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Essential Norm and Weak Compactness of Composition Operators on Weighted Banach Spaces of Analytic Functions

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Abstract. Every weakly compact composition operator between weighted Banach spaces H_v^∞ of analytic functions with weighted sup-norms is compact. Lower and upper estimates of the essential norm of continuous composition operators are obtained. The norms of the point evaluation functionals on the Banach space H_v^∞ are also estimated, thus permitting to get new characterizations of compact composition operators between these spaces.

Let G be an open connected domain in C. In the present note we are interested in operators defined on Banach spaces of analytic functions of the following form:

$$H_v^\infty(G) := \{f \in H(G) : \|f\|_v := \sup_{z\in G} v(z)|f(z)| < \infty\},$$

$$H_v^0(G) := \{f \in H_v^\infty(G) : v|f| \text{ vanishes at infinity on } G\},$$

endowed with the norm $\|\cdot\|_v$. Here $H(G)$ denotes the space of analytic functions on G and $v\colon G \to \mathrm{R}_+$ is an arbitrary *weight*, *i.e.*, bounded continuous positive (which means *strictly positive* throughout the paper) function. We recall that a function $g\colon G \to \mathrm{R}_+$ vanishes at infinity if for any $\varepsilon > 0$ there is a compact set K in G such that $g(z) < \varepsilon$ for $z \in G \setminus K$.

These spaces appear naturally in the study of growth conditions of analytic functions and have been considered in many papers, see for example, [22], [25], [26], [14], [1], [6], [16], [17], [11], [5] and others. For analogous spaces with a sequence of weights see for example [4]. The purpose of this article is to continue our investigations [7] on composition operators C_ϕ, $C_\phi(f) := f \circ \phi$, defined for an adequate analytic map ϕ, between spaces of holomorphic functions of the type defined above. Composition operators have been extensively studied on various spaces of analytic functions on the disc. See [9], [13], [23]. Our Theorem 1 shows that a composition operator C_ϕ is compact on weighted H^∞-spaces if and only if it is weakly compact. The equivalence of compactness and weak compactness for homomorphisms between certain algebras of analytic functions has been recently investigated by several authors. We refer to [2], [15] and [28]. We also estimate in Theorem 4 the distance of C_ϕ to the space of compact operators (the so-called *essential norm* of C_ϕ). This result implies the characterization of compact operators in [7]. The essential norm of

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composition operators on Hardy spaces or weighted Bergman spaces has been studied in [24], [20], and for operators on Bloch spaces in [19]. See also [9] for more references.

As in the articles [1], [6], [5], [7], many results on weighted spaces of analytic functions of the type defined above and on operators defined on them must be formulated in terms of the so-called *associated weights* and not directly in terms of the weight v. The associated weight is defined by

$$\tilde{v}(z) := 1/\sup\{|f(z)| : f \in H_v^\infty(G), \|f\|_v \leq 1\} = 1/\|\delta_z\|_v.$$

Here δ_z is the evaluation functional at the point z. This weight is better tied to the space $H_v^\infty(G)$ than v itself. The associated weights are also continuous and $\tilde{v} \geq v \geq 0$. Clearly, $H_v^\infty(G) = H_{\tilde{v}}^\infty(G)$ isometrically (and, in many cases, also $H_v^0(G) = H_{\tilde{v}}^0(G)$ isometrically, see [7]), so one could restrict oneself to the *essential weights* (like in [27] or [1]), *i.e.*, those for which $v \sim \tilde{v}$, which means that there is a constant C such that:

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for each } z \in G.$$

In many papers (like in [7]) such a restriction was not made because there was no clear way of calculating explicitly $\tilde{v}$ and no useful characterization of essential weights. In the present paper we fill this gap of the theory for radial weights on the disc (and also on some other classical domains, see the remark at the end of the paper) giving in Propositions 6 and 7 various ways of calculating directly $\tilde{v}$ from v (without a reference to holomorphic functions) and giving a characterization of essential weights as those equivalent to log-convex functions (Proposition 7). This allows to avoid associated weights in the characterizations of bounded, compact and weakly compact composition operators (Corollary 8). The calculation of the associated weight is the same as the calculation of the norm of the point evaluation functionals on $H_v^\infty(G)$. We do it for instance for radial weights and $G = \mathrm{D}$.

Our notation is standard. See [8], [9], [10], [12], [23]. By ζ and z we always denote complex variables from appropriate domains. The set of all natural numbers and all non-negative integers are denoted by N and N_0 respectively. By C^* we denote the extended complex plane.

A map $T \in L(E, F)$ from the Banach space E to the Banach space F is called *compact*, *weakly compact*, Rosenthal, if it maps the closed unit ball of E onto a relatively compact, a relatively weakly compact, a conditionally weakly compact set in F. A subset A in a Banach space E is called *conditionally weakly compact*, if every sequence in A admits a weak Cauchy subsequence. Clearly every weakly compact operator is Rosenthal. Rosenthal's l_1 theorem [10] says that a Banach space E contains no subspace isomorphic to l_1 if and only if every bounded sequence in E has a weak Cauchy subsequence. The class $\mathcal{S}$ of operators not fixing a copy of l_∞ (*i.e.*, those operators $T\colon E \to F$ such that for every subspace $E_1 \subseteq E$ isomorphic to l_∞ the map T restricted to E_1 is *not* an isomorphism) forms an operator ideal (see [21]). In fact, one can prove that $T\colon E \to F$ belongs to $\mathcal{S}$ if and only if for every operator $Q\colon l_\infty \to E$ the composition TQ is weakly compact. We recall that an operator $T\colon E \to F$ is strictly singular if, for each infinite dimensional subspace $M \subset E$, T is not an isomorphism on M. The operator T is strictly cosingular if a closed subspace $N \subset F$ has finite codimension whenever $Q \circ T$ is a surjection, where Q is the quotient map from F onto F/N. We remark that there are examples of spaces of type $H_v^\infty(\mathrm{D})$ which are not

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isomorphic to l_∞. We refer to [16] and [17]. In our next theorem compactness cannot be replaced by more restrictive properties, such as nuclearity. An example can be seen in [27].

Theorem 1 *Let G_1 and G_2 be open connected domains in $\mathbb{C}$ such that $\mathbb{C}^* \setminus G_1$ has no one-point component. Let $\phi\colon G_2 \to G_1$ be an analytic map. Let v and w be arbitrary weights on G_1 and G_2, respectively. The composition operator $C_\phi\colon H_v^\infty(G_1) \to H_w^\infty(G_2)$ is either compact or it is an isomorphism when restricted to some subspace isomorphic to l_∞. In particular, every weakly compact or Rosenthal or strictly singular or strictly cosingular composition operator C_ϕ is automatically compact.*

Proof Suppose that C_ϕ is not compact. By the weak compactness Theorem [23], Section 2.4 (in the form given in [7, Lemma 3.1]), there is a bounded sequence (f_n) in $H_v^\infty(G_1)$ converging to 0 uniformly on the compact subsets of G_1 such that the sequence $(C_\phi f_n)$ does not tend to 0 in the Banach space $H_w^\infty(G_2)$. Accordingly we can find $c > 0$ and a sequence $(z_n) \subset G_2$ with $\phi(z_n)$ tending to a point in the boundary of G_1 in $\mathbb{C}^*$ and such that

$$w(z_n)\big|f_n\big(\phi(z_n)\big)\big| > c \quad \text{for all } n \in \mathbb{N}.$$

Without loss of generality we may assume that $\phi(z_n)$ tends to a point $z_0 \in L$, where $L \subseteq \mathbb{C}^* \setminus G_1$ is a closed connected set of more than one point. By the Riemann Mapping Theorem, $U := \mathbb{C}^* \setminus L$ is biconformally equivalent with the unit disc $\mathbb{D}$. Thus, by Corollary on p. 204 in [12], there is a subsequence of $\big(\phi(z_n)\big)$ which is an interpolating sequence for $H^\infty(U)$. We denote this subsequence in the same way. By Theorem III.E.4 in [29], there is a sequence $(h_k) \subset H^\infty(U)$ such that

$$h_k\big(\phi(z_n)\big) = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{if } n \neq k \end{cases}$$

and there is a constant $M > 0$ such that

$$\sum_{k=1}^{\infty} |h_k(z)| \leq M \quad \text{for every } z \in U.$$

See the argument "b) implies c)" in [29], III.E.4.

We define a map $T\colon l_\infty \to H_v^\infty(G_1)$, $(\xi_k)_{k=1}^\infty \mapsto \big(z \mapsto \sum_{k=1}^\infty \xi_k f_k(z) h_k(z)\big)$. Since

$$\big\|T\big((\xi_k)\big)\big\|_v \leq M \sup_k \|f_k\|_v \sup_k |\xi_k| \quad \text{for all } (\xi_k)_{k=1}^\infty \in l_\infty,$$

the map T is well-defined, linear and continuous. Further, we define a map $S\colon H_w^\infty(G_2) \to l_\infty$, $f \mapsto \big(f(z_n)\frac{1}{f_n(\phi(z_n))}\big)_{n=1}^\infty$. Now,

$$\|S(f)\| \leq \frac{1}{c}\|f\|_w \quad \text{for all } f \in H_w^\infty(G_2),$$

hence S is a well-defined, continuous, linear map. It can easily be checked that $S \circ C_\phi \circ T = \mathrm{Id}_{l_\infty}$. Clearly C_ϕ is neither a Rosenthal nor weakly compact nor strictly singular nor strictly cosingular operator. ∎

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The following result was known for many radial weights on D. See [17], [25].

Corollary 2 *Let v be an arbitrary weight on* D*. Then $H_v^\infty(\mathrm{D})$ contains a copy of l^∞ and, in particular, it is not reflexive.*

Corollary 3 *Let $C_\phi\colon H_v^0(\mathrm{D}) \to H_w^0(G)$ be weakly compact. If the closed unit ball of $H_v^0(\mathrm{D})$ respectively $H_w^0(G)$ is dense with respect to the compact-open topology in the closed unit ball of $H_v^\infty(\mathrm{D})$ respectively $H_w^\infty(G)$, then C_ϕ is compact.*

Proof This follows directly from Theorem 1, since our assumptions ensure that $H_v^\infty(\mathrm{D}) = H_v^0(\mathrm{D})''$, $H_w^\infty(G) = H_w^0(G)''$ and $C_\phi'' = C_\phi$ by [6]. ■

A weight v on the unit disc D is called *radial* if $v(z) = v(|z|)$ for any $z \in \mathrm{D}$. Let us recall that for radial weights, we can take instead of the weight v its non-increasing majorant, *i.e.*, the radial function u on D, $u(r) = \sup\{v(R) : r \le R < 1\}$, and both the spaces $H_v^\infty(\mathrm{D})$, $H_v^0(\mathrm{D})$ and the norm $\|\cdot\|_v$ do not change. Accordingly *we assume from now on that radial weights on* D *are non-increasing.* Moreover [7, Cor. 1.2], $H_v^\infty(\mathrm{D})$ is not equal to $H^\infty(\mathrm{D})$ if and only if $\lim_{r\to 1} v(r) = 0$.

The essential norm of a continuous linear operator T is defined by $\|T\|_e = \inf\{\|T-K\| : K \text{ is compact}\}$. Since $\|T\|_e = 0$ if and only if T is compact, the estimates on $\|T\|_e$ lead to conditions for T to be compact. A fundamental sequence of compact sets $(L_n)_n$ in G is an increasing sequence of compact sets which covers G and such that every compact subset of G is contained in one of the sets L_n.

Theorem 4 *Let $\phi\colon G \to \mathrm{D}$ be an analytic map on an open connected domain G in* C*. Let v be a radial, continuous weight which is decreasing in $[0,1[$ and such that $\lim_{r\to 1} v(r) = 0$. Let w be weight on G which vanishes at infinity. The composition operator $C_\phi\colon H_v^\infty(\mathrm{D}) \to H_w^\infty(G)$ is continuous if and only if*

$$\sup_{z\in G} \frac{w(z)}{\tilde v\big(\phi(z)\big)} < \infty.$$

In this case, $\|C_\phi\| = \sup_{z\in G} \frac{\tilde w(z)}{\tilde v(\phi(z))}$. If the supremum is finite and $(L_n)_n$ is a fundamental sequence of compact sets in G, then

$$\lim_{n\to\infty} \sup_{z\in G\setminus L_n} \frac{w(z)}{\tilde v\big(\phi(z)\big)} \le \|C_\phi\|_e \le 2 \lim_{n\to\infty} \sup_{z\in G\setminus L_n} \frac{w(z)}{\tilde v\big(\phi(z)\big)}.$$

Proof The proof of the characterization of the continuity of C_ϕ is the same as in [7, Prop. 2.1], where it was given in the case $G = \mathrm{D}$. If C_ϕ is continuous, $\|C_\phi f\|_w \le \sup_{z\in G}\Big(\tilde w(z)/\tilde v\big(\phi(z)\big)\Big)\|f\|_v$. On the other hand, since $C_\phi'\delta_z = \delta_{\phi(z)}$ for each $z \in G$, we have

$$\|C_\phi\| = \|C_\phi'\| \ge \frac{\|C_\phi'\delta_z\|_v}{\|\delta_z\|_w} = \frac{\tilde w(z)}{\tilde v\big(\phi(z)\big)} \quad \text{for every } z \in G.$$

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We prove first the lower estimate of the essential norm. We proceed by contradiction. Assume we can find constants $b > c > 0$ and a compact operator $K\colon H_v^\infty(\mathrm{D}) \to H_w^\infty(G)$ such that

$$\lim_{n\to\infty} \sup_{z\in G\setminus L_n} \frac{w(z)}{\tilde{v}\big(\phi(z)\big)} > b > c > \|C_\phi - K\|.$$

We find a sequence $(z_n)_n$ in G with $z_n \notin L_n$ for each n and $w(z_n) > b\tilde{v}\big(\phi(z_n)\big)$ for n large enough. Since w vanishes at infinity on G, $\lim_{n\to\infty} w(z_n) = 0$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} |\phi(z_n)| = 1$, since $\tilde{v}(\zeta) > 0$ for every $\zeta \in \mathrm{D}$. We select an increasing sequence $\big(\alpha(n)\big)_n$ of natural numbers going to infinity such that $|\phi(z_n)|^{\alpha(n)} \geq c/b$ for each $n \in \mathrm{N}$.

For every $n \in \mathrm{N}$, we can apply [5, 1.2] to find f_n in the unit ball of $H_v^0(\mathrm{D})$ such that $|f_n\big(\phi(z_n)\big)| = 1/\tilde{v}\big(\phi(z_n)\big)$. We set $g_n(z) := z^{\alpha(n)} f_n(z)$, $z \in \mathrm{D}$. The sequence $(g_n)_n$ is contained in the unit ball of $H_v^\infty(\mathrm{D})$ and it converges to zero in the compact open topology. Since the operator K is compact, we have that $\lim_{n\to\infty} \|Kg_n\|_w = 0$. We conclude, for $n \in \mathrm{N}$,

$$c > \|C_\phi - K\| \geq \|(C_\phi - K)g_n\|_w \geq \|C_\phi g_n\|_w - \|Kg_n\|_w.$$

This implies

$$c > \limsup_{n\to\infty} \|C_\phi g_n\|_w \geq \limsup_{n\to\infty} w(z_n)|g_n\big(\phi(z_n)\big)| = \limsup_{n\to\infty} \frac{w(z_n)}{\tilde{v}\big(\phi(z_n)\big)} |\phi(z_n)|^{\alpha(n)} \geq c,$$

which is a contradiction.

We check the upper estimate. To do this, we first take a sequence of linear operators $C_k\colon H(\mathrm{D}) \to H(\mathrm{D})$, $k \in \mathrm{N}$, which are continuous for the compact open topology, such that $C_k f \to f$ uniformly on every compact subset of D and such that $C_k\colon H_v^\infty(\mathrm{D}) \to H_v^\infty(\mathrm{D})$ is a well-defined compact operator with $\|C_k\| = 1$. One can take, for example, $C_k f(z) = \frac{1}{k+1} \sum_{s=0}^{k} (\sum_{l=0}^{s} a_l z^l)$, $z \in \mathrm{D}$, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, the operator which gives the Cesáro means of the partial sums of the Taylor expansion of f (*cf.* [4, Section 1]) or one could also take $C_k f(z) = f\big(\frac{k}{k+1} z\big)$, $z \in \mathrm{D}$.

For $n \in \mathrm{N}$ fixed we have

$$\|C_\phi\|_e \leq \|C_\phi - C_\phi C_k\| = \|C_\phi(\mathrm{Id} - C_k)\| \leq \sup_{\|f\|_v \leq 1} \sup_{z\in G\setminus L_n} w(z)|\big((\mathrm{Id} - C_k)f\big)\big(\phi(z)\big)|$$
$$+ \sup_{\|f\|_v \leq 1} \sup_{z\in L_n} w(z)|\big((\mathrm{Id} - C_k)f\big)\big(\phi(z)\big)| =: I_n + J_n.$$

To estimate the first term I_n observe that, for $\|f\|_v \leq 1$ and $z \in G \setminus L_n$,

$$\tilde{v}\big(\phi(z)\big)|(\mathrm{Id} - C_k)f\big(\phi(z)\big)| \leq \sup_{\zeta\in\mathrm{D}} \tilde{v}(\zeta)|\big((\mathrm{Id} - C_k)f\big)(\zeta)| \leq \|\,\mathrm{Id} - C_k\| \leq 2.$$

Thus $I_n \leq 2\sup_{z\in G\setminus L_n} w(z)/\tilde{v}\big(\phi(z)\big)$.

To estimate the second term J_n, we put

$$A := \max_{z\in G} w(z)$$

and, further, $s_n := \max_{z\in L_n} |\phi(z)|$ which takes values in $[0,1[$. We have

$$J_n \le A \sup_{\|f\|_v\le 1} \sup_{|\zeta|\le s_n} \left|\big((\mathrm{Id} - C_k) f\big)(\zeta)\right|.$$

The sequence of operators $(\mathrm{Id} - C_k)_k$ satisfies $\lim_{k\to\infty} (\mathrm{Id} - C_k) g = 0$ for each g in $H(\mathrm{D})$, and the space $H(\mathrm{D})$ endowed with the compact open topology co is a Frećhet space. By the Banach-Steinhaus theorem, $(\mathrm{Id} - C_k)_k$ converges to zero uniformly on the compact subsets of $\big(H(\mathrm{D})\,\mathrm{co}\big)$. Since the unit ball of $H_v^\infty(\mathrm{D})$ is a compact subset of $\big(H(\mathrm{D}), \mathrm{co}\big)$ we conclude that

$$\lim_{k\to\infty} \sup_{\|f\|_v\le 1} \sup_{|\zeta|\le s_n} \big(\big((\mathrm{Id} - C_k) f\big)(\zeta)\big) = 0.$$

Consequently,

$$\|C_\phi\|_e \le \limsup_{k\to\infty} \|C_\phi - C_\phi C_k\| \le \limsup_{k\to\infty} I_k + \limsup_{k\to\infty} J_k \le 2 \sup_{z\in G\setminus L_n} \frac{w(z)}{\tilde{v}\big(\phi(z)\big)}.$$

This implies that $\|C_\phi\|_e \le 2 \lim_{n\to\infty} \sup_{z\in G\setminus L_n} w(z)/\tilde{v}\big(\phi(z)\big)$. ■

A function $\eta\colon [0,1) \to \mathrm{R}_+$ is called *log-convex* if $\log\eta(t)$ is a convex function of $\log t$. Our purpose is to characterize the radial weights on D which are essential. Two weights v and w are said to be *equivalent*, if there are $c, C > 0$ such that $cv \le w \le Cv$.

Lemma 5 *Let $w\colon [0,1] \to \mathrm{R}_+$ be a continuous function with $w(1) = 0$, $w(r) > 0$ for all $r \in\,]0,1[$. Let $w\colon \mathrm{D} \to \mathrm{R}_+$ be the radial extension $w(z) = w(|z|)$ and let $\bar{w}(z) := 1/\sup_{n\in\mathrm{N}} \frac{|z|^n}{\|\zeta^n\|_w}$, where the norm of the monomial ζ^n is calculated in $H_w^\infty(\mathrm{D})$. If w is log-convex, i.e., $\varphi(t) := \log w(e^t)$ is convex in $]-\infty, 0[$, then w is equivalent to both $\tilde{w}$ and $\bar{w}$.*

Remark Weights similar to $\bar{w}$ appear in [18, Satz 1.9].

Proof For $t \in [0,\infty[$, we define $b(t) := \sup_{0<r<1} w(r)r^t$. If $t = 0$, $b(0) = w(0)$. If $t > 0$, the supremum is a maximum attained at some $r_t \in\,]0,1[$, because $g_t(r) := w(r)r^t$ is continuous in $[0,1]$, $g_t(0) = g_t(1) = 0$ and $g_t(r) > 0$ for all $r \in\,]0,1[$. Thus $b(t) > 0$ for all $t > 0$ and b is a decreasing function on $[0,\infty[$. Moreover, if $n \in \mathrm{N}_0$, $b(n) = \|\zeta^n\|_w$, the norm calculated in the Banach space $H_w^\infty(\mathrm{D})$. We set $a(t) := 1/b(t)$, $t \in [0,\infty[$.

Claim $\sup_{t\in[0,\infty[} a(t)r^t = 1/w(r)$ for all $r \in\,]0,1[$.

Indeed, by the very definition, $\sup_{t\in[0,\infty[} a(t)r^t \le 1/w(r)$ holds for every $r \in\,]0,1[$. We fix $r_0 \in\,]0,1[$ and we put $t_0 := \log r_0 \in\,]-\infty, 0[$. Since φ is increasing and convex at t_0, there is $y_0 \ge 0$ such that

$$y_0(\eta - t_0) + \varphi(t_0) \le \varphi(\eta) \quad \text{for all } \eta \in\,]-\infty, 0[\,.$$

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This yields

$$b(y_0) = \sup_{0<r<1} w(r)r^{y_0} = \exp\Big(\sup_{0<r<1}\big(\log w(r) + y_0 \log r\big)\Big)$$
$$= \exp\Big(\sup_{-\infty<\eta<0}\big(y_0\eta - \varphi(\eta)\big)\Big) \le \exp\big(y_0 t_0 - \varphi(t_0)\big) = w(r_0)r_0^{y_0}.$$

Therefore $\sup_{0\le t<\infty} a(t)r_0^t \ge a(y_0)r_0^{y_0} = r_0^{y_0}/b(y_0) \ge 1/w(r_0)$, and the claim is proved.

To complete the proof we show

$$w(z) \le \tilde{w}(z) \le \bar{w}(z) \le \max\big(2, w(0)/w(1/2)\big)w(z) \quad \text{for all } z \in \mathrm{D}.$$

The first inequality is obvious. The second inequality can be seen as follows: for $z \in \mathrm{D}$ we have

$$\frac{1}{\bar{w}(z)} = \sup_{n\in\mathrm{N}} \frac{|z|^n}{\|\zeta^n\|_w} = \sup_{n\in\mathrm{N}} \Big|\frac{z^n}{\|\zeta^n\|_w}\Big| \le \sup\Big\{|f(z)| : |f| \le \frac{1}{w}, f \in H(\mathrm{D})\Big\} = \frac{1}{\tilde{w}(z)}.$$

To see the last inequality, fix first $r \in \big[\frac{1}{2}, 1\big[$. We have, by our claim above,

$$\frac{1}{w(r)} = \sup_{t\in[0,\infty[} a(t)r^t \le \sup_{n\in\mathrm{N}_0} a(n+1)r^n = \frac{1}{r}\sup_{n\in\mathrm{N}_0} a(n+1)r^{n+1} = \frac{1}{r}\frac{1}{\bar{w}(r)} \le 2\frac{1}{\bar{w}(r)}.$$

On the other hand, if $r \in \big[0, \frac{1}{2}\big]$, we have $w(1/2) \le w(r) \le \tilde{w}(r) \le \bar{w}(r) \le w(0)$, since $1/\bar{w}(r) = \sup_{n\in\mathrm{N}_0} a(n)r^n \ge a(1) \ge a(0) = 1/w(0)$. This implies that $\bar{w}(r) \le \big(w(0)/w(1/2)\big)w(r)$ for all $r \in [0, 1/2]$ as desired. ■

Let $v\colon \mathrm{D} \to \mathrm{R}_+$ be a radial, continuous weight which is decreasing in $[0, 1[$. We set $v(1) := \lim_{r\to 1} v(r)$. If $v(1) > 0$, then v is essential and $\tilde{v}$ is equivalent to the constant function 1. If we assume that $v(1) = 0$, we can apply [5, 1.2, 1.5] and [7, 1.1] to obtain that the associated weight $\tilde{v}$ is radial, continuous, decreasing in $[0, 1[$, $\lim_{r\to 1} \tilde{v}(r) = 0$ and $1/\tilde{v}$ is subharmonic. By Hadamard three circles theorem $1/\tilde{v}$ is log-convex. See [3, 4.419 and 4.4.26]. The next two propositions follow now directly from Lemma 5.

Proposition 6 *Let v be a radial, continuous weight on D such that $\lim_{r\to 1} v(r) = 0$. Then $1/\tilde{v}(z) = \|\delta_z\|_v$, $z \in \mathrm{D}$, the norm calculated in $H_v^\infty(\mathrm{D})'$, is a log-convex function of $r \in [0, 1[$ and it is equivalent to $u(z) := 1/\bar{v}(z) = \sup_{n\in\mathrm{N}} \frac{|z|^n}{\|\zeta^n\|_v}$, $z \in \mathrm{D}$.*

Proposition 7 *The following conditions are equivalent for a radial, continuous weight v: $\mathrm{D} \to \mathrm{R}_+$ which satisfies $\lim_{r\to 1} v(r) = 0$.*

(a) *v is essential,*
(b) *$1/v(r)$ is equivalent to a log-convex function,*
(c) *$v \sim u$, with $u(z) := \bar{v}(z) = 1/\sup_{n\in\mathrm{N}} \frac{|z|^n}{\|\zeta^n\|_v}$, $z \in \mathrm{D}$.*

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Remarks 1. Analyzing the proof of Lemma 5 one observes easily that

$$\tilde{v}(z) \le u(z) \le C_v \tilde{v}(z),$$

where

$$C_v := \max\left(\frac{v(0)}{v(\rho)}, \frac{1}{\rho}\right),$$

with $\rho \in (0,1)$ satisfying

$$\frac{1}{v(t)} \ge (t-\rho) + \frac{1}{v(\rho)} \quad \text{for all } t \in [0,1).$$

2. By Example 3.3 of [5], which is based on an example by Clunie and Kövari, there are radial, continuous weights $v\colon \mathbb{C} \to \mathbb{R}_+$, which are decreasing in $[0,+\infty[$, such that $\lim_{r\to\infty} r^n v(r) = 0$ for all $n \in \mathbb{N}$ and for which $1/v$ is log-convex in $[0,+\infty[$, but still v is not an essential weight. Accordingly the characterization given in Proposition 7 is not valid for weights defined on $\mathbb{C}$.

Proposition 6 above permits to obtain further characterizations of continuity and compactness of composition operators as follows.

Corollary 8 *Let $\phi\colon G \to \mathbb{D}$ be an analytic map. Let $v\colon \mathbb{D} \to \mathbb{R}_+$ be a radial, continuous weight, decreasing on $[0,1[$ and such that $\lim_{r\to 1} v(r) = 0$. Let w be a weight vanishing at infinity on G.*

(a) The map $C_\phi\colon H_v^\infty(\mathbb{D}) \to H_w^\infty(G)$ is bounded if and only if

$$\sup_{n\in\mathbb{N}} \frac{\|\phi^n(z)\|_w}{\|\zeta^n\|_v} < \infty.$$

(b) The map $C_\phi\colon H_v^\infty(\mathbb{D}) \to H_w^\infty(G)$ is (weakly) compact if and only if

$$\lim_{n\to\infty} \frac{\|\phi^n(z)\|_w}{\|\zeta^n\|_v} = 0.$$

Proof The necessity of (a) follows trivially, since $f_n(z) = z^n/\|\zeta^n\|_v$ belongs to the unit ball of $H_v^\infty(\mathbb{D})$ for each $n \in \mathbb{N}$. To prove the converse, by Proposition 6 there is $C > 0$ such that $1/\tilde{v}(r) \le C \sup_{n\in\mathbb{N}} r^n/\|\zeta^n\|_v$ for every $r \in\,]0,1[$. Then

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}\big(\phi(z)\big)} \le C \sup_{z\in\mathbb{D}} \sup_{n\in\mathbb{N}} \frac{w(z)|\phi(z)|^n}{\|\zeta^n\|_v} \le C \sup_{n\in\mathbb{N}} \frac{\|\phi(z)^n\|_w}{\|\zeta^n\|_v} < \infty.$$

The conclusion follows now from Theorem 4.

To prove the necessity of part (b) we only have to show that the sequence of polynomials $(f_n)_n$ converges to zero for the compact open topology. Indeed, if $0 < R < r < 1$ and $|z| < R$, we have

$$|f_n(z)| \le \frac{R^n}{\|\zeta^n\|_v} = \frac{R^n}{r^n} \frac{r^n}{\|\zeta^n\|_v} \le \left(\frac{R}{r}\right)^n \frac{1}{\tilde{v}(r)},$$

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which tends to zero as $n \to \infty$.

To complete the proof, we assume the condition in part (b) and take a bounded sequence $(g_k)_k$ in $H_v^\infty(\mathbb{D})$ which converges to zero for the compact open topology. By Proposition 6 there is $C > 0$ such that

$$|g_k(z)| \le C \sup_{n\in\mathbb{N}} \frac{|z^n|}{\|\zeta^n\|_v} \quad \text{for all } z \in \mathbb{D}.$$

To prove that $\lim_{k\to\infty} \|C_\phi g_k\|_w = 0$, we fix $\varepsilon > 0$. By assumption, there is $N \in \mathbb{N}$ with $w(z)|\phi(z)|^n/\|\zeta^n\|_v < \varepsilon/C$ for each $z \in \mathbb{D}$, $n \ge N$. We find a compact subset K of G such that, for $z \in G \setminus K$, $w(z) \le (\varepsilon/C) \max_{1\le n\le N} \|\zeta^n\|_v$. If $z \in G \setminus K$, we have

$$w(z)|g_k\bigl(\varphi(z)\bigr)| \le C \sup_{n\in\mathbb{N}} w(z) \frac{|\phi(z)|^n}{\|\zeta^n\|_v} < \varepsilon.$$

Thus, for all $k \in \mathbb{N}$,

$$\|C_\phi g_k\|_w \le \bigl(\max_{z\in G} w(z)\bigr)\bigl(\sup_{\zeta\in\phi(K)} |g_k(\zeta)|\bigr) + \varepsilon,$$

from where the conclusion follows. ■

It is worth noting that there are analogues of Propositions 6, 7 and Corollary 8 for other domains G than $\mathbb{D}$. For instance, if $G = \Pi_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ then a weight v on Π_+ depending only on the real part of z, tending to zero at the imaginary axis with $\lim_{t\to\infty} v(t) = a < +\infty$ is essential if and only if $1/v$ is equivalent to a function with convex logarithm if and only if $v \sim u$, $u(z) := 1/\sup_{n\in\mathbb{N}} \frac{|e^{-z}|^n}{\|e^{-n\zeta}\|_v}$. Under the above assumptions an analogue of Corollary 8 is true whenever the n-th monomial function is replaced by $z \mapsto e^{-nz}$.

Analogously, if $G = S := \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$ and v is a weight depending only on the real part of the argument and tending to zero at the boundary, then the above result holds whenever we take sup and lim over all integers n (both positive and negative!).

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