K]_1^\Lambda)^c}Q_{\le k}^{([K]_1^\Lambda)^c}P_-^{]u[_q^{([K]_1^\Lambda)^c}}R_E^{([K]_1^\Lambda)^c}P_+^{[j]_r^\Lambda\cap([K]_1^\Lambda)^c}Q_{\le k}^{([K]_1^\Lambda)^c}\Big\|_{HS}^s\Big)\Big)\\
&\le C_k^s\left(|\Lambda|^{2sk+2}\mathrm{e}^{-sm_0(r-6kN-2)}+\sum_{q=-1}^{r-6kN-3}\mathrm{e}^{-sm_0(q)_+}f_N^\Lambda(r-6kN-q-3)\right)\\
&= C_k^s\left(|\Lambda|^{2sk+2}\mathrm{e}^{-sm_0(r-6kN-2)}+\sum_{p=0}^{r-6kN-2}\mathrm{e}^{-sm_0(r-p-6kN-3)_+}f_N^\Lambda(p)\right).
\end{aligned}\tag{4.12}$$

Combining (4.8)–(4.12), we get

$$G_1(r) \le C\left(\lambda\Delta^2\right)^{-s}\mathrm{e}^{-m_k' kN}\left(|\Lambda|^{2sk+2}\mathrm{e}^{-m_k' r}+\sum_{p=0}^{r}\mathrm{e}^{-m_k'(r-p)}f_N^\Lambda(p)\right),\tag{4.13}$$

for an appropriate $m_k' > 0$.

To estimate $G_2(r)$, we note that it follows from Lemma 3.8(ii), letting

$$K(a) = [j]_{ad_\rho}^\Lambda \quad\text{and}\quad S(a) = [\partial^\Lambda K(a)]_{d_\rho-1}^\Lambda \quad\text{for}\quad a\in\mathbb{N},\tag{4.14}$$

that

$$G_2(r) \le \sum_{a=1}^{3k-1} G_2^{(a)}(r),\quad G_2^{(a)}(r) = \mathbb{E}\left(\left\|\chi_N^\Lambda Q_{\le k}^\Lambda P_+^{S(a)}P_-^{K(a)}P_-^{(K(a))^c}\mathcal{N}_j R_E^\Lambda P_+^{[j]_r^\Lambda}Q_{\le k}^\Lambda\right\|_{HS}^s\right).\tag{4.15}$$

To estimate $G_2^{(a)}(r)$, we use (3.71) and (3.72), the Cauchy-Schwarz inequality and Hölder's inequality (recall $3s < 1$) to get (we mostly omit a from the notation)

$$\begin{aligned}
G_2^{(a)}(r) &\le C\Delta^{-s}\left(\mathbb{E}\|Y\|^{2s}\right)^{1/2}\left(\mathbb{E}\|Z\|_{HS}^{2s}\right)^{1/2}\\
&\le C\Delta^{-s}\left(\mathbb{E}\|Y\|^{s}\right)^{1/4}\left(\mathbb{E}\|Y\|^{3s}\right)^{1/4}\left(\mathbb{E}\|Z\|_{HS}^{2s}\right)^{1/2},
\end{aligned}\tag{4.16}$$

where

$$Y = \chi_N^\Lambda Q_{\le k}^\Lambda P_+^{S(a)}P_-^{K(a)}P_-^{(K(a))^c}\mathcal{N}_j R_E^{K(a),(K(a))^c}P_-^{\partial^\Lambda K(a)} \text{ and } Z = P_-^{\partial^\Lambda K(a)}R_E^\Lambda P_+^{[j]_r^\Lambda}Q_{\le k}^\Lambda\chi_N^\Lambda.\tag{4.17}$$

It follows immediately from (3.38) that

$$\mathbb{E}\|Z\|_{HS}^{2s} \le C|\Lambda|^{4sk+3} \quad\text{and}\quad \mathbb{E}\|Y\|^{3s} \le C|\Lambda|^{6sk+3},\tag{4.18}$$

where we used $\left|\partial^\Lambda K(a)\right| \le 4$ since $K(a)$ is connected, and hence, we have

$$G_2^{(a)}(r) \le C\Delta^{-s}|\Lambda|^{\frac{7}{2}sk+\frac{9}{4}}\left(\mathbb{E}\|Y\|^s\right)^{1/4}.\tag{4.19}$$

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To estimate $E\|Y\|^s$, we use (the dependence on a is being ommitted)

$$\|Y\| \le \sum_{x\in\partial^\Lambda K} \|Y_x\|, \quad \text{with} \quad Y_x = \chi_N^\Lambda Q_{\le k}^\Lambda P_+^S P_-^K P_-^{K^c} \mathcal{N}_j R_E^{K,K^c} \mathcal{N}_x. \tag{4.20}$$

We consider first the case $x \in \partial_{in}^\Lambda K$. Using (3.74), we can further decompose Y_x as

$$Y_x = \sum_{\nu\in\sigma(H^{K^c})\cap[1-\frac{1}{\Delta},\infty)} Y_{x,\nu}, \quad Y_{x,\nu} = \chi_N^\Lambda Q_{\le k}^\Lambda P_+^S P_-^K P_-^{K^c} \mathcal{N}_j \left(R_{E-\nu}^K \otimes \pi_{K_\nu}\right)\mathcal{N}_x. \tag{4.21}$$

Note that

$$\|Y_x\| = \max_\nu \left\|Y_{x,\nu}\right\| \le \sum_{\nu\in\sigma(H^{K^c})\cap[1-\frac{1}{\Delta},k(1-\frac{1}{\Delta}))} \left\|Y_{x,\nu}\right\| + \max_{\nu\in\sigma(H^{K^c})\cap[k(1-\frac{1}{\Delta}),\infty)} \left\|Y_{x,\nu}\right\|. \tag{4.22}$$

Clearly, we can bound

$$\left\|Y_{x,\nu}\right\| \le \left\|P_+^S\left(R_{E-\nu}^K \otimes \pi_{K_\nu}\right)\mathcal{N}_x\right\| \le \left\|P_+^{S\cap K} R_{E-\nu}^K \mathcal{N}_x\right\|. \tag{4.23}$$

For $\nu \ge 1-\frac{1}{\Delta}$, we have $E-\nu \in I_{\le k-1}$ for $E \in I_{\le k}$ (recall (2.14)). For $\nu \in \sigma(H^{K^c})\cap[1-\frac{1}{\Delta}, k(1-\frac{1}{\Delta}))$, we use the induction hypothesis for Theorem 2.4 and the statistical independence of H^{K^c} and $\{\omega_i\}_{i\in K}$ to conclude that

$$\mathbb{E}\left\|Y_{x,\nu}\right\|^s \le \mathbb{E}_K \left\|P_+^{S\cap K} R_{E-\nu}^K \mathcal{N}_x\right\|^s \le C_{k-1}|\Lambda|^{\xi_{k-1}} \mathrm{e}^{-\theta_{k-1}\frac{r}{6k}}, \tag{4.24}$$

where we used (3.65).

For $\nu \in \sigma(H^{K^c}) \cap [k(1-\frac{1}{\Delta}),\infty)$, $E-\nu \le \frac{3}{4}\left(1-\frac{1}{\Delta}\right)$, and in this case,

$$P_+^{S\cap K} R_{E-\nu}^K \mathcal{N}_x = P_+^{S\cap K} \widehat{R}_{E-\nu}^K \mathcal{N}_x, \tag{4.25}$$

so it follows from (3.23) with $k=0$, using (3.65), that

$$\left\|Y_{x,\nu}\right\| \le C_0 \mathrm{e}^{-m_0\frac{r}{6k}}. \tag{4.26}$$

Using (4.22), (4.24), (4.26) and (3.27), we get

$$\mathbb{E}\|Y_x\|^s \le C|\Lambda|^{\xi_{k-1}} \mathrm{e}^{-\theta_{k-1}\frac{r}{6k}} \operatorname{tr} \chi_{[1-\frac{1}{\Delta},k(1-\frac{1}{\Delta}))}(H^{K^c}) \le C_k |\Lambda|^{\xi_{k-1}+2k} \mathrm{e}^{-\frac{\theta_{k-1}}{6k}r}. \tag{4.27}$$

Similar considerations show that the estimate (4.27) holds also for $x \in \partial_{ex}^\Lambda K$.

Combining (4.20) and (4.27) and recalling $\left|\partial^\Lambda K\right| \le 4$, we get

$$\mathbb{E}\|Y\|^s \le C_k |\Lambda|^{\xi_{k-1}+2k} \mathrm{e}^{-\frac{\theta_{k-1}}{6k}r}. \tag{4.28}$$

Combining (4.19) and (4.28), we see that

$$G_2^{(a)}(r) \le C_k \Delta^{-s} |\Lambda|^{\zeta_k} \mathrm{e}^{-\frac{\theta_{k-1}}{24k}r}. \tag{4.29}$$

It now follows from (4.15) and (4.29) that

$$G_2(r) \le C_k \Delta^{-s} |\Lambda|^{\zeta_k} \mathrm{e}^{-\frac{\theta_{k-1}}{24k}r} \le C_k \Delta^{-s} |\Lambda|^{\zeta_k} \mathrm{e}^{-\theta''_{k-1}r}. \tag{4.30}$$

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To estimate $G_3(r)$, given $4kN < \gamma < \frac{r}{2}$, we let $d_\gamma := \lfloor \frac{\gamma}{3k} \rfloor$. Given $a \in \{1, 2, \ldots, 3k-1\}$, we let $K(a,\gamma)$ be as in (3.66) with $A = \{j\}$, and let $K_1(a,\gamma) = [j]^\Lambda_{ad_\gamma}$, the connected component of $K(a,\gamma)$ that contains j. We also set $K_2(a,\gamma) = K(a,\gamma) \setminus K_1(a,\gamma)$, $S(a,\gamma) = [\partial^\Lambda K(a,\gamma)]^\Lambda_{d_\gamma - 1}$, and $T(a,\gamma) = [j]^\Lambda_{\gamma_{\{j\}}(M)}$. It follows from Lemma 3.8(iii) that

$$\pi_M = \pi_M P_+^{T(a,\gamma_{\{j\}}(M))} P_+^{S(a,\gamma_{\{j\}}(M))} P_-^{K_1(a,\gamma_{\{j\}}(M))} P_-^{K_2(a,\gamma_{\{j\}}(M))}, \tag{4.31}$$

for some $a \in \{1, 2, \ldots, 3k-1\}$, and hence,

$$\begin{aligned} G_3(r) &\leq \sum_{\gamma=4kN+1}^{\lfloor \frac{r}{2} \rfloor} \sum_{a=1}^{3k-1} G_3^{(a,\gamma)}(r), \quad \text{where} \\ G_3^{(a,\gamma)}(r) &= \mathbb{E}\left(\left\| \chi_N^\Lambda Q_{\leq k}^\Lambda P_+^{T(a,\gamma)} P_+^{S(a,\gamma)} P_-^{K_1(a,\gamma)} P_-^{K_2(a,\gamma)} \mathcal{N}_j R_E^\Lambda P_+^{[j]_r^\Lambda} Q_{\leq k}^\Lambda \right\|_{HS}^s \right). \end{aligned} \tag{4.32}$$

To estimate $G_3^{(a,\gamma)}(r)$, we start with the following analogue of (4.8) (we mostly omit (a,γ) from the notation):

$$\begin{aligned} G_3^{(a,\gamma)}(r) &\leq C\left(\lambda \Delta^2\right)^{-s} \mathbb{E}_K(\|Y\|^s)\, \mathbb{E}_{[K]_1^c}(\|Z\|_{HS}^s); \\ Y &:= \chi_N^K Q_{\leq k}^K P_+^{T\cap K} P_+^S P_-^{K_1} P_-^{K_2} R_E^K P_-^{\partial_{in}^\Lambda K}, \\ Z &:= P_-^{\partial_{ex}^\Lambda [K]_1} R_E^{\left([K]_1^\Lambda\right)^c} P_+^{[j]_r^\Lambda \cap \left([K]_1^\Lambda\right)^c} Q_{\leq k}^{\left([K]_1^\Lambda\right)^c} \chi_N^{\left([K]_1^\Lambda\right)^c}. \end{aligned} \tag{4.33}$$

Proceeding exactly as in (4.11)–(4.12), we get

$$\mathbb{E}_{\left([K]_1^\Lambda\right)^c}(\|Z\|_{HS}^s) \leq C_k^s \left(|\Lambda|^{\xi_k} \mathrm{e}^{-sm_0(r-\gamma-d_\gamma)} + \sum_{p=0}^{r-(\gamma+d_\gamma)-2} \mathrm{e}^{-sm_0\left(r-p-(\gamma+d_\gamma)-3\right)_+} f_N^\Lambda(p) \right). \tag{4.34}$$

We estimate $E\|Y\|^s$ similarly to (4.20)–(4.28). We have

$$\|Y\| \leq \sum_{x \in \partial_{in} K} \|Y_x\|, \quad \text{where} \quad Y_x = \chi_N^K Q_{\leq k}^K P_+^{T\cap K} P_+^S P_-^{K_1} P_-^{K_2} R_E^K \mathcal{N}_x. \tag{4.35}$$

We consider first the case $x = x_i \in \partial_{in}\left([K]_1^\Lambda\right)^c K_i$, $i \in \{1,2\}$, and $i' = \{1,2\} \setminus \{i\}$. Using (3.74), we can further decompose Y_x as

$$Y_{x_i} = \sum_{\nu \in \sigma(H^{K_{i'}}) \cap [1-\frac{1}{\Delta}, \infty)} Y_{x_i,\nu}, \quad Y_{x_i,\nu} = P_+^S P_-^{K_{i'}} \left(R_{E-\nu}^{K_i} \otimes \pi_{K_\nu} \right) \mathcal{N}_x. \tag{4.36}$$

Note that

$$\left\|Y_{x_i}\right\| = \max_\nu \left\|Y_{x_i,\nu}\right\| \leq \sum_{\nu \in \sigma(H^{K_{i'}}) \cap [1-\frac{1}{\Delta}, k(1-\frac{1}{\Delta}))} \left\|Y_{x_i,\nu}\right\| + \max_{\nu \in \sigma(H^{K_{i'}}) \cap [k(1-\frac{1}{\Delta}), \infty)} \left\|Y_{x_i,\nu}\right\|. \tag{4.37}$$

Clearly, we can bound

$$\left\|Y_{x_i,\nu}\right\| \leq \left\| P_+^S \left(R_{E-\nu}^{K_i} \otimes \pi_{K_\nu} \right) \mathcal{N}_{x_i} \right\| \leq \left\| P_+^{S\cap K_i} R_{E-\nu}^{K_i} \mathcal{N}_{x_i} \right\|. \tag{4.38}$$

For $\nu \geq 1 - \frac{1}{\Delta}$, we have $E - \nu \in I_{\leq k-1}$ for $E \in I_{\leq k}$ (recall (2.14)). For $\nu \in \sigma(H^{K_{i'}}) \cap [1-\frac{1}{\Delta}, k(1-\frac{1}{\Delta}))$, we use the induction hypothesis for Theorem 2.4 and the statistical independence of $H^{K_{i'}}$ and $\{\omega_i\}_{i \in K_i}$

to conclude that

$$\mathbb{E}\left\|Y_{x,\nu}\right\|^s \le \mathbb{E}_{K_i}\left\|P_+^{S\cap K_i} R_{E-\nu}^{K_i} \mathcal{N}_{x_i}\right\|^s \le C_{k-1}|K_i|^{\xi_{k-1}} \mathrm{e}^{-\theta_{k-1} d_\gamma} \le C_{k-1}|\gamma|^{\xi_{k-1}} \mathrm{e}^{-\theta_{k-1} d_\gamma}. \tag{4.39}$$

For $\nu \in \sigma(H^{K_{i'}}) \cap [k(1-\frac{1}{\Delta}), \infty)$, $E - \nu \le \frac{3}{4}\left(1 - \frac{1}{\Delta}\right)$, and in this case,

$$P_+^{S\cap K_i} R_{E-\nu}^{K_i} \mathcal{N}_x = P_+^{S\cap K_i} \widehat{R}_{E-\nu}^{K_i} \mathcal{N}_{x_i},$$

so it follows from (3.23) with $k = 0$ that

$$\left\|Y_{x_i,\nu}\right\| \le C_0 \mathrm{e}^{-m_0 d_\gamma}. \tag{4.40}$$

Using (4.37), (4.39), (4.40) and (3.27), we get

$$\mathbb{E}\left\|Y_{x_i}\right\|^s \le C\gamma^{\xi_{k-1}} \mathrm{e}^{-\frac{\theta_{k-1}}{3k}\gamma} \operatorname{tr} \chi_{[1-\frac{1}{\Delta}, k(1-\frac{1}{\Delta}))}(H^{K_{i'}}) \le C_k \gamma^{\xi_{k-1}+2k} \mathrm{e}^{-\frac{\theta_{k-1}}{3k}\gamma}. \tag{4.41}$$

Combining (4.35) and (4.41) and recalling $\left|\partial_{in}^{\Lambda} K_i\right| \le 4$, we get

$$\mathbb{E}\|Y\|^s \le C_k |\gamma|^{\xi_{k-1}+2k} \mathrm{e}^{-\frac{\theta_{k-1}}{3k}\gamma} \le C_k \mathrm{e}^{-\theta'_{k-1}\gamma}. \tag{4.42}$$

Combining (4.33), (4.34) and (4.42), we get

$$G_3^{(a,\gamma)}(r) \le C_k \left(\lambda\Delta^2\right)^{-s} \mathrm{e}^{-\theta'_{k-1}\gamma} \left(|\Lambda|^{2sk+2} \mathrm{e}^{-s\theta'_{k-1} r} + \sum_{p=0}^{r} \mathrm{e}^{-s\theta'_{k-1}(r-p)} f_N^{\Lambda}(p)\right). \tag{4.43}$$

It follows from (4.32) and (4.43) that

$$G_3(r) \le C_k \left(\lambda\Delta^2\right)^{-s} \mathrm{e}^{-\widehat{m}N} \left(|\Lambda|^{2sk+2} \mathrm{e}^{-s\theta'_{k-1} r} + \sum_{p=0}^{r} \mathrm{e}^{-s\theta'_{k-1}(r-p)} f_N^{\Lambda}(p)\right). \tag{4.44}$$

Putting together (4.7), (4.13), (4.30) and (4.44), we obtain (4.6). □

We can now prove Lemma 4.1.

Proof of Lemma 4.1. For $\Lambda \subset \mathbb{Z}$ finite, $E \in I_{\le k}$, $N \in \mathbb{N}$, and $r \in \mathbb{N}^0$, we set

$$f_N^{\Lambda}(r) = f_N^{\Lambda}(k, E, r) = \max_{\Theta\subset\Lambda} \max_{j\in\Theta} \mathbb{E}\left(\left\|\chi_N^{\Theta} Q_{\le k}^{\Theta} \mathcal{N}_j R_E^{\Theta} P_+^{[j]_r^{\Theta}} Q_{\le k}^{\Theta}\right\|_{HS}^s\right). \tag{4.45}$$

Note that $f_N^{\Lambda}(r)$ is monotone increasing in Λ, and it follows from (3.33) that

$$\max_{r\in\mathbb{N}^0} f_N^{\Lambda}(r) \le C\lambda^{-s} k^s |\Lambda|^{2sk+1}. \tag{4.46}$$

Moreover, if $8kN \ge r$, it follows from (3.49) that

$$f_N^{\Lambda}(r) \le C_{k,s} |\Lambda|^{2(sk+1)} e^{-m_{0,\mu} r}. \tag{4.47}$$

If $8kN < r$, we use Lemma 4.2. Since this lemma holds for arbitrary finite subsets of $\mathbb{Z}$, it follows from (4.6) that for $8kN < r$, we have

$$f_N^{\Lambda}(r) \le C_k \left(|\Lambda|^{\zeta_k} \mathrm{e}^{-m_k r} + \mathrm{e}^{-m_k N} \left(\lambda\Delta^2\right)^{-s} \sum_{p=0}^{r} \mathrm{e}^{-m_k(r-p)} f_N^{\Lambda}(p)\right), \tag{4.48}$$

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for all $\Lambda \subset \mathbb{Z}$ finite. Combining with (4.47), we get (with possibly slightly different constants $C, m_k > 0$, $\zeta_k > 0$)

$$f^{\Lambda}(r) \leq \sum_{N=1}^{|\Lambda|} f_N^{\Lambda}(r) \leq C\left(|\Lambda|^{\zeta_k} \mathrm{e}^{-m_k r} + \left(\lambda\Delta^2\right)^{-s} \sum_{p=0}^{r} \mathrm{e}^{-m_k(r-p)}\right). \tag{4.49}$$

The proof can now be completed by a standard subharmonicity argument. Let $h^{\Lambda}(r) = f^{\Lambda}(r) - 2C|\Lambda|^{\zeta_k}\mathrm{e}^{-m_k \frac{r}{2}}$, and take $\Delta \geq \Delta_0$ and $\lambda \geq \lambda_0$ such that

$$2C\left(\lambda\Delta^2\right)^{-s} \sum_{q=-\infty}^{\infty} \mathrm{e}^{-m_k \frac{|q|}{2}} \leq 1. \tag{4.50}$$

Then (4.49) implies that

$$\begin{aligned} h^{\Lambda}(r) &\leq C|\Lambda|^{\zeta_k}\mathrm{e}^{-m_k r} - 2C|\Lambda|^{\zeta_k}\mathrm{e}^{-m_k \frac{r}{2}} \\ &\quad + C\left(\lambda\Delta^2\right)^{-s} \sum_{p=0}^{r} \mathrm{e}^{-m_k(r-p)} \left(h^{\Lambda}(p) + 2C|\Lambda|^{\zeta_k}\mathrm{e}^{-m_k \frac{p}{2}}\right) \\ &\leq C|\Lambda|^{\zeta_k}\left(\mathrm{e}^{-m_k r} - \mathrm{e}^{-m_k \frac{r}{2}}\right) + C\left(\lambda\Delta^2\right)^{-s} \sum_{p=0}^{r} \mathrm{e}^{-m_k(r-p)} h^{\Lambda}(p), \end{aligned} \tag{4.51}$$

for all $r \in \mathbb{N}^0$. In addition, it follows from (4.46) that

$$R = \sup_{r \in \mathbb{N}^0} h^{\Lambda}(r) \leq \sup_{r \in \mathbb{N}^0} f^{\Lambda}(r) \leq C|\Lambda|^{2sk+3} < \infty. \tag{4.52}$$

We claim that $R \leq 0$, which implies that (4.4) holds (with different constants), finishing the proof of Lemma 4.1. Indeed, suppose that $R > 0$. Then it follows from (4.51) and (4.50) that

$$R \leq C\left(\lambda\Delta^2\right)^{-s} \sup_{r \in \mathbb{N}^0}\left(\sum_{p=0}^{|\Lambda|} \mathrm{e}^{-m_k|r-p|}\right) R \leq C\left(\lambda\Delta^2\right)^{-s}\left(\sum_{q=-\infty}^{\infty} \mathrm{e}^{-m_k \frac{|q|}{2}}\right) R \leq \tfrac{1}{2} R, \tag{4.53}$$

a contradiction. □

The proof of Theorem 2.4 is complete.

5. Quasi-locality in expectation

In this section, we prove Corollary 2.6. To do so, we first extract from Theorem 2.4 a probabilistic statement (cf. [20, Proposition 5.1] and [19, Lemma 7.2]).

We fix $k \in \mathbb{N}$ and let s, θ_k, ξ_k be as in (2.17), slightly modified so (2.17) holds with $\rho^{\Lambda}(A, B)$ substituted for $\mathrm{dist}_{\Lambda}(A, B^c)$ (recall (3.6)).

We fix a finite subset Λ of $\mathbb{Z}$. Given $\emptyset \neq K \subset \Lambda$, we let $H^{K'}$ be the restriction of H^K to $\mathrm{Ran}\, P_-^K = \mathrm{Ran}\, \chi_{\mathbb{N}}(\mathcal{N}_K)$, $K^c = \Lambda \setminus K$ (we allow $K^c = \emptyset$), and consider $H^{K',K^c} = H^{K'} + H^{K^c}$, $\Gamma^{K',K^c} = H^{\Lambda} - H^{K',K^c}$, $R_E^{K',K^c} = (H^{K',K^c} - E)^{-1}$, operators on $\mathrm{Ran}\, P_-^K \oplus \mathcal{H}_{K^c}$. Given an interval I and an operator H, we set $\sigma_I(H) = \sigma(H) \cap I$.

We start by proving Wegner-like estimates for the XXZ model.

Lemma 5.1. *Let* $\emptyset \neq K \subset \Lambda$.

1. *Consider the open interval* $I \subset I_k$. *Then*

$$\mathbb{P}_K\left\{\sigma_I(H^{K',K^c}) \neq \emptyset\right\} \leq C_k \lambda^{-1}|I||\Lambda|^{2k+1}. \tag{5.1}$$

2. *Let* $0 < \delta < \frac{1}{4}\left(1 - \frac{1}{\Delta}\right)$. *Then (recall (2.14))*

$$\mathbb{P}\left\{\operatorname{dist}\left\{\sigma_{\widehat{I}_k}(H^{K',K^c}), \sigma_{\widehat{I}_k}(H^{K^c})\right\} < \delta\right\} \leq C_k \lambda^{-1}\delta|\Lambda|^{4k+1}. \tag{5.2}$$

Proof. To prove Part (i), recall (3.27) (it applies to $H^{(K',K^c)}$), and let $E_1 \leq E_2 \leq \ldots$ be the at most $Ck|\Lambda|^{2k}$ eigenvalues of H^{K',K^c} in $\widehat{I}_{\leq k}$, counted with multiplicity, which we consider as functions of ω_K for fixed ω_{K^c}. Since $\mathcal{N}_K \geq 1$, each $E_n(\omega_K)$ is a monotone function on $\mathbb{R}^{|K|}$. Let $e = (1, 1, \ldots, 1) \in \mathbb{R}^{|K|}$. We have $E_n(\omega_K + te) - E_n(\omega_K) \geq \lambda t$ for all $t > 0$ and all n by the min-max principle, so we can apply Stollmann's Lemma [46] to get

$$\mathbb{P}_K\{E_n(\omega_K) \in I\} \leq C|I|\lambda^{-1}|K|. \tag{5.3}$$

In view of (3.27), (5.1) follows using (5.3) for each one of the eigenvalues E_n.

Part (ii) follows from Part (i) and (3.27) for H^{K^c}, since the random variables ω_K and ω_{K^c} are independent. □

Let $E \in \mathbb{R}$, $m > 0$, $r \in \mathbb{N}$, $\emptyset \neq K \subset \Lambda$, and let $H^\sharp$ denote either H^K or $H^{(K',K^c)}$. Then the operator $H^{K^\sharp}$ is said to be (m, E, r)-regular if

$$\begin{aligned} &F_E^{K^\sharp} \leq \mathrm{e}^{-mr} \quad \text{and} \quad \operatorname{dist}(E, \sigma(H^{K^\sharp})) > \mathrm{e}^{-mr}, \\ &\text{where} \quad F_E^{K^\sharp} = \max_{i \in K} F_E^{K^\sharp}(i) \quad \text{with} \quad F_E^{K^\sharp}(i) = \left\| \mathcal{N}_i R_E^{K^\sharp} P_+^{[i]_r^K} \right\|. \end{aligned} \tag{5.4}$$

In addition, consider the probabilistic event

$$\mathcal{F}_k^\Lambda(K, m, r) = \left\{E \in I_k \implies \text{ either } H^{(K',K^c)} \text{ or } H^{K^c} \text{ is } (m, E, r)\text{-regular}\right\}. \tag{5.5}$$

Lemma 5.2. *Let* $\emptyset \neq K \subsetneq \Lambda$, *and let* $r \in \mathbb{N}$, $r \geq \frac{18}{\theta_k}$. *Then*

$$\mathbb{P}\left\{\left(\mathcal{F}_k^\Lambda(K, \tfrac{\theta_k}{9}, r)\right)^c\right\} \leq C|\Lambda|^{\xi_k'} \mathrm{e}^{-\frac{\theta_k}{9}r}. \tag{5.6}$$

Proof. Let $\emptyset \neq K \subsetneq \Lambda$, $r \geq \frac{18}{\theta_k}$, and set $m = \frac{\theta_k}{9}$, so $\mathrm{e}^{mr} \geq 4$. Let S denote either the pair K', K^c or K^c, and let $S' = K$ if $S = K', K^c$, or $S' = K^c$ if $S = K^c$. Consider the (random) energy sets

$$D_S = \left\{E \in I_k : F_E^S > \mathrm{e}^{-mr}\right\} \quad \text{and} \quad J_S = \left\{E \in I_k : F_E^S > \mathrm{e}^{-2mr}\right\}, \tag{5.7}$$

and the event

$$\mathcal{J}_S = \left\{|J_S| > e^{-5mr}\right\}. \tag{5.8}$$

Using (2.17), we get

$$\begin{aligned} \mathbb{P}\{\mathcal{J}_S\} &\leq \mathrm{e}^{5mr}\,\mathbb{E}\{|J_S|\} \leq \mathrm{e}^{5mr}\,\mathbb{E}\left\{\int_{I_k} \mathrm{e}^{2smr}\left(F_E^S\right)^s dE\right\} \\ &\leq \mathrm{e}^{7mr}\int_{I_k}\sum_{i \in S'}\mathbb{E}\left\{\left(F_E^S(i)\right)^s\right\} dE \leq C_k|\Lambda|^{\xi_k+1}e^{-2mr}. \end{aligned} \tag{5.9}$$

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We now consider the (random) energy set

$$Y_S = \left\{E \in I_k :\ \mathrm{dist}(E, \sigma(H^S)) \leq \mathrm{e}^{-mr}\right\} \tag{5.10}$$

and claim that $D_S \subset Y_S$ on the complementary event $\mathcal{J}_S^c = \left\{|J_S| \leq e^{-5mr}\right\}$.

To see this, suppose $|J_S| \leq e^{-5mr}$ and $E \in D_S \setminus Y_S$. Since $E \in D_S$, there exists $i \in S'$ such that $F_E^S(i) > e^{-mr}$. Let $E' \in I_k$ such that $|E' - E| \leq 2e^{-5mr}$. Using $E \in Y_S$, we get $\mathrm{dist}(E', \sigma(H^S) > e^{-mr} - 2e^{-5mr} \geq \frac{1}{2}e^{-mr}$. Thus, using the resolvent identity and $r \geq \frac{18}{\theta_k}$, we have

$$F_{E'}^S(i) \geq F_E^S(i) - |E' - E|\left\|R_E^S\right\|\left\|R_{E'}^S\right\| > e^{-mr} - (2e^{-5mr})e^{mr}(2e^{mr}) \geq e^{-2mr}. \tag{5.11}$$

It follows that $[E - 2e^{-5mr}, E + 2e^{-5mr}] \cap I_k \subset J_S$. Since $|I_k| \geq 2e^{-5mr}$ as $r \geq \frac{18}{\theta_k}$, we conclude that $|J_S| \geq 2e^{-5mr} > e^{-5mr}$, a contradiction.

We proved that $|J_S| \leq e^{-5mr}$ implies $D_S \subset Y_S$, so $\widehat{Y}_S = I_k \setminus Y_S \subset I_k \setminus D_S$. In particular, outside the event $\mathcal{J}_S$, $E \in \widehat{Y}_S$ implies that H^S is (m, E, r)-regular.

We now consider the event

$$\begin{aligned} \mathcal{E}_K &= \left\{I_k \setminus (\widehat{Y}_{K',K^c} \cup \widehat{Y}_{K^c}) \neq \emptyset\right\} = \left\{I_k \cap Y_{K',K^c} \cap Y_{K^c} \neq \emptyset\right\} \\ &\subset \left\{\mathrm{dist}\left\{\sigma_{\widehat{I}_k}(H^{K',K^c}), \sigma_{\widehat{I}_k}(H^{K^c})\right\} \leq 2e^{-mr}\right\} \end{aligned} \tag{5.12}$$

and note that it follows from Lemma 5.1(ii) that

$$\mathbb{P}\{\mathcal{E}_K\} \leq C_k|\Lambda|^{4k+1}e^{-mr}. \tag{5.13}$$

Since

$$\mathbb{P}\left\{\mathcal{E}_K \cup \mathcal{J}_{K',K^c} \cup \mathcal{J}_{K^c}\right\} \leq C_k|\Lambda|^{4k+1}e^{-mr} + 2C_k|\Lambda|^{\xi_k+1}e^{-2mr} \leq C|\Lambda|^{\xi_k'}\mathrm{e}^{-mr}, \tag{5.14}$$

and on the complementary event, we have $I_k = \widehat{Y}_{K',K^c} \cup \widehat{Y}_{K^c}$, so for $E \in I_k$, either H^{K',K^c} or H^{K^c} is (m, E, r)-regular, the lemma is proved. □

Proof of Corollary 2.6. Let $A \subset B \subset \Lambda$, A connected in Λ, let $r = \rho^\Lambda(A, B)$, and recall $\left\|P_-^A f(H^\Lambda)P_+^B\right\| \leq \left\|P_-^A f(H^\Lambda)P_+^{[A]_r^\Lambda}\right\|$.

We set

$$\Theta^\Lambda(A, r) = \sup_{\substack{f \in B(I_{\leq k}): \\ \|f\|_\infty \leq 1}} \left\|P_-^A f(H^\Lambda)P_+^{[A]_r^\Lambda}\right\| \leq 1. \tag{5.15}$$

To estimate $\mathbb{E}\left\{\Theta^\Lambda(A, r)\right\}$, note that

$$\Theta^\Lambda(A, r) \leq \sum_{E \in \sigma_{I_k}(H^\Lambda)} \left\|P_-^A P_{\{E\}} P_+^{[A]_r^\Lambda}\right\|, \quad \text{where} \quad P_{\{E\}} = \chi_{\{E\}}(H^\Lambda). \tag{5.16}$$

The spectrum of H^Λ is simple almost surely, as commented in [19, Section 3], so we assume this on what follows for simplicity. (Otherwise, we just need to label the eigenvalues taking into account multiplicity.) For $E \in \sigma(H^\Lambda)$, we let ϕ_E denote the corresponding eigenfunction, and let $N_E \in \mathbb{N}^0$ be given by $\mathcal{N}_\Lambda \phi_E = N_E \phi_E$.

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For $E \in I_k$, we have

$$\begin{aligned} P_{\{E\}} &= \widehat{R}^{\Lambda}_{k,E}\left(\widehat{H}^{\Lambda}_k - E\right)P_{\{E\}} = \widehat{R}^{\Lambda}_{k,E}\left(\widehat{H}^{\Lambda}_k - H^{\Lambda} + (H^{\Lambda} - E)\right)P_{\{E\}} \\ &= k\left(1 - \tfrac{1}{\Delta}\right)\widehat{R}^{\Lambda}_{k,E}\, Q^{\Lambda}_{\le k} P_{\{E\}}. \end{aligned} \tag{5.17}$$

Let $r \ge R_k = 6k(\lceil \frac{18}{\theta_k} \rceil + 2)$. Using (A.7) and (3.23), we obtain

$$\begin{aligned} \left\| P^{A}_{-} P_{\{E\}} P^{[A]^{\Lambda}_r}_{+} \right\| &= k\left(1 - \tfrac{1}{\Delta}\right)\left\| P^{A}_{-} \widehat{R}^{\Lambda}_{k,E}\, Q^{\Lambda}_{\le k} P_{\{E\}} P^{[A]^{\Lambda}_r}_{+} \right\| \\ &= k\left(1 - \tfrac{1}{\Delta}\right)\left\| P^{A}_{-} \widehat{R}^{\Lambda}_{k,E} P^{[A]^{\Lambda}_\infty}_{-}\, Q^{\Lambda}_{\le k} P_{\{E\}} P^{[A]^{\Lambda}_r}_{+} \right\| \\ &\le k \sum_{q=-|A|}^{|\Lambda|} \left\| P^{A}_{-} \widehat{R}^{\Lambda}_{k,E} P^{[A]^{\Lambda}_q}_{+} \right\| \left\| P^{]A[^{\Lambda}_q}_{-} Q^{\Lambda}_{\le k} P_{\{E\}} P^{[A]^{\Lambda}_r}_{+} \right\| \\ &\le C_0 \sum_{q=-|A|}^{|\Lambda|} \mathrm{e}^{-m_0 (q)_+} \left\| P^{]A[^{\Lambda}_q}_{-} Q^{\Lambda}_{\le k} P_{\{E\}} P^{[A]^{\Lambda}_r}_{+} \right\| \\ &\le 2C_0 \sum_{q=-|A|}^{r-1-R_k} \mathrm{e}^{-m_0 (q)_+} \sum_{u \in]A[^{\Lambda}_q} \left\| Q^{\Lambda}_{\le k} \mathcal{N}_u P_{\{E\}} P^{[u]^{\Lambda}_{r-q-1}}_{+} \right\| + C_k |\Lambda| \mathrm{e}^{-m_0 r}. \end{aligned} \tag{5.18}$$

Let $u \in \Lambda$ and $p \ge R_k$. If $8kN_E \ge p$, it follows from (3.53)–(3.55) that

$$\begin{aligned} &\left\| \chi^{\Lambda}_{N_E} Q^{\Lambda}_{\le k} \mathcal{N}_u P_{\{E\}} P^{[u]^{\Lambda}_p}_{+} \right\| \le \chi_{\mathcal{B}^{N_E}_k}, \\ &\mathbb{P}_{\Lambda}\left(\mathcal{B}^{N_E}_k\right) \le C_k |\Lambda|^{2k} e^{-c_\mu N_E} \le C_k |\Lambda|^{2k} \mathrm{e}^{-\frac{c_\mu}{8k} p}. \end{aligned} \tag{5.19}$$

If $p > 8kN_E$, we set (cf. (4.14))

$$\begin{aligned} &K(0) = [u]^{\Lambda}_{\frac{3p}{4}} \quad \text{and} \quad K(a) = [u]^{\Lambda}_{a\lfloor \frac{p}{6k} \rfloor} \;\; \text{for } \; a = 1, 2, \ldots, 3k-1, \\ &S(a) = [\partial^{\Lambda} K(a)]^{\Lambda}_{\lfloor \frac{p}{6k} \rfloor - 1} \quad \text{for} \quad a = 0, 1, \ldots, 3k-1. \end{aligned} \tag{5.20}$$

Using Lemma 3.8, we get

$$\left\| \chi^{\Lambda}_{N_E} Q^{\Lambda}_{\le k} \mathcal{N}_u P_{\{E\}} P^{[u]^{\Lambda}_p}_{+} \right\| \le \sum_{a=0}^{3k-1} \left\| \chi^{\Lambda}_{N_E} Q^{\Lambda}_{\le k} \mathcal{N}_u Y(a) P_{\{E\}} P^{[u]^{\Lambda}_p}_{+} \right\|, \tag{5.21}$$

where $Y(0) = P_{+}^{\Lambda \setminus [u]^{\Lambda}_{\frac{p}{2}}}$ and $Y(a) = P^{S(a)}_{+} P^{K(a)}_{-} P^{K^c(a)}_{-}$ for $a > 0$.

We now consider the event (see (5.5))

$$\mathcal{J}_k(u,p) = \bigcap_{a=0}^{3k-1} \mathcal{F}^{\Lambda}_k(K(a), \widehat{\theta}_k, \widehat{p}), \;\; \text{where } \widehat{\theta}_k = \tfrac{\theta_k}{9} \text{ and } \widehat{p} = \lfloor \tfrac{p}{6k} \rfloor - 1 \ge \tfrac{18}{\theta_k} \tag{5.22}$$

and note that it follows from Lemma 5.2 that

$$\mathbb{P}\{(\mathcal{J}_k(u,p))^c\} \le 3kC|\Lambda|^{\xi'_k} \mathrm{e}^{-\widehat{\theta_k}\widehat{p}}. \tag{5.23}$$

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For $\omega \in \mathcal{J}_k(u,p)$ and $a \in \{0,1,\ldots,3k-1\}$, either $H^{(K(a),K^c(a))}$ or $H^{K^c(a)}$ is $(\widehat{\theta_k}, E, \widehat{p})$-regular ($K^c(a) = (K(a))^c$). If $H^{K^c(a)}$ is $(\widehat{\theta_k}, E, \widehat{p})$-regular, we note that

$$\begin{aligned} P_{\{E\}} P_+^{[u]_p^\Lambda} &= P_{\{E\}} \left(H^{K(a)} + H^{K^c(a)} - E \right) R_E^{K^c(a)} P_+^{[u]_p^\Lambda} \\ &= -P_{\{E\}} \Gamma^{(K(a),K^c(a))} P_-^{\partial_{ex}^\Lambda K(a)} P_+^{K(a)} R_E^{K^c(a)} P_+^{[u]_p^\Lambda}, \end{aligned} \tag{5.24}$$

where we have used $R_E^{K^c(a)} P_+^{[u]_p^\Lambda} = P_+^{K(a)} R_E^{K^c(a)} P_+^{[u]_p^\Lambda}$ due to $K(a) \subset [u]_p^\Lambda$. We deduce that

$$\left\| \chi_{N_E}^\Lambda Q_{\le k}^\Lambda \mathcal{N}_u Y(a) P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \le \left\| P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \le \tfrac{1}{\Delta} \left\| P_-^{\partial_{ex}^\Lambda K(a)} R_E^{K^c(a)} P_+^{[u]_p^\Lambda \cap (K^c(a))} \right\| \le \frac{2 \mathrm{e}^{-\widehat{\theta_k} \widehat{p}}}{\Delta}, \tag{5.25}$$

using (A.3), (5.4) and the definition of $K(a)$. If $H^{(K(a),K^c(a))}$ is $(\widehat{\theta_k}, E, \widehat{p})$-regular, we use

$$\begin{aligned} \mathcal{N}_u P_{\{E\}} P_+^{[u]_p^\Lambda} &= \mathcal{N}_u R_E^{(K(a),K^c(a))} \left(H^{(K(a),K^c(a))} - E \right) P_{\{E\}} P_+^{[u]_p^\Lambda} \\ &= -\mathcal{N}_u R_E^{(K(a),K^c(a))} P_-^{\partial^\Lambda K(a)} \Gamma^{(K(a),K^c(a))} P_{\{E\}} P_+^{[u]_p^\Lambda}. \end{aligned} \tag{5.26}$$

Thus,

$$\begin{aligned} \left\| \chi_{N_E}^\Lambda Q_{\le k}^\Lambda \mathcal{N}_u Y(a) P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| &\le \left\| \mathcal{N}_u Y(a) P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \le \tfrac{1}{\Delta} \left\| \mathcal{N}_u Y(a) R_E^{(K(a),K^c(a))} P_-^{\partial^\Lambda K(a)} \right\| \\ &\le \tfrac{1}{\Delta} \left\| P_+^{S(a)} R_E^{(K(a),K^c(a))} P_-^{\partial^\Lambda K(a)} \right\| \le \tfrac{2}{\Delta} \mathrm{e}^{-\widehat{\theta_k} \widehat{p}}, \end{aligned} \tag{5.27}$$

using (A.3), (5.4) and the definition of $S(a)$.

Combining (5.21), (5.25) and (5.27), we conclude that for $p > 8kN_E$ and $\omega \in \mathcal{J}_k(u,p)$, we have

$$\left\| \chi_{N_E}^\Lambda Q_{\le k}^\Lambda \mathcal{N}_u P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \le \tfrac{12k}{\Delta} \mathrm{e}^{-\widehat{\theta_k} \widehat{p}}. \tag{5.28}$$

Since $\left\| \chi_{N_E}^\Lambda Q_{\le k}^\Lambda \mathcal{N}_u P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \le 1$, it follows that for $p > 8kN_E$, we have

$$\left\| \chi_{N_E}^\Lambda Q_{\le k}^\Lambda \mathcal{N}_u P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \le \tfrac{12k}{\Delta} \mathrm{e}^{-\widehat{\theta_k} \widehat{p}} + \chi_{\mathcal{J}_k(u,p)^c}. \tag{5.29}$$

It follows that for $u \in \Lambda$ and $p \ge R_k$, using (5.19), (5.29) and (3.27), we conclude that

$$\mathbb{E}\Bigg(\sum_{E \in \sigma_{I_k}(H^\Lambda)} \left\| Q_{\le k}^\Lambda \mathcal{N}_u P_{\{E\}} P_+^{[u]_p^\Lambda} \right\| \Bigg) \le C_k |\Lambda|^{\xi_k'} \mathrm{e}^{-\theta_k' p}. \tag{5.30}$$

Combining with (5.16), (5.18) and (3.27), we obtain

$$\mathbb{E}\left\{ \Theta^\Lambda(A,r) \right\} \le C_k |\Lambda|^{\xi_k'} \mathrm{e}^{-\theta_k' r}. \tag{5.31}$$

The estimate (5.31) holds for $r \ge R_k$. Since $\mathbb{E}\left\{ \Theta^\Lambda(A,r) \right\} \le 1$ for all $r \ge 0$, it holds for all $r \ge 0$ if the constant C_k is replaced by the constant $\widetilde{C}_k = C_k \mathrm{e}^{\theta_k' \widehat{R}_k}$. □

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A. Useful identities

In this appendix, we list some useful identities. Their derivations are straightforward, so we leave out the proofs.

We fix $\Lambda \subset \mathbb{Z}$ finite,

- For all $i, j \in \Lambda$, we have (recall (2.15))

$$P_-^{\{i\}} = \mathcal{N}_i,$$
$$P_-^{\{i,j\}} = \mathcal{N}_i + \mathcal{N}_j - \mathcal{N}_i\mathcal{N}_j = \mathcal{N}_i\left(1 - \mathcal{N}_j\right) + \mathcal{N}_j = P_+^{(\{j\})}\mathcal{N}_i + \mathcal{N}_j. \tag{A.1}$$

- Consider the self-adjoint operator $h_{i,i+1}$ (recall (2.8)) on the four-dimensional Hilbert space $\mathcal{H}_{\{i,i+1\}} = \mathbb{C}_i^2 \otimes \mathbb{C}_{i+1}^2$. An explicit calculation shows that $h_{i,i+1}$ has eigenvalues $-1, 0, \pm\frac{1}{\Delta}$. It follows that if $\{i, i+1\} \subset \Lambda$, we have

$$\left\|h_{i,i+1}\right\| = 1 \quad \text{on} \quad \mathcal{H}_\Lambda. \tag{A.2}$$

- The following identities hold on $\mathcal{H}_\Lambda$ for $\{i, i+1\} \subset \Lambda$:

$$\begin{aligned}
&h_{i,i+1}P_+^{\{i,i+1\}} = P_+^{\{i,i+1\}}h_{i,i+1} = 0,\\
&\left\|P_+^{\{i\}}h_{i,i+1}\right\| = \left\|P_+^{\{i+1\}}h_{i,i+1}\right\| = \tfrac{1}{2\Delta},\\
&P_+^{(\{i\})}h_{i,i+1}P_+^{(\{i\})} = P_+^{(\{i+1\})}h_{i,i+1}P_+^{(\{i+1\})} = 0,\\
&h_{i,i+1}\mathcal{N}_i\mathcal{N}_{i+1} = \mathcal{N}_i\mathcal{N}_{i+1}h_{i,i+1} = \mathcal{N}_i\mathcal{N}_{i+1}h_{i,i+1}\mathcal{N}_i\mathcal{N}_{i+1}.
\end{aligned} \tag{A.3}$$

In particular, the first identity above implies

$$h_{i,i+1} = h_{i,i+1}P_-^{\{i,i+1\}} = P_-^{\{i,i+1\}}h_{i,i+1} = P_-^{\{i,i+1\}}h_{i,i+1}P_-^{\{i,i+1\}}. \tag{A.4}$$

- Let $K \subset \Lambda$, and recall (3.10). It follows from (A.4) that

$$\Gamma^K = P_-^{\partial^\Lambda K}\Gamma^K P_-^{\partial^\Lambda K}. \tag{A.5}$$

If K is connected in Λ, it follows from (A.5) that

$$\left\|P_+^K\Gamma^K\right\| \le \tfrac{1}{\Delta} \quad \text{and} \quad \left\|P_+^{K^c}\Gamma^K\right\| \le \tfrac{1}{\Delta}. \tag{A.6}$$

- The following identities hold for any nonempty $M \subset \Lambda$ (recall (2.16)):

$$\begin{aligned}
P_-^{[M]_\infty}P_+^M &= \sum_{q=0}^{|\Lambda|} P_+^{[M]_q^\Lambda}P_-^{]M[_q^\Lambda} = \sum_{q=0}^{|\Lambda|} P_+^{[M]_q^\Lambda}P_-^{\partial_{ex}^\Lambda[M]_q},\\
P_-^M &= \sum_{q=-|M|}^{-1} P_+^{[M]_q^\Lambda}P_-^{]M[_q^\Lambda} = \sum_{q=-|M|}^{-1} P_+^{[M]_q^\Lambda}P_-^{\partial_{in}^\Lambda[M]_{q+1}},\\
P_-^{[M]_\infty^\Lambda} &= \sum_{q=-|M|}^{|\Lambda|} P_+^{[M]_q^\Lambda}P_-^{]M[_q^\Lambda}.
\end{aligned} \tag{A.7}$$

B. Many-body quasi-locality

In this appendix, we prove (1.6). Recall we only consider finite subsets of $\mathbb{Z}$. We fix $\Lambda \subset \mathbb{Z}$ and consider the Hilbert space $\mathcal{H}_\Lambda$.

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Lemma B.1. *Suppose that $H \in \mathcal{A}_\Lambda$ satisfies*

1. *For all $K \subset \Lambda$, we have $[P_-^K, H]P_+^{[K]_1^\Lambda} = 0$.*
2. *For all connected $K \subset \Lambda$, we have $\|[P_-^K, H]\| \le \gamma$.*

Then for all $A \subset B \subset \Lambda$, A connected in Λ, we have

$$\|P_-^A e^{itH} P_+^B\| \le \gamma^r \frac{|t|^r}{r!}, \quad \textit{where} \quad r = \mathrm{dist}_\Lambda(A, B^c) \ge 1. \tag{B.1}$$

Proof. We note that $[A]_s^\Lambda \subset B$ for $s = 0, 1, \ldots, r-1$. We have

$$P_-^A e^{itH} P_+^B = ie^{itH} \int_0^t K(s) P_+^B ds, \tag{B.2}$$

where $K(s) = e^{-isH} [P_-^A, H] e^{isH}$. If $r \ge 2$, condition (i) of the Lemma yields $K(s) = e^{-isH} [P_-^A, H]P_-^{[A]_1^\Lambda} e^{isH}$. Proceeding recursively, we get

$$\begin{aligned} P_-^A e^{itH} P_+^B &= i^r \int_0^t \int_0^{s_1} \ldots \int_0^{s_{r-1}} \prod_{j=1}^r K_{j-1}(s_j) ds_j P_+^B, \\ K_j(s) &= e^{-isH} [P_-^{[A]_j}, H] e^{isH}. \end{aligned} \tag{B.3}$$

Using assumption (ii), we get

$$\|P_-^A e^{itH} P_+^B\| \le \gamma^r \frac{|t|^r}{r!}. \tag{B.4}$$

□

Lemma B.2. *Let $f \in C_0^n$ (i.e., f is compactly supported and n times differentiable function on $\mathbb{R}$ (with $n \ge 2$)). Then for A, B, H as in Lemma B.1 and $r = \mathrm{dist}_\Lambda(A, B^c)$, we have*

$$\|P_-^A f(H) P_+^B\| \le \widetilde{C}(f, n) r^{-(n-1)\min(1, \frac{r}{n})} \le \widetilde{C}(f, n) r^{-n}. \tag{B.5}$$

Proof. Let $\hat{f}$ denote the Fourier transform of f. Then we have $|\hat{f}(t)| \le C(f, n)\langle t\rangle^{-n}$ for $t \in \mathbb{R}$ (we recall that $\langle t\rangle := \sqrt{1+t^2}$). We can bound

$$\|P_-^A f(H) P_+^B\| \le \int_{\mathcal{R}} \|P_-^A e^{itH} P_+^B\| |\hat{f}(t)| dt + \int_{\mathcal{R}^c} |\hat{f}(t)| dt, \tag{B.6}$$

where $\mathcal{R} := [-R, R]$, where $R > 0$ will be chosen later.

We can bound the first integral on the right-hand side of (B.6) using (B.1) as

$$\begin{aligned} \int_{\mathcal{R}} \|P_-^A e^{itH} P_+^B\| |\hat{f}(t)| dt &\le C(f, n) \frac{\gamma^r}{r!} \int_{\mathcal{R}} |t|^r \langle t\rangle^{-n} dt \le C_n C(f, n) \frac{\gamma^r R^{1+(r-n)_+}}{r!} \\ &\le C_n' C(f, n) \left(\frac{e\gamma}{r}\right)^r R^{1+(r-n)_+}, \end{aligned} \tag{B.7}$$

where we used $r! \ge e^{1-r} r^r$.

We can bound the second integral in (B.6) as

$$\int_{\mathcal{R}^c} |\hat{f}(t)| dt \le C(f, n) \int_{\mathcal{R}^c} \langle t\rangle^{-n} dt \le C_n C(f, n)(1+R)^{1-n} \le C_n C(f, n) R^{1-n}. \tag{B.8}$$

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Choosing $R = \left(\frac{r}{e\gamma}\right)^{\frac{r}{n+(r-n)_+}}$, we get (B.5). □

Competing interest. The authors have no competing interest to declare.

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