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Embeddings and Duality Theorems for Weak Classical Lorentz Spaces

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Abstract. We characterize the weight functions u, v, w on $(0, \infty)$ such that

$$\left(\int_0^\infty f^*(t)^q w(t)\,dt\right)^{1/q} \le C \sup_{t\in(0,\infty)} f_u^{**}(t)v(t),$$

where

$$f_u^{**}(t) := \left(\int_0^t u(s)\,ds\right)^{-1} \int_0^t f^*(s)u(s)\,ds.$$

As an application we present a new simple characterization of the associate space to the space $\Gamma^\infty(v)$, determined by the norm

$$\|f\|_{\Gamma^\infty(v)} = \sup_{t\in(0,\infty)} f^{**}(t)v(t),$$

where

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds.$$

1 Introduction and Statement of Results

Let $(\mathcal{R}, \mu)$ be a totally σ-finite measure space with a non-atomic measure μ, and let $\mathcal{M}(\mathcal{R}, \mu)$ be the set of all extended complex-valued μ-measurable functions on $\mathcal{R}$. By $\mathcal{M}^+(\mathcal{R}, \mu)$ we denote the set of all non-negative functions from $\mathcal{M}(\mathcal{R}, \mu)$. For $f \in \mathcal{M}(\mathcal{R}, \mu)$, let $f_*(t) = \mu(\{x \in \mathcal{R}\,;\, |f(x)| > t\})$, $t > 0$, be the *distribution function* of f. The *non-increasing rearrangement* of f is defined by

$$f^*(t) = \inf\{s > 0\,;\, f_*(s) \le t\}, \quad t \in [0, \mu(\mathcal{R})).$$

We will assume that $\mu(\mathcal{R}) = \infty$. Throughout the paper, u, v and w will denote weights, that is, locally integrable non-negative functions on $(0, \infty)$. We set, once and for all,

$$U(t) = \int_0^t u(s)\,ds, \quad V(t) = \int_0^t v(s)\,ds, \quad W(t) = \int_0^t w(s)\,ds.$$

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We assume that u is such that $U(t) > 0$ for every $t \in (0,\infty)$. We then denote

$$f_u^{**}(t) = \frac{1}{U(t)} \int_0^t f^*(s)u(s)\,ds, \quad t \in (0,\infty).$$

When $u \equiv 1$ (hence $U(t) = t$), we will omit the subscript u. We thus write, in such a case, f^{**}, $\Gamma^p(v)$ and $\Gamma^\infty(v)$ in place of f_u^{**}, $\Gamma_u^p(v)$ and $\Gamma_u^\infty(v)$.

We shall now define four types of the so-called *classical Lorentz spaces*.

Definition 1.1 Let $p \in (0,\infty)$ and let u, v be weights on $(0,\infty)$. We define

$$\Lambda^p(v) = \Big\{ f \in \mathcal{M}(R,\mu)\,;\, \|f\|_{\Lambda^p(v)} := \Big(\int_0^\infty f^*(t)^p v(t)\,dt \Big)^{1/p} < \infty \Big\};$$

$$\Lambda^\infty(v) = \Big\{ f \in \mathcal{M}(R,\mu)\,;\, \|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{t\in(0,\infty)} f^*(t)v(t) < \infty \Big\};$$

$$\Gamma_u^p(v) = \Big\{ f \in \mathcal{M}(R,\mu)\,;\, \|f\|_{\Gamma_u^p(v)} := \Big(\int_0^\infty f_u^{**}(t)^p v(t)\,dt \Big)^{1/p} < \infty \Big\};$$

$$\Gamma_u^\infty(v) = \Big\{ f \in \mathcal{M}(R,\mu)\,;\, \|f\|_{\Gamma_u^\infty(v)} := \operatorname*{ess\,sup}_{t\in(0,\infty)} f_u^{**}(t)v(t) < \infty \Big\}.$$

The spaces $\Lambda^p(v)$ were introduced by Lorentz in 1951 in [13]. The spaces $\Gamma_u^p(v)$ with $0 < p < \infty$ are a simple modification of the spaces $\Gamma^p(v)$, which were first defined by Sawyer in [15]. Carro and Soria introduced in [4] weak classical Lorentz spaces $\Gamma^{p,\infty}(v)$, with $p \in (0,\infty)$, determined by the norm

$$\|f\|_{\Gamma^{p,\infty}(v)} := \sup_{t\in(0,\infty)} f^{**}(t)V(t)^{\frac{1}{p}} < \infty.$$

In our notation, $\Gamma^{p,\infty}(v) = \Gamma^\infty(V^{\frac{1}{p}})$. Weak Lorentz spaces were further investigated in [3, 5–7].

During the last two decades, many authors spent enormous efforts in order to find necessary and sufficient conditions on pairs of possibly different weights such that various embeddings exist between all the above-defined four types of function spaces. Such embeddings proved to be indispensable in several important areas of analysis, including the theory of interpolation and the modern study of Sobolev spaces. A state of the art survey of the area up the end of 1990's can be found in [7].

Characterization of embeddings turned out to be, in some particular cases, a rather difficult problem. It required the development of new techniques, for example the *discretization method* of Gol'dman, Heinig and Stepanov [9–11], or the use of *Halperin's level function* [7, 16]. These techniques enabled the authors to characterize the desired embeddings, and thereby brought significant progress to development of the theory. On the other hand, the resulting characterizing criteria were often expressed in a way that was not very satisfactory from a practical point of view. It was quite difficult, if not impossible, to verify some of the conditions in concrete, practical examples. Later, further progress was achieved thanks to the so-called *blocking* *technique* developed by K.-G. Grosse-Erdmann [12]. In our previous paper [8], similar ideas were used to create an *anti-discretization* method, an inverse technique to discretization of function norms. This method enabled us to formulate criteria for embeddings between $\Gamma^p(v)$ spaces and $\Lambda^q(w)$ spaces in the desired integral form, far more manageable than the discretized form that had been known before.

In this paper, our goal is to characterize embeddings of type

$$\Gamma_u^\infty(v) \hookrightarrow \Lambda^q(w).$$

Such embeddings have not yet been been characterized in a satisfactory way, although certain partial results were obtained in [7].

Thus, given $q \in (0, \infty)$, our aim will be to establish necessary and sufficient conditions on the weight functions u, v, w such that the inequality

$$\Big(\int_0^\infty f^*(t)^q w(t)\,dt\Big)^{\frac{1}{q}} \lesssim \operatorname*{ess\,sup}_{t\in(0,\infty)} f_u^{**}(t)v(t) \tag{1.1}$$

holds for every $f \in \mathcal{M}(\mathcal{R}, \mu)$.

Here, and throughout the paper, we write $A \lesssim B$ if there exists a positive constant C, independent of appropriate quantities such as functions, satisfying $A \leq CB$. We write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

First, we have to recall some definitions and basic facts.

Definition 1.2 Let $\{a_k\}$ be a sequence of positive real numbers. We say that $\{a_k\}$ is *strongly increasing* or *strongly decreasing* and write $a_k \upuparrows$ or $a_k \downdownarrows$ when

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \text{or} \quad \sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} < 1,$$

respectively.

Definition 1.3 Let θ be a continuous strictly increasing function on $[0, \infty)$ such that $\theta(0) = 0$ and $\lim_{t\to\infty} \theta(t) = \infty$. Then we say that θ is *admissible*.

Let θ be an admissible function. We say that a function h is θ-*quasiconcave* if h is equivalent to a non-decreasing function on $[0, \infty)$ and $\frac{h}{\theta}$ is equivalent to a non-increasing function on $(0, \infty)$. We say that a θ-quasiconcave function h is *non-degenerate* if

$$\lim_{t\to 0+} h(t) = \lim_{t\to\infty} \frac{1}{h(t)} = \lim_{t\to\infty} \frac{h(t)}{\theta(t)} = \lim_{t\to 0+} \frac{\theta(t)}{h(t)} = 0.$$

The family of non-degenerate θ-quasiconcave functions will be denoted by Ω_θ.

We say that h is *quasiconcave* when $h \in \Omega_\theta$ with $\theta(t) = t$.

Definition 1.4 Assume that θ is admissible and $h \in \Omega_\theta$. We say that $\{\mu_k\}_{k\in\mathbb{Z}}$ is a *discretizing sequence for h with respect to θ* if

(i) $\mu_0 = 1$ and $\theta(\mu_k) \upuparrows$;
(ii) $h(\mu_k) \upuparrows$ and $\frac{h(\mu_k)}{\theta(\mu_k)} \downdownarrows$;

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(iii) there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$ such that $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \varnothing$ and, for every $t \in [\mu_k, \mu_{k+1}]$,

$$h(\mu_k) \approx h(t) \text{ if } k \in \mathbb{Z}_1, \quad \frac{h(\mu_k)}{\theta(\mu_k)} \approx \frac{h(t)}{\theta(t)} \text{ if } k \in \mathbb{Z}_2.$$

Lemma 1.5 *Let u, v be weights as above and let φ be defined by*

$$\varphi(t) := \operatorname*{ess\,sup}_{s\in(0,t)} U(s) \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{U(\tau)}, \quad t \in (0,\infty). \tag{1.2}$$

Then φ is the least U-quasiconcave majorant of v, and

$$\Gamma_u^\infty(v) = \Gamma_u^\infty(\varphi) \tag{1.3}$$

with identical norms. Further, for $t \in (0,\infty)$,

$$\varphi(t) = \operatorname*{ess\,sup}_{\tau\in(0,\infty)} v(\tau) \min\left\{1, \frac{U(t)}{U(\tau)}\right\} = U(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{1}{U(s)} \operatorname*{ess\,sup}_{\tau\in(0,s)} v(\tau) \tag{1.4}$$

and

$$\varphi(t) \approx \operatorname*{ess\,sup}_{s\in(0,\infty)} v(s) \frac{U(t)}{U(s)+U(t)}. \tag{1.5}$$

Proof Let f be a locally integrable function. First, note that the functions $U(t)$ and $\int_0^t f^*(s)u(s)\,ds$ are non-decreasing in $t \in (0,\infty)$, while the function $f_u^{**}(t)$ is non-increasing in $t \in (0,\infty)$, being a weighted average of a non-increasing function. We will use these facts combined with the following simple observation. Whenever F, G are non-negative functions on $(0,\infty)$ and F is non-increasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{s\in(0,t)} G(s), \quad t \in (0,\infty);$$

likewise, when F is non-decreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} G(s), \quad t \in (0,\infty).$$

Altogether, we have by the Fubini theorem,

$$\begin{aligned}
\|f\|_{\Gamma_u^\infty(v)} &= \operatorname*{ess\,sup}_{t\in(0,\infty)} \frac{v(t)}{U(t)} \int_0^t f^*(s)u(s)\,ds \\
&= \operatorname*{ess\,sup}_{t\in(0,\infty)} \int_0^t f^*(s)u(s)\,ds \operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{v(s)}{U(s)} \\
&= \operatorname*{ess\,sup}_{t\in(0,\infty)} f_u^{**}(t)U(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{v(s)}{U(s)} \\
&= \operatorname*{ess\,sup}_{t\in(0,\infty)} f_u^{**}(t) \operatorname*{ess\,sup}_{s\in(0,t)} U(s) \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{U(\tau)} = \|f\|_{\Gamma_u^\infty(\varphi)},
\end{aligned}$$

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proving (1.3). Next, observe that

$$\begin{aligned}\varphi(t) &= \operatorname*{ess\,sup}_{s\in(0,t)} U(s) \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{U(\tau)}\\ &= \operatorname*{ess\,sup}_{\tau\in(0,\infty)} \frac{v(\tau)}{U(\tau)} \operatorname*{ess\,sup}_{s\in(0,\min\{\tau,t\})} U(s)\\ &= \operatorname*{ess\,sup}_{\tau\in(0,\infty)} \frac{v(\tau)}{U(\tau)} \min\{U(t),U(\tau)\}\\ &= \operatorname*{ess\,sup}_{\tau\in(0,\infty)} v(\tau)\min\Big\{1,\frac{U(t)}{U(\tau)}\Big\},\end{aligned}$$

showing the first equality in (1.4), and that

$$\begin{aligned}U(t)\operatorname*{ess\,sup}_{s\in(t,\infty)} \frac{1}{U(s)} \operatorname*{ess\,sup}_{\tau\in(0,s)} v(\tau) &= U(t)\operatorname*{ess\,sup}_{\tau\in(0,\infty)} v(\tau) \operatorname*{ess\,sup}_{s\in(\max\{\tau,t\},\infty)} \frac{1}{U(s)}\\ &= U(t)\operatorname*{ess\,sup}_{\tau\in(0,\infty)} v(\tau)\min\Big\{\frac{1}{U(t)},\frac{1}{U(\tau)}\Big\}\\ &= \operatorname*{ess\,sup}_{\tau\in(0,\infty)} v(\tau)\min\Big\{1,\frac{U(t)}{U(\tau)}\Big\},\end{aligned}$$

proving the remaining part of (1.4). Again, (1.5) follows from the obvious relation

$$\min\Big\{1,\frac{U(t)}{U(s)}\Big\} \approx \frac{U(t)}{U(s)+U(t)}, \tag{1.6}$$

which is valid for any values of $s,t \in (0,\infty)$. Finally, the U-quasiconcavity of φ follows from (1.4). ∎

Remark 1.6 It follows from Lemma 1.5 that we can always assume that the weight v in the definition of the space $\Gamma_u^\infty(v)$ is U-quasiconcave.

In the sequel it will be useful to note that if u and φ are from Lemma 1.5 and $q \in (0,\infty)$, then $U^q/\varphi^q \in \Omega_{U^q}$.

Definition 1.7 Let θ be an admissible function and let ν be a non-negative Borel measure on $[0,\infty)$. We say that the function h defined as

$$h(t) = \theta(t)\int_{[0,\infty)} \frac{d\nu(s)}{\theta(s)+\theta(t)}, \quad t\in(0,\infty), \tag{1.7}$$

is the *fundamental function of the measure* ν with respect to θ. We will also say that ν is a *representation measure* of h with respect to θ.

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Let us recall [8, Remark 2.10(ii)] that if $h \in \Omega_\theta$, then there always exists a representation measure ν of h with respect to θ.

We can now state our main results.

Theorem 1.8 *Let $q \in (0,\infty)$ and let u, v, w be weights. Assume that u is such that U is admissible. Let φ, defined by* (1.2)*, be non-degenerate with respect to U. Let ν be the representation measure of U^q/φ^q with respect to U^q.*

(i) *If $1 \le q < \infty$, then* (1.1) *holds for all f if and only if*

$$A(1) = \Big(\int_0^\infty \sup_{s\in(t,\infty)} \frac{W(s)}{U(s)^q}\, d\nu(t) \Big)^{\frac{1}{q}} < \infty.$$

Moreover, the optimal constant C in (1.1) *satisfies $C \approx A(1)$.*

(ii) *If $0 < q < 1$, then* (1.1) *holds for all f if and only if*

$$A(2) = \Big(\int_0^\infty \frac{\zeta(t)}{U(t)^q}\, d\nu(t) \Big)^{\frac{1}{q}} < \infty,$$

where

$$\zeta(t) = W(t) + U(t)^q \Big(\int_t^\infty \Big(\frac{W(s)}{U(s)} \Big)^{\frac{q}{1-q}} w(s)\, ds \Big)^{1-q}, \quad t \in (0,\infty). \tag{1.8}$$

Moreover, the optimal constant C in (1.1) *satisfies $C \approx A(2)$. Furthermore, $A(2) \approx A(3)$, where*

$$A(3) := \Big(\int_0^\infty \zeta(t)^{\frac{q}{1-q}} \varphi(t)^{-q} W(t)^{\frac{q}{1-q}} w(t)\, dt \Big)^{\frac{1}{q}} < \infty.$$

Corollary 1.9 *Let v be a weight and let φ be given by* (1.2) *with $U(s) = s$. Then, the associate space to $\Gamma^\infty(v)$ is equivalent to $\Lambda^1(d\frac{t}{\varphi(t)})$.*

Remark 1.10 Let us finally note that the method presented here also works for the case when $q = \infty$ (with usual modifications), but, in that case, the corresponding characterization is known (see [7]), and therefore we omit it here.

2 Proofs of the Main Results

Proof of Theorem 1.8 *Sufficiency.* Using the argument from [18], we can easily verify that it suffices to check (1.1) for functions f in the form $f^*(t) = \int_t^\infty h(s)\, ds$, where $h \in \mathcal{M}^+(0,\infty)$. For such f, we have by the Fubini theorem and trivial estimates,

$$f_u^{**}(t) \approx \int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\, ds.$$

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Next, recall that by (1.3), we can replace v by φ on the right side of (1.1). Altogether, (1.1) is equivalent to

$$\Big(\int_0^\infty\Big(\int_t^\infty h(s)\,ds\Big)^q w(t)\,dt\Big)^{\frac{1}{q}} \lesssim \sup_{t\in(0,\infty)}\varphi(t)\int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\,ds \tag{2.1}$$

for $h\in\mathcal{M}^+(0,\infty)$.

Since $\varphi\in\Omega_U$, there exists a discretizing sequence $\{x_k\}$ of φ with respect to U. By [8, Lemma 3.8], applied to $q=1$, $\varphi=$ our φ, $f=Uh$ and $u=U$, we have

$$\begin{aligned}\sup_{t\in(0,\infty)}\varphi(t)\int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\,ds &\approx \sup_{k\in\mathbb{Z}}\varphi(x_k)\int_0^\infty \frac{U(s)h(s)}{U(x_k)+U(s)}\,ds\\ &\approx \sup_{k\in\mathbb{Z}}\int_{x_k}^{x_{k+1}} h(s)\varphi(s)\,ds, \quad h\in\mathcal{M}^+(0,\infty).\end{aligned}\tag{2.2}$$

Now let us discretize the left side of (2.1). We get

$$\begin{aligned}\int_0^\infty\Big(\int_t^\infty h(s)\,ds\Big)^q w(t)\,dt &\approx \sum_{k\in\mathbb{Z}}\int_{x_k}^{x_{k+1}}\Big(\int_t^{x_{k+1}} h(s)\,ds\Big)^q w(t)\,dt\\ &\quad+\sum_{k\in\mathbb{Z}}\int_{x_k}^{x_{k+1}} w(t)\,dt\Big(\int_{x_{k+1}}^\infty h(s)\,ds\Big)^q = \mathrm{I}+\mathrm{II},\end{aligned}\tag{2.3}$$

say.

Now let us distinguish two cases. Assume first that $1\le q<\infty$. Then, by the weighted Hardy inequality due to Bradley [2],

$$\int_{x_k}^{x_{k+1}}\Big(\int_t^{x_{k+1}} h(s)\,ds\Big)^q w(t)\,dt \le \Big(\int_{x_k}^{x_{k+1}} h(s)\varphi(s)\,ds\Big)^q \sup_{t\in[x_k,x_{k+1}]}\frac{\int_{x_k}^t w(s)\,ds}{\varphi(t)^q}.$$

We obtain

$$\begin{aligned}\mathrm{I} &\lesssim \sum_{k\in\mathbb{Z}}\Big(\int_{x_k}^{x_{k+1}} h(s)\varphi(s)\,ds\Big)^q \sup_{t\in[x_k,x_{k+1}]}\frac{\int_{x_k}^t w(y)\,dy}{\varphi(t)^q}\\ &\le \Big(\sup_{k\in\mathbb{Z}}\int_{x_k}^{x_{k+1}} h(s)\varphi(s)\,ds\Big)^q \sum_{k\in\mathbb{Z}}\sup_{t\in[x_k,x_{k+1}]}\frac{W(t)}{\varphi(t)^q}.\end{aligned}\tag{2.4}$$

Since W is non-decreasing and W/U^q is continuous, we clearly have for $t\in(0,\infty)$,

$$\begin{aligned}\sup_{s\in(0,\infty)}\frac{W(s)}{U(s)^q+U(t)^q} &= \max\Big\{\sup_{s\in(0,t)}\frac{W(s)}{U(s)^q+U(t)^q}, \sup_{s\in(t,\infty)}\frac{W(s)}{U(s)^q+U(t)^q}\Big\}\\ &\approx \max\Big\{\frac{W(t)}{U(t)^q}, \sup_{s\in(t,\infty)}\frac{W(s)}{U(s)^q}\Big\}\\ &= \sup_{s\in(t,\infty)}\frac{W(s)}{U(s)^q}.\end{aligned}$$

Thus, applying [8, Lemma 3.7] to $f = W^{\frac{1}{q}}$, $u = U$ and $h = U^q/\varphi^q$, we get

$$\sum_{k\in\mathbb{Z}} \sup_{t\in[x_k,x_{k+1}]} \frac{W(t)}{\varphi(t)^q} \approx \int_0^\infty \sup_{s\in(0,\infty)} \frac{W(s)}{U(s)^q + U(t)^q}\, d\nu(t) \tag{2.5}$$
$$\approx \int_0^\infty \sup_{s\in(t,\infty)} \frac{W(s)}{U(s)^q}\, d\nu(t)$$
$$= A(1)^q.$$

Plugging this into (2.4) and using (2.2), we arrive at

$$\mathrm{I} \lesssim A(1)^q \Big(\sup_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} h(s)\varphi(s)\, ds \Big)^q \tag{2.6}$$
$$\lesssim A(1)^q \Big(\sup_{t\in(0,\infty)} \varphi(t) \int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\, ds \Big)^q.$$

Now, let us estimate II. Using (2.2) and (2.5), we have

$$\mathrm{II} \le \sup_{k\in\mathbb{Z}} \varphi(x_{k+1})^q \Big(\int_{x_{k+1}}^\infty h(s)\, ds \Big)^q \sum_{k\in\mathbb{Z}} \frac{W(x_{k+1})}{\varphi(x_{k+1})^q} \tag{2.7}$$
$$\lesssim \Big(\sup_{t\in(0,\infty)} \varphi(t) \int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\, ds \Big)^q \sum_{k\in\mathbb{Z}} \frac{W(x_{k+1})}{\varphi(x_{k+1})^q}.$$
$$\lesssim A(1)^q \Big(\sup_{t\in(0,\infty)} \varphi(t) \int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\, ds \Big)^q,$$

and (1.1) follows from (2.3), (2.6) and (2.7).

Now assume $0 < q < 1$. By a simple modification of [17, Theorem 3.3], we have

$$\int_{x_k}^{x_{k+1}} \Big(\int_t^{x_{k+1}} h(s)\, ds \Big)^q w(t)\, dt$$
$$\lesssim \Big(\int_{x_k}^{x_{k+1}} h(s)\varphi(s)\, ds \Big)^q \Big(\int_{x_k}^{x_{k+1}} \Big(\frac{\int_{x_k}^t w(y)\, dy}{\varphi(t)} \Big)^{\frac{q}{1-q}} w(t)\, dt \Big)^{1-q}$$

with constant independent of x_k. We get

$$\mathrm{I} \lesssim \sum_{k\in\mathbb{Z}} \Big(\int_{x_k}^{x_{k+1}} h(s)\varphi(s)\, ds \Big)^q \Big(\int_{x_k}^{x_{k+1}} \Big(\frac{\int_{x_k}^t w(y)\, dy}{\varphi(t)} \Big)^{\frac{q}{1-q}} w(t)\, dt \Big)^{1-q}$$
$$\lesssim \Big(\sup_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} h(s)\varphi(s)\, ds \Big)^q \sum_{k\in\mathbb{Z}} \Big(\int_{x_k}^{x_{k+1}} \Big(\frac{W(t)}{\varphi(t)} \Big)^{\frac{q}{1-q}} w(t)\, dt \Big)^{1-q}.$$

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Thus, using (2.2) and applying [8, Lemma 3.6] to $q = 1 - q$, $f = W^{\frac{q}{1-q}} w$, $u = U^{\frac{q}{1-q}}$ and $h = U^q/\varphi^q$, we obtain

$$\text{(2.8)} \qquad \mathrm{I} \lesssim \Big(\sup_{t\in(0,\infty)} \varphi(t) \int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\, ds \Big)^q \sum_{k\in\mathbb{Z}} \frac{\zeta(x_k)}{\varphi(x_k)^q}.$$

Now, since (2.7) holds for any $q \in (0,\infty)$ and $W \le \zeta$, we have, altogether with (2.8),

$$\text{(2.9)} \qquad \mathrm{I} + \mathrm{II} \lesssim \Big(\sup_{t\in(0,\infty)} \varphi(t) \int_0^\infty \frac{U(s)h(s)}{U(s)+U(t)}\, ds \Big)^q \sum_{k\in\mathbb{Z}} \frac{\zeta(x_k)}{\varphi(x_k)^q}.$$

Denote $\theta(t) = \frac{U(t)^q}{\varphi(t)^q}$. Note that $\varphi \in \Omega_U$ implies $\theta \in \Omega_{U^q}$ and that $\{x_k\}$, being a discretizing sequence of φ with respect to U, is also a discretizing sequence of θ with respect to U^q. Therefore, we can apply [8, Theorem 2.11] to the following set of parameters:

$$p = r = q = \text{our } q, \quad h = \theta, \quad u = U, \quad \{\mu_k\} = \{x_k\} \quad \text{and} \quad \sigma = \frac{U^q}{\zeta}.$$

We note that the assumption $\sigma \in \Omega_{u^p}$ of [8, Theorem 2.11] (meaning here $\frac{U^q}{\zeta} \in \Omega_{U^q}$) is satisfied, as follows immediately from

$$\zeta(t) \approx \Big(\int_0^\infty W(s)^{\frac{q}{1-q}} w(s) \min\Big\{ 1, \frac{U(t)^{\frac{q}{1-q}}}{U(s)^{\frac{q}{1-q}}} \Big\}\, ds \Big)^{1-q}.$$

We get

$$\text{(2.10)} \qquad \sum_{k\in\mathbb{Z}} \frac{\zeta(x_k)}{\varphi(x_k)^q} \approx A(2)^q,$$

which, together with (2.9) yields (1.1).

Finally, using [8, Theorem 2.11] with the parameters

$$p = q = \frac{q}{1-q}, \quad r = 1-q, \quad h = \zeta^{\frac{1}{1-q}}, \quad u = U, \quad \{\mu_k\} = \{x_k\}, \quad \sigma = \varphi^{\frac{q}{1-q}},$$

observing that

$$\zeta^{\frac{1}{1-q}} \in \Omega_{U^{\frac{q}{1-q}}}, \quad \varphi^{\frac{1}{1-q}} \in \Omega_{U^{\frac{q}{1-q}}}$$

and that the representation measure of $\zeta^{\frac{1}{1-q}}$ with respect to $U^{\frac{q}{1-q}}$ is $W(t)^{\frac{q}{1-q}} w(t)dt$, we see that $A(2) \approx A(3)$.

Necessity. Let still $\{x_k\}$ be a discretizing sequence for φ with respect to U. By (2.2) and (2.1),

$$\text{(2.11)} \qquad \Big(\sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} \Big(\int_t^{x_{k+1}} h(s)\, ds \Big)^q w(t)\, dt \Big)^{\frac{1}{q}} \lesssim \sup_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} h(s)\varphi(s)\, ds, \quad h \in \mathcal{M}^+(0,\infty).$$

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Let $1 \le q < \infty$. For $k \in \mathbb{Z}$, let h_k be functions that saturate the Hardy inequality (*cf.* [14]), that is, functions satisfying $\operatorname{supp} h_k \subset [x_k, x_{k+1})$, $\int_{x_k}^{x_{k+1}} h_k(s)\varphi(s)\,ds = 1$, and

$$\int_{x_k}^{x_{k+1}} \Big(\int_t^{x_{k+1}} h_k(s)\,ds\Big)^q w(t)\,dt \gtrsim \sup_{t\in[x_k,x_{k+1}]} \varphi(t)^{-q} \int_{x_k}^t w(s)\,ds. \tag{2.12}$$

We define the test function

$$h(s) = \sum h_k(s).$$

Plugging this into (2.11) and using (2.12), we get

$$\Big(\sum_{k\in\mathbb{Z}} \sup_{t\in[x_k,x_{k+1}]} \varphi(t)^{-q} \int_{x_k}^t w(s)\,ds\Big)^{\frac{1}{q}} < \infty. \tag{2.13}$$

Using (2.5) and [8, Lemma 3.1(ii)] with $\tau_k = \frac{1}{\varphi(x_k)^q}$ and $a_m = \int_{x_{m-1}}^{x_m} w(s)\,ds$, we get

$$\begin{aligned}
A(1) &\approx \Big(\sum_{k\in\mathbb{Z}} \sup_{s\in[x_k,x_{k+1}]} \frac{W(s)}{\varphi(s)^q}\Big)^{\frac{1}{q}}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{W(x_k)}{\varphi(x_k)^q}\Big)^{\frac{1}{q}} + \Big(\sum_{k\in\mathbb{Z}} \sup_{s\in[x_k,x_{k+1}]} \frac{\int_{x_k}^s w(y)\,dy}{\varphi(s)^q}\Big)^{\frac{1}{q}}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{\int_{x_{k-1}}^{x_k} w(s)\,ds}{\varphi(x_k)^q}\Big)^{\frac{1}{q}} + \Big(\sum_{k\in\mathbb{Z}} \sup_{s\in[x_k,x_{k+1}]} \frac{\int_{x_k}^s w(y)\,dy}{\varphi(s)^q}\Big)^{\frac{1}{q}}\\
&= A + B,
\end{aligned}$$

say. Now obviously $A \le B$ and, moreover, by (2.13), $B < \infty$. Altogether, we get $A(1) < \infty$, as desired.

Let $0 < q < 1$. For $k \in \mathbb{Z}$, define h_k so that $\operatorname{supp} h_k \in [x_k, x_{k+1}]$,

$$\int_{x_k}^{x_{k+1}} h_k(s)\varphi(s)\,ds = 1,$$

and

$$\int_{x_k}^{x_{k+1}} \Big(\int_t^{x_{k+1}} h_k(s)\,ds\Big)^q w(t)\,dt \gtrsim \Big(\int_{x_k}^{x_{k+1}} \Big(\frac{\int_{x_k}^t w(s)\,ds}{\varphi(t)}\Big)^{\frac{q}{1-q}} w(t)\,dt\Big)^{1-q}.$$

This is possible since the Hardy inequality is saturated and due to a simple modification of [17, Theorem 3.3]. We define

$$h(s) = \sum_{k\in\mathbb{Z}} h_k(s).$$

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From (2.11) we get, using the definition of x_k and h,

$$\Big(\sum_{k\in\mathbb{Z}}\Big(\int_{x_k}^{x_{k+1}}\Big(\int_{x_k}^{t} w(s)\,ds\Big)^{\frac{q}{1-q}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}}<\infty. \tag{2.14}$$

Now, we use (2.10), then an obvious analogue of (1.6), and then [8, Lemma 3.6] applied to the parameters $q=1-q$, $f=W^{\frac{q}{1-q}}$, $u=U^{\frac{q}{1-q}}$ and $h=\frac{U^q}{\varphi^q}$. We thereby obtain

(2.15)

$$\begin{aligned}
A(2)&\approx\Big(\int_0^\infty\Big(\int_0^\infty W(s)^{\frac{q}{1-q}}w(s)\min\Big\{\frac{1}{U(t)^{\frac{q}{1-q}}},\frac{1}{U(s)^{\frac{q}{1-q}}}\Big\}\,ds\Big)^{1-q}d\nu(t)\Big)^{\frac{1}{q}}\\
&\approx\Big(\int_0^\infty\Big(\int_0^\infty\frac{W(s)^{\frac{q}{1-q}}w(s)}{U(t)^{\frac{q}{1-q}}+U(s)^{\frac{q}{1-q}}}\,ds\Big)^{1-q}d\nu(t)\Big)^{\frac{1}{q}}\\
&\approx\Big(\sum_{k\in\mathbb{Z}}\Big(\int_{x_k}^{x_{k+1}}W(t)^{\frac{q}{1-q}}w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}}\\
&\approx\Big(\sum_{k\in\mathbb{Z}}W(x_k)^q\Big(\int_{x_k}^{x_{k+1}}w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}}\\
&\quad+\Big(\sum_{k\in\mathbb{Z}}\Big(\int_{x_k}^{x_{k+1}}\Big(\int_{x_k}^{t}w(s)\,ds\Big)^{\frac{q}{1-q}}w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}}.
\end{aligned}$$

By (2.14), the second term is finite. Let us estimate the first one. Note that

$$\begin{aligned}
&W(x_k)^q\Big(\int_{x_k}^{x_{k+1}}w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\\
&\qquad\lesssim\varphi(x_k)^{-q}W(x_k)+\Big(\int_{x_k}^{x_{k+1}}\Big(\int_{x_k}^{t}w(s)\,ds\Big)^{\frac{q}{1-q}}w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}.
\end{aligned}$$

Indeed, if $\int_{x_k}^{x_{k+1}}w(s)\,ds<W(x_k)$, then, by the monotonicity of φ,

$$\begin{aligned}
&W(x_k)^q\Big(\int_{x_k}^{x_{k+1}}w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\\
&\qquad\le\varphi(x_k)^{-q}W(x_k)^q\Big(\int_{x_k}^{x_{k+1}}w(t)\,dt\Big)^{1-q}\le\varphi(x_k)^{-q}W(x_k).
\end{aligned}$$

Now assume that $\int_{x_k}^{x_{k+1}}w(s)\,ds\ge W(x_k)$. Let $t_k\in[x_k,x_{k+1}]$ be such that $\int_{x_k}^{t_k}w(s)\,ds=$

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$W(x_k)$. Then we have

$$\begin{aligned}
W(x_k)^q&\Big(\int_{x_k}^{x_{k+1}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\\
&\le W(x_k)^q\Big(\int_{x_k}^{t_k} w(t)\,dt\Big)^{1-q}\varphi(x_k)^{-q}\\
&\quad+\Big(\int_{t_k}^{x_{k+1}}\Big(\int_{x_k}^{t} w(s)\,ds\Big)^{\frac{q}{1-q}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\\
&\lesssim \varphi(x_k)^{-q}W(x_k)+\Big(\int_{x_k}^{x_{k+1}}\Big(\int_{x_k}^{t} w(s)\,ds\Big)^{\frac{q}{1-q}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{(2.16)}\qquad &\Big(\sum_{k\in\mathbb{Z}} W(x_k)^q\Big(\int_{x_k}^{x_{k+1}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{-q}W(x_k)\Big)^{\frac{1}{q}}\\
&\quad+\Big(\sum_{k\in\mathbb{Z}}\Big(\int_{x_k}^{x_{k+1}}\Big(\int_{x_k}^{t} w(s)\,ds\Big)^{\frac{q}{1-q}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}},
\end{aligned}$$

and, using [8, Lemma 3.1 (ii)] with $\tau_k=\varphi(x_k)^{-q}$, $p=1$, $a_k=\int_{x_{k-1}}^{x_k} w(s)\,ds$,

$$\begin{aligned}
\text{(2.17)}\qquad \Big(\sum_{k\in\mathbb{Z}} W(x_k)\varphi(x_k)^{-q}\Big)^{\frac{1}{q}} &\approx \Big(\sum_{k\in\mathbb{Z}}\varphi(x_k)^{-q}\int_{x_{k-1}}^{x_k} w(s)\,ds\Big)^{\frac{1}{q}}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}}\Big(\int_{x_k}^{x_{k+1}}\Big(\int_{x_k}^{t} w(s)\,ds\Big)^{\frac{q}{1-q}} w(t)\varphi(t)^{-\frac{q}{1-q}}\,dt\Big)^{1-q}\Big)^{\frac{1}{q}},
\end{aligned}$$

which is finite by (2.14).

Finally, combining (2.15), (2.17) and (2.16), we obtain $A(2)<\infty$. ∎

Proof of Corollary 1.9 By the definition of the associate norm,

$$\|g\|_{(\Gamma^\infty(v))'}=\sup_{f\not\equiv 0}\frac{\int_0^\infty f^*(t)g^*(t)\,dt}{\|f\|_{\Gamma^\infty(v)}}.$$

This supremum is equal to the optimal constant in (1.1) with $q=1$, $u\equiv 1$ and $w=g^*$. By Theorem 1.8(i), this optimal constant is comparable to the corresponding

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value of $A(1)$. Thus, by the monotonicity of g^{**} and the Fubini theorem,

$$\|g\|_{(\Gamma^\infty(\nu))'} \approx \int_0^\infty \sup_{s\in[t,\infty)} \frac{1}{s}\int_0^s g^*(y)\,dy d\nu(t) \tag{2.18}$$
$$= \int_0^\infty \frac{1}{t}\int_0^t g^*(s)\,ds d\nu(t)$$
$$= \int_0^\infty g^*(t)\int_t^\infty \frac{d\nu(s)}{s}\,dt.$$

Now, note that, since ν is the representation measure of $\frac{t}{\varphi(t)}$ with respect to t,

$$\int_0^t d\Big(\frac{s}{\varphi(s)}\Big) = \frac{t}{\varphi(t)} \approx \int_0^t d\nu(s) + t\int_t^\infty \frac{d\nu(s)}{s} \approx \int_0^t \int_s^\infty \frac{d\nu(y)}{y}\,d\tau.$$

Thus, by Hardy's lemma (*cf.* [1]), we have

$$\int_0^\infty g^*(t)\int_t^\infty \frac{d\nu(s)}{s}\,dt \approx \int_0^\infty g^*(t)\,d\Big(\frac{t}{\varphi(t)}\Big).$$

Inserting into (2.18), we obtain

$$\|g\|_{(\Gamma^\infty(\nu))'} \approx \int_0^\infty g^*(t)\,d\Big(\frac{t}{\varphi(t)}\Big). \qquad \blacksquare$$

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