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On a unified approach to the law of the iterated logarithm for martingales

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There are two distinct approaches in the literature to framing a version of the law of the iterated logarithm for martingales. One involves norming by constants, using the martingale variance and the other involves norming by random variables, using the sums of conditional variances of the increments, given their past. In this paper a portmanteau approach is provided, still based on the Skorokhod representation of the martingale, but involving normalization by more general random variables. This extends the functional forms of all the previously existing results.

1. Introduction and results

The literature contains various laws of the iterated logarithm for martingales which are not directly comparable. There are versions based on norming by constants (for example, Heyde and Scott [1]) and versions based on norming by random variables (for example, Strassen [7], Stout [5]). The former use variances and the latter, conditional variances. Here we employ a Skorokhod representation approach, based on the law of the iterated logarithm for brownian motion, to relate the earlier results.

Let ϕ be the real-valued function on (e, ∞) defined by

$$\phi(t) = (2t \log \log t)^{\frac{1}{2}} .$$

If $W(t)$ $(t \geq 0)$ is standard brownian motion then (Lévy [3])

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$$\lim_{t\to\infty} \sup W(t)/\phi(t) = +1 \quad \text{almost surely,}$$

(1)

$$\lim_{t\to\infty} \inf W(t)/\phi(t) = -1 \quad \text{almost surely.}$$

If $\{t_n, n \ge 1\}$ is a sequence of constants increasing to ∞ then (1) implies that

$$\limsup_{n\to\infty} W(t_n)/\phi(t_n) \le +1 \quad \text{almost surely,}$$

(2)

$$\liminf_{n\to\infty} W(t_n)/\phi(t_n) \ge -1 \quad \text{almost surely.}$$

If the points of the sequence $\{t_n\}$ are close enough together then equality holds throughout (2) with probability 1. In fact, if $\{T_n, n \ge 1\}$ is any sequence of random variables satisfying

(3) $$T_n > e \ , \quad T_n \uparrow \infty \text{ almost surely and } T_{n+1}^{-1}T_n \xrightarrow{\text{a.s.}} 1 \ ,$$

then (1) implies that

$$\limsup_{n\to\infty} W(T_n)/\phi(T_n) = 1 \quad \text{almost surely}$$

(4)

$$\liminf_{n\to\infty} W(T_n)/\phi(T_n) = -1 \quad \text{almost surely,}$$

and so the variables $S_n = W(T_n)$ satisfy the law of the iterated logarithm with a random norming.

Now let $\{S_n, F_n, n \ge 1\}$ be a zero-mean, square-integrable martingale on the probability space (Ω, F, P), where $F_0 = \{\phi, \Omega\}$ and F_n is the σ-field generated by S_j, $0 \le j \le n$. Let $S_0 = X_0 = 0$ and $S_n = \sum_1^n X_k$. We shall write

$$V_n^2 = \sum_1^n E\left(X_k^2 \mid F_{k-1}\right) , \quad U_n^2 = \sum_1^n X_k^2 , \quad s_n^2 = ES_n^2 = EV_n^2 ,$$

and $\{W_n, n \ge 1\}$ will denote a non-decreasing sequence of positive random variables such that $W_1^2 > e$. By extending the original probability space

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if necessary, we may suppose that there exists a brownian motion W and a non-decreasing sequence of non-negative variables $\{T_n, n \geq 1\}$ defined on our probability space such that $S_n = W(T_n)$ almost surely for all n (see the statement of the Skorokhod representation theorem in Strassen [7], Theorem 4.3). If the T_n satisfy condition (3) then (4) immediately gives us a law of the iterated logarithm for $\{S_n, n \geq 0\}$.

Unfortunately the T_n are quite difficult to compute from the martingale $\{S_n, F_n\}$, and the law is not very meaningful unless we can replace T_n by a variable W_n^2 which can be expressed as a function of the differences $X_1, X_2, \ldots$; W_n^2 should satisfy

(5) $$T_n^{-1}W_n^2 \xrightarrow{\text{a.s.}} 1 .$$

Strassen [6] formulated a functional law of the iterated logarithm which extends the law in (4). Consider the metric space (C, ρ) of all real-valued continuous functions on $[0, 1]$ with

$$\rho(x, y) = \sup_{0 \leq t \leq 1} |x(t)-y(t)| \quad \text{for} \quad x, y \in C .$$

Let K be the set of absolutely continuous $x \in C$ such that $x(0) = 0$ and

$$\int_0^1 \dot{x}(t)^2 dt \leq 1$$

where $\dot{x}$ denotes the derivative of x determined almost everywhere with respect to Lebesgue measure. For $u \in [0, 1]$ define

$$l = l(n, u) = \max\left\{j \leq n \mid W_j^2 \leq uW_n^2\right\} \quad (= 0 \text{ if the set is empty})$$

and

$$\mu_n(u) = \left[\phi\left(W_n^2\right)\right]^{-1}\left[S_l+\left(uW_n^2-W_l^2\right)\left(W_{l+1}^2-W_l^2\right)^{-1}X_{l+1}\right] ,$$

where $S_n = W(T_n)$ and $X_n = S_n - S_{n-1}$ $(S_0 = 0)$.

THEOREM A. *If* (3) *and* (5) *hold then* $\{\mu_n,\ n \geq 1\}$ *is relatively compact in* $C[0,1]$ *and the set of its almost sure limit points coincides with* K .

(Here we do not assume that $\{S_n,\ F_n,\ n \geq 0\}$ is a martingale, although we do in all the work which follows.)

If we are to use Theorem A to establish a law of the iterated logarithm for martingales, we must find conditions under which T_n can be approximated by functions W_n^2 of the martingale differences. It is convenient to work with martingales whose differences are truncated, and hence our technique is to approximate to $\{S_n\}$ by a truncated martingale $\{S_n^*\} = \{W(T_n^*)\}$, and then find functions W_n^2 which are close to T_n^* . Our main result is given in the following theorem.

THEOREM 1. *Let* $\{Z_n,\ n \geq 1\}$ *be a sequence of non-negative random variables and suppose that* Z_n *and* W_n *are* F_{n-1}*-measurable. If*

$$(6)\quad \lim_{n\to\infty} \left[\phi\left(W_n^2\right)\right]^{-1} \sum_1^n \{X_j I(|X_j| > Z_j) - E(X_j I(|X_j| > Z_j) \mid F_{j-1})\} = 0$$

almost surely,

$$(7)\quad \lim_{n\to\infty} W_n^{-2} \sum_1^n \left[E\left(X_j^2 I(|X_j| \leq Z_j) \mid F_{j-1}\right) - \{E(X_j I(|X_j| \leq Z_j) \mid F_{j-1})\}^2\right] = 1$$

almost surely,

$$(8)\quad \sum_1^\infty W_j^{-4} E\left(X_j^4 I(|X_j| \leq Z_j) \mid F_{j-1}\right) < \infty \text{ almost surely,}$$

and

$$(9)\quad \lim_{n\to\infty} W_{n+1}^{-1} W_n = 1 \text{ almost surely and } W_n \to \infty \text{ almost surely,}$$

then the conclusions of Theorem A *hold.*

Two obvious candidates for W_n for use in this theorem are $V_n \vee 2$ and $U_{n-1} \vee 2$. These lead to equivalent results if $\lim_{n\to\infty} V_n^{-2} U_n^2 = 1$ almost

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surely, which would commonly be the case; but interestingly enough this would not universally be true. When V_n is used in the norming the condition (7) simplifies usefully, giving the following result.

COROLLARY 1. *Let* $\{Z_n, n \geq 1\}$ *be as in Theorem* 1. *The conclusions of Theorem* A *hold for* $W_n = V_n \vee 2$ *if*

$$(10)\quad \lim_{n\to\infty} \left[\phi\left(V_n^2\right)\right]^{-1} \sum_1^n \{X_j I(|X_j| > Z_j) - E(X_j I(|X_j| > Z_j) \mid F_{j-1})\} = 0 \text{ almost surely,}$$

$$(11)\quad \lim_{n\to\infty} V_n^{-2} \sum_1^n E\left(X_j^2 I(|X_j| > Z_j) \mid F_{j-1}\right) = 0 \text{ almost surely,}$$

$$(12)\quad \sum_1^\infty V_j^{-4} E\left(X_j^4 I(|X_j| \leq Z_j) \mid F_{j-1}\right) < \infty \text{ almost surely,}$$

and

$$(13)\quad \lim_{n\to\infty} V_{n+1}^{-1} V_n = 1 \text{ almost surely, } V_n \to \infty \text{ almost surely.}$$

The next corollary extends the main result (Theorem 1) of Heyde and Scott [1] which deals with the case $\eta^2 = 1$ almost surely. It highlights the restrictiveness of a constant norming form.

COROLLARY 2. *If for some almost surely finite and non-zero random variable* η^2 *we have*

$$(14)\quad s_n^{-2} U_n^2 \xrightarrow{\text{a.s.}} \eta^2 \text{ and } s_n \to \infty ,$$

$$(15)\quad \sum_1^\infty s_j^{-1} E[|X_j| I(|X_j| > \varepsilon s_j)] < \infty , \text{ for all } \varepsilon > 0 ,$$

and

$$(16)\quad \sum_1^\infty s_j^{-4} E\left[X_j^4 I(|X_j| \leq \delta s_j)\right] < \infty , \text{ for some } \delta > 0 ,$$

then the conclusions of Theorem A *hold for* $W_n = U_{n-1} \vee 2$.

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The final corollary presents a portmanteau form for the sufficient conditions.

COROLLARY 3. *Let* f *and* g *be positive non-decreasing functions on* $(0, \infty)$ *such that* $t^{-1}f(t) \downarrow 0$, $t^{-1}g(t) \downarrow 0$ *as* t *increases and*

$$\int_1^\infty t^{-2}f(t)dt < \infty . \tag{17}$$

If $V_n \to \infty$ *almost surely as* $n \to \infty$ *and*

$$\sum_1^\infty \left[g\left(V_j^2\right)\right]^{-1} E\left(X_j^2 I\left(X_j^2 > f\left(V_j^2\right)\right) \mid F_{j-1}\right) < \infty \text{ almost surely,} \tag{18}$$

then the conclusions of Theorem A *hold for* $W_n = V_n \vee 2$.

This result can be compared with Corollary 4.5 of Strassen [7] which deals with the case $f(t) = g(t) = t(\log t)^{-5}$, $t > e$, and with Theorem 5.1 of Jain, Jogdeo and Stout [2] which deals with the case $f(t) = t(\log t)^{-1}(\log \log t)^{-6}$, $g(t) = t(\log \log t)^{-2}$, $t > e$. These results, however, are concerned with the integral test for upper and lower functions which, even in the case of independent variables, requires the imposition of slightly more stringent conditions than for the classical law of the iterated logarithm itself. Theorem 3 of Stout [5] essentially deals with the case $f(t) = g(t) = o\left(t(\log \log t)^{-1}\right)$, $t > e$, but obtains the classical and not the functional law of the iterated logarithm.

2. Proofs

Proof of Theorem A. Define $\mu(u)$ on $[0, \infty)$ by

$$\mu(u) = S_p + \left(u-W_p^2\right)\left(W_{p+1}^2-W_p^2\right)^{-1} X_{p+1} ,$$

where

$$p = p(u) = \max\left\{j \mid W_j^2 \le u\right\} .$$

Then

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$$\mu_n(u) = \left[\phi\left(W_n^2\right)\right]^{-1} \mu\left(W_n^2 u\right) ,$$

and so in view of Corollary 1 of Strassen [6], it suffices to prove that

(19) $$\lim_{t\to\infty} [\phi(t)]^{-1} \sup_{u\le t} |\mu(u)-W(u)| = 0 \text{ almost surely.}$$

Now, (3) and (5) imply that $W_{n+1}^{-2}W_n^2 \xrightarrow{\text{a.s.}} 1$ and hence

$$1 \ge u^{-1}W_{p(u)}^2 \ge W_{p(u)+1}^{-2}W_{p(u)}^2 \xrightarrow{\text{a.s.}} 1 \text{ as } u \to \infty ,$$

so that

$$u^{-1}W_{p(u)}^2 \xrightarrow{\text{a.s.}} 1 \text{ as } u \to \infty .$$

Similarly,

$$u^{-1}W_{p(u)+1}^2 \xrightarrow{\text{a.s.}} 1 \text{ as } u \to \infty .$$

Combining these and (5),

$$u^{-1}T_{p(u)} \xrightarrow{\text{a.s.}} 1 \text{ and } u^{-1}T_{p(u)+1} \xrightarrow{\text{a.s.}} 1 \text{ as } u \to \infty .$$

Since

$$|\mu(u)-W(u)| \le \max\left\{\left|W\left(T_{p(u)}\right)-W(u)\right|, \left|W\left(T_{p(u)+1}\right)-W(u)\right|\right\} ,$$

then result (19) follows as in Strassen's proof on page 217 of [6].

Proof of Theorem 1. The proof is based on that of Strassen [6] and follows that of Heyde and Scott [1].

Define

(20) $$\tilde{X}_j = X_j I\left(c_j < |X_j| \le Z_j\right) + \tfrac{1}{2}X_j I\left(|X_j| \le c_j\right) + \tfrac{1}{2}\operatorname{sgn}\left(X_j\right)c_j\left(1+Z_j|X_j|^{-1}\right)I\left(|X_j| > Z_j\right)$$

where $\{c_j, j \ge 1\}$ is a monotone sequence of positive constants with $c_j \to 0$ as $j \to \infty$ so fast that

$$\sum_1^\infty c_j < \infty , \quad \sum_1^\infty c_j Z_j W_j^{-2} < \infty \text{ almost surely.}$$

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(If $Z_j(\omega) < c_j$, let $I(c_j < |X_j| \le Z_j)(\omega) = 0$.) Set also

$$X_j^* = \tilde{X}_j - E(\tilde{X}_j \mid F_{j-1}) .$$

It is easily checked that $\tilde{X}_1, \ldots, \tilde{X}_n$ and hence $X_1^*, \ldots, X_n^*$ also generate the σ-field F_n .

Write $S_n^* = \sum_1^n X_j^*$, $V_n^{*2} = \sum_1^n E\left(X_j^{*2} \mid F_{j-1}\right)$, and if $u \in [0, 1]$, let

$$\mu_n^*(u) = \left[\phi\left(W_n^2\right)\right]^{-1}\left[S_l^* + \left(uW_n^2 - W_l^2\right)\left(W_{l+1}^2 - W_l^2\right)^{-1} X_{l+1}^*\right] .$$

We use (20) to obtain

$$|X_j - X_j^* - \{X_j I(|X_j| > Z_j) - E(X_j I(|X_j| > Z_j) \mid F_{j-1})\}| \le 2c_j ,$$

and then

$$\sup_{0 \le u \le 1} |\mu_n(u) - \mu_n^*(u)| \le \left[\phi\left(W_n^2\right)\right]^{-1} \sup_{1 \le r \le n} \left|\sum_{j=1}^r (X_j - X_j^*)\right|$$

$$\le \left[\phi\left(W_n^2\right)\right]^{-1} \sup_{1 \le r \le n} \left|\sum_1^r \{X_j I(|X_j| > Z_j) - E(X_j I(|X_j| > Z_j) \mid F_{j-1})\}\right| + $$

$$+ \left[\phi\left(W_n^2\right)\right]^{-1} \sum_1^n 2c_j$$

$$\xrightarrow{\text{a.s.}} 0$$

as $n \to \infty$, in view of (6).

Next we introduce the Skorokhod representation (see Strassen [7], Theorem 4.3). By extending the original probability space if necessary, we may suppose that there exists a brownian motion W and a sequence $\{T_n^*, n \ge 1\}$ of non-negative random variables defined on our probability space such that $S_n^* = W(T_n^*)$ almost surely for all n . Let $t_n = T_n^* - T_{n-1}^*$, $n \ge 1$ $(T_0 = 0)$. If G_n is the σ-field generated by $X_1, \ldots, X_n$ and $W(u)$ for $u < T_n^*$, then t_n is G_n-measurable,

$$E(t_n \mid G_{n-1}) = E\left(X_n^{*2} \mid G_{n-1}\right) = E\left(X_n^{*2} \mid F_{n-1}\right) \text{ almost surely,}$$

and for some constant L ,

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$$E\left(t_n^2 \mid G_{n-1}\right) \leq LE\left(X_n^{*4} \mid G_{n-1}\right) = LE\left(X_n^{*4} \mid F_{n-1}\right) \text{ almost surely.}$$

In view of (9), (21) and Theorem A, it suffices to prove that

(22) $$W_n^{-2} T_n^* \xrightarrow{\text{a.s.}} 1 .$$

To this end we first show that

(23) $$T_n^* - V_n^{*2} = o\left(W_n^2\right) \text{ almost surely as } n \to \infty .$$

Since

$$\begin{aligned} E\left(X_j^{*4} \mid F_{j-1}\right) &= E\left(\tilde{X}_j^4 \mid F_{j-1}\right) - 4E\left(\tilde{X}_j^3 \mid F_{j-1}\right)E\left(\tilde{X}_j \mid F_{j-1}\right) \\ &\quad + 6E\left(\tilde{X}_j^2 \mid F_{j-1}\right)\left[E\left(\tilde{X}_j \mid F_{j-1}\right)\right]^2 - 3\left[E\left(\tilde{X}_j \mid F_{j-1}\right)\right]^4 \\ &\leq 11E\left(\tilde{X}_j^4 \mid F_{j-1}\right) \\ &\leq 11E\left(X_j^4 I\left(|X_j| \leq z_j\right) \mid F_{j-1}\right) + 11c_j^4 , \end{aligned}$$

we have from (8) that

$$\sum_1^\infty W_j^{-4} E\left(X_j^{*4} \mid F_{j-1}\right) < \infty \text{ almost surely}$$

and hence, using Proposition IV.6.2, page 148 of Neveu [4],

$$\sum_1^n \left[t_j - E\left(t_j \mid G_{j-1}\right)\right] = o\left(W_n^2\right) \text{ almost surely,}$$

which is equivalent to (23).

Next, we have

(24) $$\left|E\left(\tilde{X}_j^2 \mid F_{j-1}\right) - E\left(X_j^2 I\left(|X_j| \leq z_j\right) \mid F_{j-1}\right)\right| \leq c_j^2 ,$$

(25) $$E\left(X_j^{*2} \mid F_{j-1}\right) = E\left(\tilde{X}_j^{*2} \mid F_{j-1}\right) - \left(E\left(\tilde{X}_j \mid F_{j-1}\right)\right)^2 ,$$

and

(26) $$\left|E\left(\tilde{X}_j \mid F_{j-1}\right) - E\left(X_j I\left(|X_j| \leq z_j\right) \mid F_{j-1}\right)\right| \leq c_j .$$

Then, using (26),

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$$(27)\quad \sum_1^n \left[\{E(\tilde{X}_j \mid F_{j-1})\}^2 - \{E(X_j I(|X_j| \le Z_j) \mid F_{j-1})\}^2\right] = $$

$$= o\left(W_n^2\right) \text{ almost surely}$$

since

$$\sum_1^n c_j |E(X_j I(|X_j| \le Z_j) \mid F_{j-1})| = o\left(W_n^2\right) \text{ almost surely,}$$

by virtue of $\sum_1^\infty c_j Z_j W_j^{-2} < \infty$ almost surely and Kronecker's Lemma. Hence, from (7) in conjunction with (24), (25), and (27), we have

$$V_n^{*2} - W_n^2 = o\left(W_n^2\right) \text{ almost surely.}$$

Combined with (23), this establishes (22) and completes the proof of Theorem 1.

Proof of Corollary 1. Take $W_n = V_n \vee 2$ for all n. Conditions (6), (8), and (9) translate immediately into (10), (12), and (13). To check condition (7) it suffices to show that

$$(28)\quad \lim_{n\to\infty} V_n^{-2} \sum_1^n \{E(X_j I(|X_j| \le Z_j) \mid F_{j-1})\}^2 = 0 \text{ almost surely.}$$

That (28) is indeed satisfied follows from

$$E(X_j I(|X_j| \le Z_j) \mid F_{j-1}) = -E(X_j I(|X_j| > Z_j) \mid F_{j-1}) \text{ almost surely,}$$

$$\{E(X_j I(|X_j| > Z_j) \mid F_{j-1})\}^2 \le E\left(X_j^2 I(|X_j| > Z_j) \mid F_{j-1}\right) \text{ almost surely,}$$

and (11).

Proof of Corollary 2. We use Theorem 1 and take $Z_j = \delta s_j$, $W_j = U_{j-1} \vee 2$, $j > 1$.

The condition (15), together with Kronecker's Lemma, gives

$$s_n^{-1} \sum_1^n |X_j| I(|X_j| > \varepsilon s_j) \xrightarrow{\text{a.s.}} 0 ,$$

(29)

$$s_n^{-1} \sum_1^n E(|X_j| I(|X_j| > \varepsilon s_j) \mid F_{j-1}) \xrightarrow{\text{a.s.}} 0 ,$$

as $n \to \infty$ for any $\varepsilon > 0$, so that in particular, for any $\varepsilon > 0$,

$$s_n^{-1} \sup_{j \le n} |X_j| \le \varepsilon + s_n^{-1} \sum_1^n |X_j| I(|X_j| > \varepsilon s_j) \xrightarrow{\text{a.s.}} \varepsilon .$$

Hence

(30) $$s_n^{-2} \sup_{j \le n} X_j^2 \xrightarrow{\text{a.s.}} 0 ,$$

and in view of (14),

$$U_n^{-2} \sup_{j \le n} X_j^2 \xrightarrow{\text{a.s.}} 0 ,$$

which implies (9). We can now observe from (29) that (6) holds, while (16) ensures (8).

Finally, (16) yields via an application of Proposition IV.6.1 of [4],

(31) $$\lim_{n\to\infty} s_n^{-2} \sum_1^n \left[X_j^2 I(|X_j| \le \delta s_j) - E\left(X_j^2 I(|X_j| \le \delta s_j) \mid F_{j-1}\right)\right] =$$

$$= 0 \quad \text{almost surely,}$$

while

(32) $$s_n^{-2} \sum_1^n X_j^2 I(|X_j| > \delta s_j) \le$$

$$\le \left(s_n^{-2} \sup_{j \le n} |X_j|\right) s_n^{-1} \sum_1^n |X_j| I(|X_j| > \delta s_j) \xrightarrow{\text{a.s.}} 0$$

using (29) and (30). We deduce from (31) and (32) that

$$\lim_{n\to\infty} s_n^{-2} \sum_1^n E\left(X_j^2 I(|X_j| \le \delta s_j) \mid F_{j-1}\right) = \eta^2 \quad \text{almost surely,}$$

and condition (7) follows in view of (29). This completes the proof.

Proof of Corollary 3. We take $Z_j^2 = f\left(V_j^2\right)$ and check the conditions of Corollary 1.

A sufficient condition for (10) is, using Proposition IV.6.2 of [4],

$$\sum_1^\infty \left[\phi\left(V_j^2\right)\right]^{-2} E\left(X_j^2 I\left(X_j^2 > f\left(V_j^2\right)\right) \mid F_{j-1}\right) < \infty \text{ almost surely,}$$

which holds in view of (18) since $t^{-1}g(t) \to 0$ as $t \to \infty$. Furthermore, that (11) holds follows immediately from (18) and an application of Kronecker's Lemma.

To check (12), we have

$$\begin{aligned}\sum_2^\infty V_j^{-4} E\left(X_j^4 I\left(X_j^2 \le f\left(V_j^2\right)\right) \mid F_{j-1}\right) &\le \sum_2^\infty V_j^{-4} f\left(V_j^2\right)\left(V_j^2 - V_{j-1}^2\right) \\ &\le \sum_2^\infty \int_{V_{j-1}^2}^{V_j^2} t^{-2} f(t)\,dt \\ &= \int_{V_1^2}^\infty t^{-2} f(t)\,dt \\ &< \infty \text{ almost surely,}\end{aligned}$$

using the fact that $t^{-1}f(t) \downarrow 0$ and (17).

Finally, $V_n \to \infty$ almost surely,

$$1 - V_n^{-2} E\left(X_n^2 \mid F_{n-1}\right) = V_n^{-2} V_{n-1}^2 \le 1 ,$$

while

$$\begin{aligned} 0 \le V_n^{-2} E\left(X_n^2 \mid F_{n-1}\right) &= \\ &= V_n^{-2} E\left(X_n^2 I\left(X_n^2 \le f\left(V_n^2\right)\right) \mid F_{n-1}\right) + V_n^{-2} E\left(X_n^2 I\left(X_n^2 > f\left(V_n^2\right)\right) \mid F_{n-1}\right) \\ &\le V_n^{-2} f\left(V_n^2\right) + V_n^{-2} E\left(X_n^2 I\left(X_n^2 > f\left(V_n^2\right)\right) \mid F_{n-1}\right) \\ &\xrightarrow{\text{a.s.}} 0 ,\end{aligned}$$

since $t^{-1}f(t) \downarrow 0$ and (11) holds. This verifies condition (13) and

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completes the proof.

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