8-DIMENSIONAL EINSTEIN-THORPE MANIFOLDS

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Abstract

We prove that a compact orientable Einstein-Thorpe manifold of dimension 8 that satisfies $6\chi = |P_2|$ must be flat.

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1. Introduction

The local geometry of a manifold provides us with information about its global topology. For instance, the generalized Gauss-Bonnet theorem [2, 5] states that the Euler-Poincaré characteristic $\chi$ of a compact oriented Riemannian manifold $M^{4k}$ can be written as an integral

$$\chi = \frac{2}{V} \frac{[(2k)!]^2}{4k} \int_M \text{trace} (\ast R_{2k} \ast R_{2k}) \, dV,$$

where $V$ is the volume of the Euclidean unit $4k$-sphere, $dV$ is the volume element of $M$, $\ast$ is the Hodge $\ast$-operator, and $R_{2k}$ is called the $2k$th curvature operator. If $R_{2k}$ commutes with $\ast$, that is, $R_{2k} \ast = \ast R_{2k}$, we call this condition a Thorpe condition and this metric a Thorpe metric and this manifold a Thorpe manifold. If Ricci curvature, $\text{ric}$, is a constant multiple of the metric, equivalently, traceless Ricci curvature, $\text{ric}_0$, vanishes then we call this condition an Einstein condition and this metric an Einstein metric and this manifold an Einstein manifold. In the 4-dimensional case, the Thorpe condition is equivalent to the Einstein condition [1]. And so, in 4 dimensions there is another way of stating the Einstein equation, namely the Thorpe condition. Moreover,
in case of a compact oriented Einstein manifolds of dimension 4 with $2\chi = |P_1|$, Hitchin in [3] has classified these manifolds. On the other hand, Thorpe metrics need not be Einstein in dimensions higher than 4 and the following metrics satisfy the Thorpe condition but they are not Einstein metrics [4]:

(i) $S^{4k} \times H^{4k}$, product metric of standard metrics;
(ii) $CP^2 \times CH^2$, product metric of standard metrics.

The following examples also provide us with Einstein manifolds but not Thorpe manifolds [4]: the canonical quaternion projective space $HP^n$ with $n \geq 3$.

We say that a Riemannian $4k$ manifold is Einstein-Thorpe if it is both Einstein and Thorpe. The purpose of the present note is to see what happens to Hitchin’s result if both conditions are imposed in dimension eight.

**THEOREM 1.1.** Suppose that $(M^8, g)$ is a compact orientable Einstein-Thorpe manifold and that

$$\chi = \left( \frac{2!2!}{4!} \right) |P_2|.$$

Then $(M^8, g)$ must be a flat manifold.

The crucial ingredient in the proof is Lemma 1.1.

**LEMMA 1.1.** Let $(M, g)$ be a Riemannian manifold of dimension 8. Then

$$\text{trace } R_4 = \frac{1}{2^2} \left( \frac{1}{6} \right) \left\{ \frac{1}{2} S^2 - 4 |\text{ric}_0|^2 + 4 |R|^2 \right\}$$

where $S$ is the scalar curvature, $\text{ric}_0$ is the traceless Ricci curvature and $R$ is the curvature.

From Lemma 1.1 we can observe that trace $R_4$ is nonnegative when the Riemannian manifold is Einstein.

2. The $p$th curvature operator and Thorpe manifolds

Let $M$ be a Riemannian manifold of dimension $n$ and let $\bigwedge^p(M)$ denote the bundle of $p$-vectors of $M$; $\bigwedge^p(M)$ is a Riemannian vector bundle, with inner product on the fiber $\bigwedge^p(x)$ over the point $x$ [4]. Let $R$ denote the covariant curvature tensor of $M$. For each even $p > 0$, we define the $p$th curvature tensor $R_p$ of $M$ to be the covariant curvature...
tensor field of order $2p$ given by

$$ R_p(u_1, \ldots, u_p, v_1, \ldots, v_p) $$

$$ = \frac{1}{2^{p^2} p!} \sum_{\alpha, \beta \in S_p} \varepsilon(\alpha) \varepsilon(\beta) R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \cdots $$

$$ R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)}), $$

where $u_i, v_j \in T_x M$ and $S_p$ denotes the group of permutations of $(1, \ldots, p)$ and, for $\alpha \in S_p$, $\varepsilon(\alpha)$ is the sign of the permutation $\alpha$.

The tensor $R_p$ has the following properties: it is alternating in the first $p$ variables, alternating in the last $p$ variables and it is invariant under the operation of interchanging the first $p$ variables with the last $p$ variables. Hence, at each point $x \in M$, $R_p$ can be regarded as a symmetric bilinear form on $\Lambda^p(x)$. By use of the inner product on $\Lambda^p(x)$, $R_p$ at $x$ may then be identified with a self-adjoint linear operator $R_p$ on $\Lambda^p(x)$. Explicitly, this identification is given by

$$ \langle R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p \rangle = R_p(u_1, \ldots, u_p, v_1, \ldots, v_p) $$

with $u_i, v_j \in T_x M$. From now on, we use the same notations for the $p$th curvature operators and the $p$th curvature tensors. The tensor $R_p$ satisfies the Bianchi identity which can be expressed in the following way [3]:

$$ \text{Alt } R_p = 0, $$

where $\text{Alt}$ is the skew symmetrization operator given by

$$ \text{Alt } R_p(v_1, \ldots, v_{2p}) = \frac{1}{(2p)!} \sum_{r \in S_p} \varepsilon(r) R_p(v_{r(1)}, \ldots, v_{r(2p)}) $$

with $v_i \in T_x M$.

When $n$ is a multiple of 4, $p = n/2$ and $M$ is oriented, the Bianchi identity for $R_p$ admits another interpretation in terms of the Hodge star operator on $\Lambda^p(M)$:

$$ \text{Alt } R_p(e_1, \ldots, e_n) = \frac{p! p!}{n!} \text{trace } R_p $$

and hence for the case $p = n/2$, the Bianchi identity for $R_p$ reduces to

$$ \text{trace } R_p = 0. $$

Taking $p = n$, the space $\Lambda^n(x)$ is one dimensional and hence the self-adjoint linear operator $R_n : \Lambda^n(x) \to \Lambda^n(x)$ is a scalar multiple of the identity. More explicitly, when expressed globally, the line bundle homomorphism $R_n : \Lambda^n(M) \to \Lambda^n(M)$ is

$$ R_n = K I $$
where $I$ is the identity automorphism of $\bigwedge^n(M)$ and $K$ is the Lipschitz-Killing curvature of $M$. Furthermore, for $x \in M$,

$$K(x) = R_n(e_1, \ldots, e_n, e_1, \ldots, e_n),$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis for $T_x M$. The generalized Gauss-Bonnet theorem [5] expresses the Euler-Poincaré characteristic $\chi$ of a compact oriented Riemannian manifold of even dimension $n$ as an integral

$$\chi = \frac{2}{c_n} \int_M K \, dV,$$

where $K$ is the Lipschitz-Killing curvature of $M$, $c_n$ is the volume of Euclidean unit $n$-sphere and $dV$ is the volume element of $M$. Now we show that the Lipschitz-Killing curvature $K$ of $M$ can be expressed in terms of $R_p$ and the Hodge $*$-operator. Let $M$ be an oriented Riemannian manifold of even dimension $n$, then according to [5], the Lipschitz-Killing curvature $K$ of $M$ is the function whose value at $x \in M$ is

$$\frac{p!(n-p)!}{n!} \text{trace} \left( *R_{n-p} * R_p \right).$$

For an oriented Riemannian manifold of dimension $n = 4k$, we can consider the middle curvature operator $R_{2k}$, and if this operator satisfies the condition

$$R_{2k} * = * R_{2k},$$

then, since $*^2 = \text{Identity}$, the trace formula for $K$ reduces to

$$K = \frac{(2k)!^2}{(4k)!} \text{trace} R_{2k}^2 \geq 0.$$

Next we consider a necessary condition for the existence of a Thorpe metric [5]:

**THEOREM 2.1.** Let $M$ be a compact orientable $4k$-dimensional Riemannian manifold which admits a Thorpe metric. Then

$$\chi \geq \frac{k!k!}{(2k)!} |P_k|,$$

where $\chi$ is the Euler characteristic of $M$ and $P_k$ is the $k$th Pontrjagin number of $M$. In particular, $\chi \geq 0$. Furthermore, $\chi = 0$ if and only if $R_{2k} = 0$.

**PROOF.** The de Rham representation for the $k$th Pontrjagin class of $M$ [2] is the differential $4k$-form

$$\frac{(2k)!^3}{(2k!)^2 (2\pi)^{2k}} \text{trace} (R_{2k} * R_{2k}) \, dV.$$
Since $R_{2k}$ commutes with $*$ it also commutes with $I \pm *$, where $I$ denotes the identity operator on $\wedge^{2k}$. Hence $R_{2k}(I \pm *)$ is self-adjoint and

$$0 \leq \text{trace} [R_{2k}(I \pm *)]^2 = 2 \left[ \text{trace}(R_{2k})^2 \pm \text{trace}(R_{2k} \ast R_{2k}) \right],$$

and so

$$\text{trace}(R_{2k})^2 \geq |\text{trace} (R_{2k} \ast R_{2k})|.$$ 

This means that

$$\chi \geq \frac{k!k!}{(2k)!} |P_k|,$$

and since $K \geq 0$, we have $\chi = 0$ if and only if $K$ is identically zero. $K \equiv 0$ is equivalent to $R_{2k} = 0$ and this completes the proof. $\square$

3. The case of $\chi = ((2!2!)/4!)|P_2|$

In this section we prove that a compact orientable Einstein-Thorpe manifold of dimension 8 that satisfies the above topological equality must be flat.

**Lemma 3.1.** Let $(M, g)$ be a Riemannian manifold of dimension 8. Then

$$\text{trace } R_4 = \frac{1}{2^2} \left( \frac{1}{6} \right) \left\{ \frac{1}{2} S^2 - 4 |\text{ric}_0|^2 + 4 |R|^2 \right\},$$

where $S$ is the scalar curvature, $\text{ric}_0$ is the traceless Ricci curvature and $R$ is the curvature.

**Proof.** For 4-forms $\{e_a \wedge e_b \wedge e_c \wedge e_d\}$ and with the Einstein summation,

$$\text{trace } R_4 = \frac{1}{2^2} R^{[ab} R_{cd]} = \frac{1}{2^2} R^{[ab} R_{cd]} = \frac{1}{2^2} \left( \frac{1}{6} \right) \left\{ R^{[ab} R_{cd]} + R^{ac} R_{db} + R^{ad} R_{bc} + R^{ab} R_{cd} + R^{ab} R_{cd} + R^{cd} R_{ab} \right\},$$

where $[ ]$ is a skew symmetrization, and $\{e_k\}_{k=1}^8$ is an orthonormal frame. We analyze the terms of this sum individually:

(i) $R^{ab} R_{cd} = 1/2 S^2 - 4 \text{ric}_{0c} \text{ric}_{0c} + 2 R_{cd} R_{cd}$;

(ii) $R^{ab} R_{cd} = - \text{ric}_{0b} \text{ric}_{0c} + R_{db} R_{dc} - R_{dc} R_{db}$. 

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Thus we obtain
\[ \text{trace } R_4 = \frac{1}{2^2} \left( \frac{1}{6} \right) \left\{ \frac{1}{2} S^2 - 4 \text{ric}_{0c}^c \text{ric}_{0c}^c - 4 \text{ric}_{0b}^b \text{ric}_{0b}^b + 2 R_{cd}^c R_{cd}^d + 4 R_{db}^d R_{ab}^d + R_{cd}^a R_{cd}^b \right\} \]
and this completes the proof.

Now we are ready to prove the main result.

**THEOREM 3.1.** Suppose that \((M^8, g)\) is a compact orientable Einstein-Thorpe manifold and that
\[ \chi = \left( \frac{212!}{4!} \right) |P_2|. \]
Then \((M^8, g)\) must be a flat manifold.

**PROOF.** By Theorem 2.1, we see that the above topological condition together with the Thorpe condition can be expressed as
\[ \text{trace } R_4 R_4 = |\text{trace } R_4 \ast R_4|. \]
We consider any orthonormal basis \(\{A_i\}_{i=1}^{14}\) in \(\wedge^+(M^8)\), and any orthonormal basis \(\{B_i\}_{i=1}^{14}\) in \(\wedge^-(M^8)\), where \(\wedge^+(M^8)\) and \(\wedge^-(M^8)\) denote the self dual space and the anti-self-dual space with respect to the Hodge \(\ast\) operator, respectively. Then we have
\[
\langle R_4(A_i), R_4(A_j) \rangle = \sum_{j=1}^{14} |R_4(A_i, A_j)|^2 + \sum_{j=1}^{14} |R_4(A_i, B_j)|^2,
\]
\[
\langle R_4(B_i), R_4(B_j) \rangle = \sum_{j=1}^{14} |R_4(B_i, B_j)|^2 + \sum_{j=1}^{14} |R_4(B_i, A_j)|^2,
\]
\[
\langle \ast R_4(A_i), R_4(A_j) \rangle = \sum_{j=1}^{14} |R_4(A_i, A_j)|^2 - \sum_{j=1}^{14} |R_4(A_i, B_j)|^2,
\]
\[
\langle \ast R_4(B_i), R_4(B_j) \rangle = \sum_{j=1}^{14} |R_4(B_i, A_j)|^2 - \sum_{j=1}^{14} |R_4(B_i, B_j)|^2,
\]
for each \(i = 1, 2, \ldots, 14\).

If we assume \(\text{trace } R_4 \ast R_4 \geq 0\), then by the given condition
\[ R_4(A_i, B_j) = R_4(B_i, B_j) = 0 \quad \text{for } i, j = 1, 2, \ldots, 14 \]
and this means that
\[ R_4^- = \frac{R_4 - \ast R_4}{2} \equiv 0. \]
Furthermore, by the Bianchi identity,

$$\text{trace } * R_4 \equiv 0,$$

and so we obtain

$$\text{trace } R_4 \equiv 0.$$ 

However, by Lemma 3.1 the Einstein condition (that means $\text{ric}_0 = 0$) implies $\text{trace } R_4 \geq 0$ and equality holds when its metric is flat and so we conclude that the given metric is flat.

On the other hand, if we assume

$$\text{trace } R_4 * R_4 < 0,$$

then we can repeat the above argument with a different choice of sign and this completes the proof. □

**Corollary 3.1.** (i) *The product manifold of $T^4$ with any compact orientable hyperbolic manifold of dimension 4 does not admit an Einstein-Thorpe metric.*

(ii) *The product manifold of $T^4$ with any compact complex hyperbolic manifold of complex dimension 2 does not admit an Einstein-Thorpe metric.*

*The manifolds described in (i) and (ii) satisfy $\chi = 0$ and $P_2 = 0.$*

**Proof.** It is easy to see that the manifolds described in part (i) and (ii) satisfy $\chi = 0$ and $P_2 = 0.$ This implies that any Einstein-Thorpe metric on the manifolds described in (i) and (ii) must be flat by Theorem 3.1, hence a contradiction. □

**References**


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