

REAL HARDY SPACES OF AN ANNULUS

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In this article we use real methods to study the Dirichlet and the boundary value problems of an annulus. Then we establish various properties of real Hardy spaces on the annulus, and give some applications.

1. Introduction

Fix r_0 , $0 < r_0 < 1$. Let \mathbb{C} be the complex numbers, and $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$ be an annulus. The boundary of A consists of two pieces $\Gamma_0 = \{z \in \mathbb{C} : |z| = r_0\}$, and $\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\}$. Write $\Gamma = \Gamma_0 \cup \Gamma_1$.

In this article, we first study the Dirichlet and the boundary value problems on the annulus A . Then we shall construct real Hardy spaces $\mathcal{H}^p(A)$. We also study the relationships between $\mathcal{H}^p(A)$ and some important operators such as the Poisson kernel, the conjugate Poisson kernel and the Hilbert transforms. Two related works should be mentioned. The first is Sarason's monograph [4], where Hardy spaces have been studied through complex methods, and then applied to study invariant theory. The second is Goldberg [3], who studied real Hardy spaces on a strip $\{z \in \mathbb{C} : 0 < \text{Im } z < 1\}$. It is noted that a Hardy space of the annulus does not go directly over to a Hardy space of the strip since circles centered at 0 do not go over into lines parallel to the x -axis.

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Actually the Cayley transform (or its inverse) takes these lines into circles tangent to the unit circle at the point -1 . Finally we give some applications to plasma physics.

2. The Poisson kernel, the conjugate Poisson kernel, and the Hilbert transform

To derive the Poisson kernel on the annulus A , we start with the closed unit disk D minus the points 1 and -1 , apply a suitable conformal mapping onto the upper half plane and then was another conformal mapping onto the strip $B = \{z \in \mathbb{C} : 0 < \text{Im } z < 1\}$. Finally, we roll B up to A .

Following Sarason [4], p. 22, consider the holomorphic function

$$F(t, r) = \frac{1}{2q_0} \tanh \left| -\frac{\pi t}{2q_0} + i \left(\frac{\pi}{4} + \frac{\pi}{2q_0} \log \frac{r}{r_0^{\frac{1}{2}}} \right) \right|$$

where $q_0 = -\log r_0$, $0 < r_0 < r < 1$, $t \in \mathbb{R}$, the real numbers. The imaginary part $P(t, r)$ and the real part $Q(t, r)$ of $F(t, r)$ are as follows:

$$P(r, t) = \frac{(1/2q_0) \cos [(\pi/q_0) \log(r/r_0^{\frac{1}{2}})]}{\cosh(\pi t/q_0) - \sin [(\pi/q_0) \log(r/r_0^{\frac{1}{2}})]},$$

$$Q(t, r) = \frac{-\sinh(\pi t/q_0)}{\cosh(\pi t/q_0) - \sin [(\pi/q_0) \log(r/r_0^{\frac{1}{2}})]}.$$

Thus P and Q are harmonic functions. Furthermore

(2.1) $P(t, r) > 0$ for $r_0 < r < 1$, $t \in \mathbb{R}$,

(2.2) $\int_{-\infty}^{\infty} P(t, r) dt + \int_{-\infty}^{\infty} P\left(t, \frac{r_0}{r}\right) dt = 1$ for $r_0 < r < 1$,

(2.3) $|P(t+2\pi k, r)| \leq \min \left\{ \frac{e}{k^4}, \frac{e \cos [(\pi/q_0) \log(r/r_0^{\frac{1}{2}})]}{|t|^4} \right\}$

for large k , $|t| \leq \pi$.

Proof of (2.3). Since

$$P(t+2\pi k, r) = \frac{(1/2q_0) \cos \left[(\pi/q_0) \log(r/r_0^{1/2}) \right]}{\cosh(\pi/q_0)(t+2\pi k) - \sin \left[(\pi/q_0) \log(r/r_0^{1/2}) \right]},$$

we have

$$\begin{aligned} |P(t+2\pi k, r)| &\leq \frac{1}{\cosh(\pi/q_0)(t+2\pi k) - 1} \\ &\leq \frac{c}{k^4} \text{ for } |t| \leq \pi \text{ and large } k, \end{aligned}$$

or

$$\begin{aligned} |P(t+2\pi k, r)| &\leq \frac{(1/2q_0) \cos \left[(\pi/q_0) \log(r/r_0^{1/2}) \right]}{\cosh(\pi/q_0)(t+2\pi k) - 1} \\ &\leq \frac{c \cos \left[(\pi/q_0) \log(r/r_0^{1/2}) \right]}{(t+2\pi k)^4} \\ &\leq \frac{c \cos \left[(\pi/q_0) \log(r/r_0^{1/2}) \right]}{|t|^4} \text{ for large } k. \end{aligned}$$

Similar arguments reveal

$$(2.3') \quad |P(t+2\pi k, r)| \leq \frac{c}{|t|} \text{ for } |t| \leq \pi \text{ and large } k.$$

By (2.3) we have

$$(2.4) \quad |P(t+2\pi k, r)| \leq \frac{c \cos \left[(\pi/q_0) \log(r/r_0^{1/2}) \right]}{k^2 |t|^2} \text{ for } |t| \leq \pi \text{ and large } k,$$

$$(2.5) \quad \left| \frac{\partial^2 P(t+2\pi k, r)}{\partial t^2} \right| \leq \frac{c}{k^2} \text{ for } |t| \leq \pi \text{ and large } k,$$

$$(2.6) \quad \left| \frac{\partial^2 P(t+2\pi k, r)}{\partial r^2} \right| \leq \frac{c}{k^2} \text{ for } |t| \leq \pi \text{ and large } k.$$

For $|t| \leq \pi$, $r_0 < r < 1$, define

$$p(t, r) = \sum_{-\infty}^{\infty} P(t+2\pi k, r).$$

By inequalities (2.1)-(2.6), we obtain

(2.7) $p(t, r)$ is harmonic on A ,

(2.8) $p(t, r) > 0$ for $r_0 < r < 1$, $|t| \leq \pi$,

(2.9)
$$\frac{1}{2\pi} \int_0^{2\pi} p(t, r) dt + \frac{1}{2\pi} \int_0^{2\pi} p\left(t, \frac{r_0}{r}\right) dt = 1,$$

(2.10) $\sup_{r_0 < r < 1} |p_r * f(t)| \leq cMf(t)$ for $f \in L^1(\Gamma)$ where

$p_r(t) = p(t, r)$ and Mf is the Hardy-Littlewood maximal function of f ,

(2.11) for any δ , $0 < \delta < \pi$, we have

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{|t| \geq \delta} p_r(t) dt = \lim_{r \rightarrow r_0} \frac{1}{2\pi} \int_{|t| \geq \delta} p_{r_0/r}(t) dt = 0.$$

Proof of (2.11). By inequality (2.4),

$$|p_r(t)| \leq \frac{c \cos\left[\left(\pi/q_0\right) \log(r/r_0^{1/2})\right]}{|t|^2} \rightarrow 0 \text{ almost everywhere as } r \rightarrow 1.$$

Similarly $p_{r_0/r}(t) \rightarrow 0$ almost everywhere as $r \rightarrow r_0$.

Using $|p_r(t)| \leq c/|t|^2$ on $\delta \leq |t| \leq \pi$ and the Lebesgue dominated convergence theorem,

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{|t| \geq \delta} p_r(t) dt = \lim_{r \rightarrow r_0} \frac{1}{2\pi} \int_{|t| \geq \delta} p_r(t) dt = 0.$$

This proves the assertion.

Set $p_r^1(t) = p_r(t)$ and $p_r^0(t) = p_{r_0/r}(t)$.

THEOREM 1. Let $f_0 \in C(\Gamma_0)$, $f_1 \in C(\Gamma_1)$. Then

$$u(t, r) = p_r^0 * f_0^1(t) + p_r^1 * f_1(t)$$

solves the Dirichlet problem on A .

Proof. By (2.7), $p(t, r)$ is harmonic on A . Write

$$\begin{aligned}
 u(t, r) - f_0(t) - f_1(t) &= \left(p_r^0 * f_0(t) - f_0(t) \right) + \left(p_r^1 * f_1(t) - f_1(t) \right) \\
 &= \frac{1}{2\pi} \int p_r^0(u) |f_0(t-u) - f_0(t)| du - \frac{1}{2\pi} \int p_r^1(u) f_0(t) du \\
 &\quad + \frac{1}{2\pi} \int p_r^1(u) |f_1(t-u) - f_1(t)| du - \frac{1}{2\pi} \int p_r^0(u) f_1(t) du .
 \end{aligned}$$

For $\epsilon > 0$ take $\delta > 0$ such that

$$\|f_0(t-u) - f_0(t)\|_\infty < \frac{\epsilon}{c+1} \text{ whenever } |u| \leq \delta .$$

But

$$\begin{aligned}
 \left| \frac{1}{2\pi} \int p_r^0(u) |f_0(t-u) - f_0(t)| du \right| &\leq \frac{1}{2\pi} \int p_r^0(u) |f_0(t-u) - f_0(t)| du \\
 &= \frac{1}{2\pi} \cdot 2 \|f_0\|_\infty \int_{|u| > \delta} p_r^0(u) du \\
 &\quad + \frac{c}{2\pi} \int_{|u| \leq \delta} |f_0(t-u) - f_0(t)| du \\
 &< c \int_{|u| > \delta} p_r^0(u) du + \epsilon \\
 &\rightarrow \epsilon \text{ as } r \rightarrow r_0 ,
 \end{aligned}$$

$$\left| \frac{1}{2\pi} \int p_r^1(u) f_0(t) du \right| \leq \frac{1}{2\pi} \|f_0\|_\infty c \cos\left(\frac{\pi}{q_0} \log \frac{r}{r_0^{\frac{1}{2}}}\right) \rightarrow 0 \text{ as } r \rightarrow 1 .$$

The estimates of the third and the fourth terms are similar. Hence

$$\lim_{r \rightarrow 1} u(t, r) = f_1(t) \text{ and } \lim_{t \rightarrow r_0} u(t, r) = f_0(t) .$$

This completes the proof.

We come to the boundary value problem on A .

THEOREM 2. *Suppose u is harmonic on A , $1 < p < \infty$. Then*

$$\sup_{r_0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |u(t, r)|^p dt < \infty$$

if and only if

$$u(t, r) = p_r^0 * f_0(t) + p_r^1 * f_1(t)$$

for some $f_0 \in L^p(\Gamma_0)$ and $f_1 \in L^p(\Gamma_1)$.

Proof. It suffices to prove the only if part.

Let

$$\sup_{r_0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |u(t, r)|^p dt < \infty .$$

By the Banach-Alaughlu theorem, there exists $f_0 \in L^p(\Gamma_0)$, $f_1 \in L^p(\Gamma_1)$ such that

$$\begin{aligned} u(t, r_k) &\rightarrow f_0(t) \text{ as } r_k \rightarrow 1, \\ u(t, s_k) &\rightarrow f_1(t) \text{ as } s_k \rightarrow r_0, \end{aligned}$$

in the w^* -topology.

Note that $g_k(t) = u(t, r_k) \in C(\Gamma_0)$ and $h_k(t) = u(t, s_k) \in C(\Gamma_1)$.

By Theorem 1,

$$\int u(x, r_k) p_r^1(t-x) dx + \int u(x, s_k) p_r^0(t-x) dx \rightarrow \begin{cases} u(t, r_k) & \text{as } r \rightarrow 1, \\ u(t, s_k) & \text{as } r \rightarrow r_0. \end{cases}$$

By the uniqueness of harmonic functions on A , and

$$u\left(t, \frac{(1-r)s_k + (r-r_0)r_k}{1-r_0}\right) \rightarrow \begin{cases} u(t, r_k) & \text{as } r \rightarrow 1, \\ u(t, s_k) & \text{as } r \rightarrow r_0, \end{cases}$$

we obtain

$$u\left(t, \frac{(1-r)s_k + (r-r_0)r_k}{1-r_0}\right) = \int u(x, r_k) p_r^1(t-x) dx + \int u(x, s_k) p_r^0(t-x) dx$$

letting $s_k \rightarrow r_0$ and $r_k \rightarrow 1$, we have

$$\begin{aligned} u(t, r) &= \int f_1(x) p_r^1(t-x) dx + \int f_0(x) p_r^0(t-x) dx \\ &= p_r^0 * f_0(t) + p_r^1 * f_1(t) . \end{aligned}$$

This completes the proof.

Theorem 2 can be extended to the Sobolev space $L_k^p(\Gamma)$ of order k .

THEOREM 3. For $1 \leq p < \infty$, k a positive integer.

(a) For $f_0 \in L_k^p(\Gamma_0)$, $f_1 \in L_k^p(\Gamma_1)$, let

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t).$$

Then $u(t, r)$ is harmonic on A ,

$$\lim_{r \rightarrow r_0} u(t, r) = f_0(t) \text{ almost everywhere,}$$

$$\lim_{r \rightarrow 1} u(t, r) = f_1(t) \text{ almost everywhere,}$$

$$\|u(t, r)\|_{L_k^p(\Gamma)} \leq \|f_0\|_{L_k^p(\Gamma_0)} + \|f_1\|_{L_k^p(\Gamma_1)},$$

for all r , $r_0 < r < 1$.

(b) Suppose $u(t, r)$ is harmonic on A . Assume there is $c > 0$, for some p , $1 \leq p \leq \infty$,

$$\|u(t, r)\|_{L_k^p(\Gamma)} \leq c \text{ for all } r, r_0 < r < 1.$$

Then

(i) for $1 < p < \infty$,

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t)$$

for some $f_0 \in L_k^p(\Gamma_0)$, $f_1 \in L_k^p(\Gamma_1)$;

(ii) for $p = 1$: if $\{f_r\}$ is Cauchy in L^1 -norm as $r \rightarrow 1$ and $r \rightarrow r_0$ where $f_r(t) = u(t, r)$, then

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t)$$

for some $f_0 \in L_k^1(\Gamma_0)$ and $f_1 \in L_k^1(\Gamma_1)$.

Proof. It suffices to prove (b) (i). By Theorem 2, take

$f_0^\alpha \in L^p(\Gamma_0)$, $f_1^\alpha \in L^p(\Gamma_1)$ with

$$\frac{\partial^\alpha}{\partial t^\alpha} \left(p_r^0 * f_0(t) \right) = p_r^0 * f_0^\alpha$$

and

$$\frac{\partial^\alpha}{\partial t^\alpha} \left(p_r^1 * f_1(t) \right) = p_r^1 * f_1^\alpha(t).$$

Taking the Fourier transforms of the above two equations, we obtain

$$f_0^\alpha = \frac{\partial^\alpha f_0}{\partial t^\alpha}, \quad f_1^\alpha = \frac{\partial^\alpha f_1}{\partial t^\alpha}.$$

This completes the proof.

Consider next the A^p -spaces, for $1 \leq p < \infty$, let

$$A^p(\Gamma) = \left\{ f \in L^1(\Gamma) : \sum_{-\infty}^{\infty} |\hat{f}(n)|^p < \infty \right\}$$

under the norm

$$\|f\|_{A^p} = \|f\|_1 + \left(\sum_{-\infty}^{\infty} |\hat{f}(n)|^p \right)^{1/p},$$

$A^p(\Gamma)$ forms a Banach space; for the basic properties of A^p -spaces, see Wang [7].

THEOREM 4. (a) For $f_0 \in A^p(\Gamma_0)$, $f_1 \in A^p(\Gamma_1)$, let

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t).$$

Then u is harmonic on A ,

$$\lim_{r \rightarrow 1} u(t, r) = f_1(t) \text{ almost everywhere,}$$

$$\lim_{r \rightarrow r_0} u(t, r) = f_0(t) \text{ almost everywhere,}$$

$$\|u(t, r)\|_{A^p} \leq \|f_0\|_{A^p} + \|f_1\|_{A^p} \text{ for } r_0 < r < 1.$$

(b) Let $u(t, r)$ be harmonic on A , and $\{f_r\}$ be Cauchy in L^1 -norm as $r \rightarrow 1$ and $r \rightarrow r_0$. Assume there is $c > 0$, for some p , $1 \leq p < \infty$;

$$\|u(t, r)\|_{A^p} \leq c \text{ for } r_0 < r < 1.$$

Then

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t)$$

for some $f_0 \in A^p(\Gamma_0)$, $f_1 \in A^p(\Gamma_1)$.

Proof. It suffices to prove (b). By Theorem 2, take $f_0 \in L^1(\Gamma_0)$, $f_1 \in L^1(\Gamma_1)$ with

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t),$$

$$\lim_{t \rightarrow 1} f_r(t) = f_1(t) \text{ in the } w^*-\text{topology,}$$

$$\lim_{r \rightarrow r_0} f_r(t) = f_0(t) \text{ in the } x^*-\text{topology.}$$

Therefore

$$\lim_{r \rightarrow 1} \hat{f}_r(n) = \hat{f}_1(n),$$

$$\lim_{r \rightarrow r_0} \hat{f}_r(n) = \hat{f}_0(n),$$

for every integer n . Therefore, by the Fatou lemma,

$$\begin{aligned} \sum |\hat{f}_1(n)|^p &= \sum \lim_{r \rightarrow 1} |\hat{f}_r(n)|^p \\ &\leq \underline{\lim}_{r \rightarrow 1} \sum |\hat{f}_r(n)|^p \\ &\leq c. \end{aligned}$$

Similarly $\sum |\hat{f}_0(n)|^p \leq c$.

Thus $f_0 \in A^p(\Gamma_0)$ and $f_1 \in A^p(\Gamma_1)$. This proves the assertion.

In Theorem 4, if p is restricted to lie between 1 and 2 then we can weaken our assumptions.

THEOREM 4'. *Let $u(t, r)$ be harmonic on A . Assume there is $c > 0$, and some p , $1 \leq p < 2$, with*

$$\|u(t, r)\|_{A^p} \leq c \text{ for } r_0 < r < 1.$$

Then

$$u(t, r) = f_0 * p_r^0(t) + f_1 * p_r^1(t)$$

for some $f_0 \in A^p(\Gamma_0)$, $f_1 \in A_p(\Gamma_1)$.

Proof. By the assumption

$$\|u(t, r)\|_{A^p} \leq c,$$

take $u_0 \in M(\Gamma_0)$, the measure algebra on Γ_0 , and $u_1 \in M(\Gamma_1)$ such that

$$u(t, r) = u_0 * p_r^0(t) + u_1 * p_r^1(t).$$

Along the same line of the proofs of Theorem 4, we obtain

$$\sum |\hat{u}_0(n)|^p < \infty,$$

$$\sum |\hat{u}_1(n)|^p < \infty.$$

By the Hausdorff-Young theorem, $u_0 \in L^q(\Gamma_0)$, $u_1 \in L^q(\Gamma_1)$, where $1/p + 1/q = 1$. Therefore $u_0 \in L^1(\Gamma_0)$, $u_1 \in L^1(\Gamma_1)$ or $u_0 \in A^p(\Gamma_0)$, $u_1 \in A^p(\Gamma_1)$. This completes the proof.

We are now in a position to describe the conjugate Poisson kernel $q(t, r)$ on A and the Hilbert transform Hf of f on Γ . Let

$$p(t, r) = p_r^0(t) + p_r^1(t)$$

then $p(t, r) = p(t, s)$ where $rs = r_0$.

By a theorem of Goldberg [3], $(1-\psi)i \operatorname{sgn}(\xi)\hat{f}(\xi)$ is the Hilbert transform of a function f on the boundary of the strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$, where ψ is a testing function with $\psi(0) = 1$. Following a well-known result of de Leeuw (see Stein and Weiss [6, p. 260]), there is a multiplier H of $L^p(\Gamma)$ to $L^p(\Gamma)$ such that

$$(Hf)^\wedge(n) = (1-\psi)i \operatorname{sgn}(n)\hat{f}(n)$$

for all integer n where

$$L^p(\Gamma) = \left\{ f : \Gamma \rightarrow \mathbb{C} : f = f_0 + f_1, \right. \\ \left. f_0(t) = f_1(t) \text{ almost everywhere, } f_0 \in L^p(\Gamma_0) \right\}.$$

We call this H the Hilbert transform of f . Note that

$$[(Hf) * p_r(t)]^\wedge(m) = \hat{f}(m)(1-\psi)i \operatorname{sgn}(n)\hat{p}_r(m)$$

for all integer m . Define

$$[q(t, r)]^\wedge(m) = (1-\psi)i \operatorname{sgn}(n)\hat{p}_r(m)$$

for all integers m . Then $q(t, r)$ is harmonic on A with

$$\|q(t, r)\|_{L^p} \leq A \|p_r\|_{L^p} \\ \leq M \quad \text{for } 1 < p < \infty.$$

$q(t, r)$ is called the conjugate Poisson kernel of $p(t, r)$.

For $p > 0$, define

$$h^p(A) = \left\{ (u, v) : \sup_{r_0 < r < 1} \int_0^{2\pi} (|u(t, r)|^2 + |v(t, r)|^2)^{p/2} dt < \infty \right\}$$

where u, v are conjugate harmonic functions on A , and $u(t, r) = u(t, s)$ with $rs = r_0$. Following Fefferman and Stein [1, p. 174], v can be recovered from u and

$$\lim_{r \rightarrow 1} u(t, r) = f$$

exists in the sense of distributions. So $h^p(A)$ can be identified with a

space $\mathcal{H}^p(\Gamma)$ of distributions on Γ . In particular, for $p > 1$,

$$\mathcal{H}^p(\Gamma) = \left\{ f : \Gamma \rightarrow \mathbb{C} : f(t) = f_0(t) + f_1(t) \text{ almost everywhere,} \right. \\ \left. f_0(t) = f_1(t) \text{ almost everywhere, } f_0 \in L^p(\Gamma_0) \right\}.$$

By Theorem 4' and Stein [5, p. 221], we obtain

PROPOSITION 5. $f \in \mathcal{H}^1(\Gamma)$ if and only if $f \in L^1(\Gamma)$, $Hf \in L^1(\Gamma)$.

Let $T : (t, r) \rightarrow re^{it}$ where $r_0 \leq r \leq 1$, $0 \leq t \leq 2\pi$.

$$\Gamma(t) = \left\{ (t', r) : |t-t'| < r-r_0, r_0 < r < \frac{1+r_0}{2} \right\} \\ \cup \left\{ (t', r) : |t-t'| < 1-r, \frac{1+r_0}{2} < r < 1 \right\},$$

$$\gamma(t) = T(\Gamma(t)),$$

$$u^*(t) = \sup_{(t', r) \in \gamma(t)} |u(t', r)|.$$

PROPOSITION 6. If $p > 0$ then $u \in \mathcal{H}^p(A)$ if and only if $u^* \in L^p(\Gamma_1)$. For each $0 < p < \infty$, there is an N so that

$$B = \left\{ \psi \in C^\infty(\Gamma) : \|\psi\|_\alpha \leq 1 \text{ for } |\alpha| \leq N \right\}$$

and if $\psi \in C^\infty(\Gamma_1)$ with $\int_0^{2\pi} \psi \neq 0$, then the L^p -norms of the following

functions are equivalent: u^* , $\sup_{(t', r) \in \gamma(t)} |\psi_r * f(t')|$,

$\sup_{r_0 < r < 1} |\psi_r * f(t)|$, and $\sup_{r_0 < r < 1} \sup_{\psi \in B} |\psi_r * f(t)|$ where $f \in \mathcal{H}^p(\Gamma)$ and

$$u(t, r) = p_r * f(t).$$

PROPOSITION 7. If $\psi \in C^\infty(\Gamma_1)$, $\frac{1}{2\pi} \int_0^{2\pi} \psi = 1$ and $\int_0^{2\pi} x^\alpha \psi(x) dx = 0$ for all $\alpha \neq 0$, then

$$\|f - \psi * f\|_{\mathcal{H}^p} \leq c \|f\|_{\mathcal{H}^p}.$$

In particular $f \in h^p \Rightarrow f - \psi * f \in H^p$ where H^p is the Hardy space on the unit disk D on the plane.

PROPOSITION 8. *The following spaces are identical:*

- (1) the dual of h^1 ;
- (2) $\{b : b = b_0 + Hb_1, b_0, b_1 \in L^\infty\}$;
- (3) $\left\{ b \in L^1 : \sup_{|Q| < 1} \frac{1}{|Q|} \int_Q \left| b - \frac{1}{|Q|} \int_Q b \right| < \infty, \right.$
 $\left. \sup_{|Q| < 1} \frac{1}{|Q|} \int_Q |b| < \infty, Q \text{ cubes on } \Gamma_1 \right\}$;
- (4) $\{b \in BMO(D) : \psi * b \in L^\infty\}$ for some $\psi \in C^\infty(\Gamma_1)$,

$$\frac{1}{2\pi} \int_0^{2\pi} \psi(x) dx = 1 , \text{ and } \int_0^{2\pi} x^\alpha \psi(x) dx = 0 .$$

Proofs of Propositions 6-8 run parallel to proofs in Goldberg [3].

3. Some applications

For r_0 , $0 < r_0 < 1$, let

$$A_p = \{z \in \mathbb{C} : |z| < r_0\} ,$$

$$A_v = \{z \in \mathbb{C} : r_0 < |z| < 1\} ,$$

$$\Gamma_0 = \{z \in \mathbb{C} : |z| = r_0\} ,$$

$$\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\} .$$

Here if we consider A_p as the domain filled by a plasma and A_v as the domain corresponding to the vacuum, then our results can be applied to solve a special problem in plasma physics.

THEOREM 9. *Given two boundary data functions $f_i \in L^p(\Gamma_i)$,*

$i = 0, 1$, $1 < p < \infty$, there is a function u defined on the closed unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$ such that

$$\begin{aligned}\Delta u &= -\lambda^2 u \quad \text{on } A_p, \\ \Delta u &= 0 \quad \text{on } A_v, \\ u &= f_i \quad \text{on } \Gamma_i, \quad i = 1, 2.\end{aligned}$$

Proof. Take a function v on the closed unit disk D satisfying

$$\begin{aligned}\Delta v &= -\lambda^2 r_0^2 v \quad \text{on } \{z \in \mathbb{C} : |z| < 1\}, \\ v &= 0 \quad \text{on } \{z \in \mathbb{C} : |z| = 1\}\end{aligned}$$

(see Folland [2, p. 137]). Let

$$w(x) = v(x/r_0)$$

and p_r^0, p_r^1 as in Theorem 2. Then

$$u = w\chi_{A_p} + \left[f_0 * p_r^0 + f_1 * p_r^1 \right] \chi_{A_v}$$

is our candidate.

Finally we ask whether our theory can be extended to general domains.

1. If we extend the annulus $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$ to a general domain B which is an elliptic annulus possessing two elliptic boundaries, then can our theory apply or not? The affirmative answer is usual for number theory.

2. For $0 < r_1 < 1 < r_2 < \infty$, let $\Gamma_0 = \{z \in \mathbb{C} : |z| = 1\}$, $\Gamma_1 = \{z \in \mathbb{C} : |z| = r_1\}$, $\Gamma_2 = \{z \in \mathbb{C} : |z| = r_2\}$. Let $C_1 = \{z \in \mathbb{C} : r_1 < |z| < 1\}$, and $C_2 = \{z \in \mathbb{C} : 1 < |z| < r_2\}$. Given boundary data functions f_i on Γ_i , $i = 0, 1, 2$, can we apply our theory to solve the generalized Dirichlet problem

$$\begin{aligned}\Delta u &= -\lambda^2 u \quad \text{on } C_1, \\ \Delta u &= 0 \quad \text{on } C_2, \\ u &= f_i \quad \text{on } \Gamma_i ?\end{aligned}$$

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