circles on BB′, CC′ as diameters become the point circle C, and the directrix is by (7) the radical axis for the circle on ab as diameter and the point circle C. Hence if C be on the directrix the circle on the chord ab must touch it at C and therefore the tangents from a point on the directrix are at right angles.

(v) Again taking d as the mid point of the chord ab and assuming that ad^2 varies as Cd, where C is the point of intersection of tangents at ends of a, b, then the distance of d from the image of C in the directrix will be constant for all tangents drawn from a point on the same diameter.

For the image of C, say e, is the second limiting point for the system C and circle on ab. Hence $de.d\mathrm{C} = ab^2$; $\therefore$ de is constant.

Eighth Meeting, June 12, 1891.

R. E. Allardice, Esq., M.A., F.R.S.E., President, in the Chair.

On a Representation of Elliptic Integrals by Curvilinear Arcs.

By John M'Cowan, M.A., B.Sc.

It is well known that the elliptic integral of the second kind may be represented by the arc of an ellipse, and mathematicians have sought with various success to represent similarly by the arc of an algebraic curve the elliptic integral of the first kind. The general solution of the problem has not been obtained, but Serret and Cayley have given solutions of a very general character.

If, however, the condition that the curve be algebraic be not imposed solutions may be obtained without difficulty. That here given has the peculiarity that the set of elliptic integrals of the first kind for all values of the modulus is represented by the arcs of one system of curves, in several distinct ways, each of a very simple geometric character. The system of curves is further interesting, as representing in the different groupings which naturally arise different systems of stream lines due to vortices in two dimensional fluid motion.

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These peculiarities have induced me to make this communication.

§ 1. *To find a curve, the arc of which will represent the integral*

$$s = \int_0^\psi \frac{a d\psi}{\sqrt{1 - k^2 \sin^2\psi}} \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

Let the curve be referred to rectangular co-ordinates x and y; let s be the length of the arc measured from the point of intersection of the curve and the axis of y to the point x, y, and let ψ be the inclination of the tangent at this point to the axis of x.

Now $$\frac{dx}{ds} = \cos\psi, \ \frac{dy}{ds} = \sin\psi$$

$$\therefore \qquad x = \int_0^\psi \cos\psi ds = \int_0^\psi \frac{a\cos\psi d\psi}{\sqrt{1 - k^2\sin^2\psi}} \quad \dots \quad \dots \quad (2)$$

$$\therefore \qquad kx = a\sin^{-1}k\sin\psi. \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

$$y = \int_0^\psi \sin\psi ds = \int_0^\psi \frac{a\sin\psi d\psi}{\sqrt{1 - k^2\sin^2\psi}} \quad \dots \quad \dots \quad (4)$$

$$\therefore \qquad ky = -a\log(k\cos\psi + \sqrt{1 - k^2\sin^2\psi})/(1 + k). \quad \dots \quad (5)$$

The equation to the curve, found by eliminating ψ between (3) and (5) is therefore

$$\cosh ky/a - \cos kx/a = k\sinh ky/a. \quad \dots \quad \dots \quad (6)$$

It may be noted that (6) holds for all real or pure imaginary values of k, and therefore for all such values this system of curves, all passing through the origin, represents the integral (1), s being measured from the origin. We proceed to show, however, that the system represents elliptic integrals of the first kind in other ways, equally simple.

§ 2. *Two other interpretations.*

For convenience put $k/a = m$, $\eta = 1/k$, and change the independent variable from ψ to x in (1) by means of (3). We thus obtain

$$s = \int_0^x \frac{dx}{\sqrt{1 - \eta^2\sin^2 mx}} \quad \dots \quad \dots \quad \dots \quad (7)$$

Hence the system (6) represents this integral also, but in an entirely different manner, the amplitude being now the abscissa, instead of the inclination of the tangent.

There is still another simple interpretation.

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Taking the common form of quadric transformation

$$2\phi = \psi + \sin^{-1}k\sin\psi = \psi + mx, \text{ by } (3)$$

(1) becomes

$$s = \frac{2a}{1+k}\int_0^{\frac{1}{2}(\psi+mx)} d\phi/\sqrt{1-c^2\sin^2\phi} \qquad \dots \qquad \dots \quad (8)$$

where $c^2 = 4k/(1+k)^2$. The transformation applied to (7) leads to the same result.

Thus in this third simple geometric manner the elliptic integrals of the first kind are represented by the system of curves (6).

§ 3. *Connexion with stream lines.*

Remembering that $\eta = 1/k$, we may write (6),

$$\eta = \frac{\sinh my}{\cosh my - \cos mx} \qquad \dots \qquad \dots \qquad \dots \quad (9)$$

whence,

$$\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} = 0.$$

Thus η may be regarded as the stream function in a case of two dimensional fluid motion. Further, η is everywhere finite except at the points on the plane $y=0$, where $\cos mx = 1$, hence where y is small, and $mx = 2n\pi + mx'$ where x' is small, (9) becomes

$$\eta = \frac{my}{m^2(x'^2+y^2)} \qquad \dots \qquad \dots \qquad \dots \quad (10)$$

Hence, η is the stream function due to a set of equal straight and parallel vortex filaments adjacent to an infinite plane wall in infinite liquid, at equal distances $2\pi/m$ apart.

The stream lines $\eta =$ constant, are thus easily pictured. When $\eta > 1$, or $k < 1$, they are closed curves, nearly circular by (10) when k is small, touching the axes of x at the vortices and always lying within the limits $x' = \pm\frac{\pi}{2m}$ on either side of each vortex. The curve for which $k=1$ is asymtotic to the planes $x' = \pm\frac{\pi}{2m}$ on either side of each vortex, and separates the closed curves for which $k<1$ from the curves for which k is >1, which pass from vortex to vortex touching the axis of x at each.

§ 4. *Second representation by stream lines.*

The system of curves given by (6) may remarkably enough be grouped in other ways, giving other stream line systems. If in the

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integration (4) we take a different inferior limit, we may take instead of (4)

$$y = -\frac{a}{k}\sinh^{-1}\frac{k}{\sqrt{1-k^2}}\cos\psi,\ k<1.... \qquad ...\ (11)$$

or

$$y - \frac{a}{k}\cosh^{-1}\frac{k}{\sqrt{k^2-1}}\cos\psi,\ k>1.... \qquad ...\ (12)$$

Hence, by (2), the system of curves is given by

$$\cos mx = \sqrt{1-k^2}\cosh my,\ k<1. \quad .. \quad ...\ (13)$$

or

$$\cos mx = \sqrt{k^2-1}\ \sinh my,\ k>1. \quad ... \quad ...\ (14)$$

The curves given by (13) and (14) are of course the same as those given by (6), but they are differently situated relatively to one another.

The symmetry of the curves is shown very obviously by these latter equations.

If in (13) we put $k = \text{sech}\,\xi$, it may be written

$$\xi = \tanh^{-1}\frac{\cos mx}{\cosh my} \quad ... \quad ... \quad ...\ (15)$$

and hence we may show, much as was done for η, that ξ is the stream function for an infinite straight vortex filament lying midway (at the origin) between two infinite plane walls, $x = \pm\frac{\pi}{m}$.

The curves (14) would cut (13) orthogonally if they were displaced through a distance $\pi/2m$ along the axis of x, and are therefore the equipotential surfaces for the same vortex shifted through this distance. In fact, putting $k = \text{cosec}\,\zeta$ in (14), it becomes

$$\zeta = \tan^{-1}\frac{\sinh my}{\cos mx} \quad ... \quad ... \quad ...\ (16)$$

and if we shift the origin through a distance $x = \pi/2m$ this becomes the velocity potential corresponding to the stream function ξ.

§ 5. The method applied to the elliptic integral of the first kind in equations (1) to (6) is obviously equally applicable to the other elliptic integrals and to a very large class of integrals depending on them, to give curves which will represent them by their arcs. The simple threefold geometric interpretation which has been given for the integral of the first kind is, however, peculiar to that integral, and in general the stream line connexion is so also.

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