SPECTRA OF IRREDUCIBLE MATRICES

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1. Introduction

A real matrix is called *non-negative* (*positive*) if all its entries are non-negative (positive). Two matrices A and B are said to be *cogredient* if there exists a permutation matrix Q such that $QAQ^{T} = B$. A square non-negative matrix is called *reducible* if it is cogredient to a matrix of the form

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix},$$

where the blocks X and Y are square. Otherwise it is called *irreducible*.

Frobenius (**1**) proved *inter alia* (see Section 3 below) that an irreducible matrix is cogredient to a matrix in the form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & A_{23} & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & \cdot & & & \cdot \\ \cdot & \cdot & & & & \cdot & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & A_{h-1,h} \\ A_{h1} & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}, \qquad (1)$$

where the zero blocks along the main diagonal are square and h is the index of imprimitivity of A, i.e. the number of eigenvalues of A of maximal modulus (see Lemma 1 (*c*) in Section 3 below).

Mirsky (**5**) showed that if $A_{12}, A_{23}, \ldots, A_{h1}$ are any complex m-square matrices (here h is an arbitrary positive integer) and the eigenvalues of the product $A_{12}A_{23}\ldots A_{h1}$ are $\omega_1, \ldots, \omega_m$, then the eigenvalues of the hm-square matrix in the form (1) with the $A_{i,i+1}$ in the indicated superdiagonal positions consist of all the hth roots of $\omega_1, \ldots, \omega_m$ (a hth root of zero being counted h times).

In this paper I extend Mirsky's result to all complex matrices in the form (1) where the superdiagonal blocks $A_{12}, \ldots, A_{h1}$ are not necessarily square, and I use this theorem to gain new information about the structure of irreducible matrices and their spectra.

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2. Main results

Theorem 1. *Let A be an n-square complex matrix in the superdiagonal block form*

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & A_{23} & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & A_{k-1,k} \\ A_{k1} & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}, \tag{2}$$

where the zero blocks along the main diagonal are square. Let $\omega_1, \ldots, \omega_m$ be the non-zero eigenvalues of the product $A_{12}A_{23}\ldots A_{k1}$. Then the spectrum of A consists of $n-km$ zeros and the km kth *roots of the numbers $\omega_1, \ldots, \omega_m$.*

In order to exploit significantly Theorem 1 via the result of Frobenius to the case of irreducible non-negative matrices, we establish the following two auxiliary theorems which may be of interest in themselves.

Theorem 2. *Let $B_1, \ldots, B_s$ and $C_1, \ldots, C_t$ be irreducible non-negative matrices. The direct sums*

$$G = \sum_{i=1}^{s}{}^{\cdot}\, B_i$$

and

$$H = \sum_{i=1}^{t}{}^{\cdot}\, C_i$$

are cogredient if and only if $s = t$ and there exists a permutation σ such that B_i and $C_{\sigma(i)}$ are cogredient for $i = 1, \ldots, s$.

Theorem 3. *If A is an irreducible non-negative matrix and if A^k is cogredient to a direct sum of irreducible matrices $C_1, \ldots, C_k$, then k divides the index of imprimitivity of A, and all the C_i have the same non-zero eigenvalues.*

By an application of the above theorems we obtain the following result.

Theorem 4. *Let A be an irreducible non-negative n-square matrix and suppose that A^k is cogredient to a direct sum of irreducible matrices $C_1, \ldots, C_k$. If the non-zero eigenvalues of C_1 are $\omega_1, \ldots, \omega_m$, then the spectrum of A consists of $n-km$ zeros and the km* kth *roots of $\omega_1, \ldots, \omega_m$.*

3. Preliminaries

Some known results are first stated for reference purposes.

Lemma 1 (Frobenius (**1**)). *If A is an irreducible non-negative matrix, then:*

(*a*) *A has a real simple positive eigenvalue r which is greater than or equal to the moduli of its other eigenvalues* (the number r is called the *maximal eigenvalue* of A);

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(*b*) *there exists a positive eigenvector corresponding to r;*

(*c*) *if A has h eigenvalues of modulus r, then these are the distinct roots of* $\lambda^h - r^h = 0$ (the number h is called the *index of imprimitivity* of A. If $h = 1$, then A is said to be *primitive*);

(*d*) *A is cogredient to a matrix in the form* (1).

Lemma 2. *If A is a complex matrix in the form* (2), *then*

$$A^k = \sum_{t=1}^{k}{}^{\cdot}\, B_t,$$

where $B_t = A_{t,t+1}A_{t+1,t+2}\dots A_{t-1,t}$, $t = 1, \dots, k$.

Lemma 3 (Sylvester (6)). *All the matrices* B_t *defined in Lemma* 2 *have the same nonzero eigenvalues.*

Lemma 4 (Minc (4)). *Let A be an irreducible non-negative matrix with index of imprimitivity h. Then A is cogredient to a matrix in the form* (2) *with k non-zero blocks in the superdiagonal if and only if k divides h.*

Lemma 5 (Minc (4)). *If A is an irreducible non-negative matrix in the form* (2) *with k non-zero blocks in the superdiagonal, then*

$$A^k = \sum_{t=1}^{k}{}^{\cdot}\, B_t,$$

where the blocks $B_t = A_{t,t+1}A_{t+1,t+2}\dots A_{t-1,t}$ *are irreducible.*

The last auxiliary result is an extension to complex matrices of a theorem of Frobenius (1) on non-negative matrices.

Lemma 6. *Let A be a complex* $n \times n$ *matrix in the form* (2), *and let*

$$\lambda^n + \Sigma b_t \lambda^{m_t},$$

where the coefficients b_t *are non-zero, be the characteristic polynomial of A. Then k divides* $n - m_t$ *for all t.*

Proof of Lemma 6. Let $p(\lambda, M)$ denote the characteristic polynomial of M. Suppose that A is in the form (2), where the block $A_{t,t+1}$ is $n_t \times n_{t+1}$, $t = 1, \dots, n-1$, and A_{k1} is $n_k \times n_1$, and let

$$D = \sum_{t=1}^{k}{}^{\cdot}\, \theta^t I_{n_t},$$

where $\theta = \exp(2\pi i/k)$. Then

$$D^{-1}AD = \theta A,$$

and therefore

$$D^{-1}(\theta\lambda I_n - A)D = \theta(\lambda I_n - A),$$

so that

$$p(\theta\lambda, A) = \theta^n p(\lambda, A).$$

Hence

$$\theta^n\lambda^n + \sum_t b_t \theta^{m_t}\lambda^{m_t} = \theta^n\lambda^n + \sum_t b_t \theta^n \lambda^{m_t},$$

i.e.

$$\theta^{m_t} = \theta^n$$

for all t. Thus

$$\exp(2\pi i(n-m_t)/k) = 1$$

for all t. The result follows.

4. Proofs

Proof of Theorem 1. The proof is similar to that of Mirsky's theorem (5). By Lemma 3, the spectrum of A^k consists of the numbers $\omega_1, \ldots, \omega_m$, each counted k times, and $n-km$ zeros. Thus

$$p(\lambda, A^k) = \lambda^{n-km} \prod_{j=1}^{m} (\lambda-\omega_j)^k, \tag{3}$$

and therefore

$$p(\lambda, A) = \lambda^{n-km}\phi(\lambda),$$

where

$$\phi(\lambda) = \sum_{t=1}^{km} c_t\lambda^t.$$

By Lemma 6, a coefficient c_t must vanish unless k divides

$$n-(n-km+t) = km-t.$$

It follows that $c_t = 0$ whenever k does not divide t. In other words, $\phi(\lambda)$ is a polynomial in λ^k:

$$\phi(\lambda) = \prod_{t=1}^{m} (\lambda^k-\zeta_t)$$

for some numbers $\zeta_1, \ldots, \zeta_m$. Hence

$$p(\lambda, A) = \lambda^{n-km} \prod_{t=1}^{m} (\lambda^k-\zeta_t)$$

$$= \lambda^{n-km} \prod_{\substack{1 \leqq t \leqq m \\ 1 \leqq j \leqq k}} (\lambda-\zeta_t^{1/k}\theta^j), \tag{4}$$

where $\theta = \exp(2\pi i/k)$ and $\zeta_t^{1/k}$ denotes any fixed kth root of ζ_t. Therefore the characteristic polynomial of A^k is

$$p(\lambda, A^k) = \lambda^{n-km} \prod_{t=1}^{m} (\lambda-\zeta_t)^k. \tag{5}$$

Comparing (3) and (5) it can be concluded that the numbers $\zeta_1, \ldots, \zeta_m$ are the same as the numbers $\omega_1, \ldots, \omega_m$, in some order. Thus the characteristic equation (4) of A reads

$$p(\lambda, A) = \lambda^{n-km} \prod_{\substack{1 \leqq t \leqq m \\ 1 \leqq j \leqq k}} (\lambda-\omega_t^{1/k}\theta^j),$$

and the theorem is established.

Proof of Theorem 2. The sufficiency of the conditions is quite obvious. To prove the necessity let P be a permutation matrix such that

$$P^{\mathsf{T}}GP = H$$

and let τ be the permutation corresponding to P, so that the (i, j) entry of G is permuted into the $(\tau(i), \tau(j))$ position of $H = P^{\mathrm{T}}GP$. For brevity the notation $\bar{\imath}$ is used in place of $\tau(i)$. Denote by $A[\mu_1, \ldots, \mu_a \mid \nu_1, \ldots, \nu_b]$ the submatrix of A lying in rows numbered $\mu_1, \ldots, \mu_a$ and columns numbered $\nu_1, \ldots, \nu_b$; the rows $\mu_1, \ldots, \mu_a$ of A (and the columns $\nu_1, \ldots, \nu_b$) are said to *intersect* the submatrix. Now suppose that for some v, $1 \leqq v \leqq$ t,

$$C_v = H[\bar{\alpha}_1, \ldots, \bar{\alpha}_p, \bar{\beta}_{p+1}, \ldots, \bar{\beta}_q \mid \bar{\alpha}_1, \ldots, \bar{\alpha}_p, \bar{\beta}_{p+1}, \ldots, \bar{\beta}_q],$$

and that rows and columns $\alpha_1, \ldots, \alpha_p$ of G intersect block B_u but none of rows nor columns $\beta_{p+1}, \ldots, \beta_q$ of G intersect B_u. However, the only non-zero entries in the rows $\alpha_1, \ldots, \alpha_p$ of G are in the columns $\alpha_1, \ldots, \alpha_p$. Thus

$$G[\alpha_1, \ldots, \alpha_p \mid \beta_{p+1}, \ldots, \beta_q] = 0,$$

and therefore

$$H[\bar{\alpha}_1, \ldots, \bar{\alpha}_p \mid \bar{\beta}_{p+1}, \ldots, \bar{\beta}_q] = 0.$$

But this would imply that C_v is reducible. Hence the supposition is impossible, and each of the C_j can intersect only rows and columns corresponding to rows and columns that intersect a single B_i. Since $\sum_{i=1}^{t}{}^{\cdot}\, C_i$ and $\sum_{i=1}^{s}{}^{\cdot}\, B_i$ are cogredient, the result follows.

Proof of Theorem 3. It is first shown that k must divide the index of imprimitivity h of A. Let r be the maximal eigenvalue of A and let x be a positive eigenvector corresponding to r. Then x is an eigenvector of A^k corresponding to r^k. Now, A^k is cogredient to $\sum_{t=1}^{k}{}^{\cdot}\, C_t$ and therefore r^k is an eigenvalue (clearly of maximal modulus) of each C_t. Since the C_t are irreducible, the eigenvalue r^k is simple and therefore A^k has exactly k eigenvalues equal to r^k. But Lemma 1 (*c*) implies that there are $d = \gcd(h, k)$ such eigenvalues. Hence $d = k$ and thus k divides h.

It now follows from Lemma 4 in conjunction with Lemma 2 and Lemma 3 that A^k is cogredient to

$$\sum_{t=1}^{k}{}^{\cdot}\, B_t,$$

where the B_t are irreducible and all the B_t have the same non-zero eigenvalues. But then $\sum_{t=1}^{k}{}^{\cdot}\, B_t$ and $\sum_{t=1}^{k}{}^{\cdot}\, C_t$ are cogredient, and all the B_t and all the C_t are irreducible. Thus by Theorem 2 the $B_1, \ldots, B_k$ are cogredient to the $C_1, \ldots, C_k$, in some order, and the result follows.

Proof of Theorem 4. By Theorem 3, k divides the index of imprimitivity of A, and thus by Lemma 4, the matrix A is cogredient to a matrix in the form (2) with blocks $A_{12}, A_{23}, \ldots, A_{k1}$ in the superdiagonal. Then A^k is cogredient to $\sum_{t=1}^{k}{}^{\cdot}\, B_t$, where $B_t = A_{t,t+1}A_{t+1,t+2}\ldots A_{t-1,t}$, $t = 1, \ldots, k$, and all the B_t have the same non-zero eigenvalues. Hence by Theorem 2 and Theorem 3, the matrices B_1 and C_1 have the same non-zero eigenvalues. The result now follows by virtue of Theorem 1.

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