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# Regular completions of uniform convergence spaces

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A regular completion with universal property is obtained for each member of the class of $u$-embedded uniform convergence spaces, a class which includes the Hausdorff uniform spaces. This completion is obtained by embedding each $u$-embedded uniform convergence space $(X, I)$ into the dual space of a complete function algebra composed of the uniformly continuous functions from $(X, I)$ into the real line.

## 1. Introduction

Let $(X, I)$ be a uniform convergence space as defined by Cook and Fischer in [2], and let $U(X)$ be the set of all uniformly continuous functions from $(X, I)$ into $(R, \mathsf{U})$, where $R$ denotes the real line and $\mathsf{U}$ the usual uniformity. We wish to assign to $U(X)$ the coarsest uniform convergence structure $J$ relative to which the evaluation map $\omega : \big(U(X), J\big) \times (X, I) \to (R, \mathsf{U})$, defined by $\omega(f, x) = f(x)$, is uniformly continuous. Unfortunately, $U$ will not exist in the class of uniform convergence spaces as that concept is defined in [2]. However Wyler, [6], has introduced an axiom system for uniform convergence spaces which is precisely suited to our needs. Wyler's definition of a uniform convergence space, which is given in the next paragraph, will be used throughout the remainder of this paper.

A *uniform convergence space* $(X, I)$ is a set $X$ along with a set $I$ of filters on $X \times X$ which satisfy the following conditions:

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(1) $\dot{x} \times \dot{x} \in I$ for each $x \in X$ ;

(2) $\Phi^{-1} \in I$ whenever $\Phi \in I$ ;

(3) $\Phi \wedge \Psi \in I$ whenever $\Phi, \Psi \in I$ ;

(4) $\Psi \in I$ whenever $\Phi \leq \Psi$ and $\Phi \in I$ ;

(5) $\Phi \circ \Psi \in I$ whenever $\Phi, \Psi$ and the composition $\Phi \circ \Psi$ is a filter.

Compositions and inverses of filters are defined in the natural way. For any point $x$ in a set $X$, $\dot{x}$ denotes the fixed ultrafilter generated by $\{x\}$. Let $\Delta = \{(x, x) : x \in X\}$ be the *diagonal* of $X \times X$, and let $\dot{\Delta}$ be the filter on $X \times X$ consisting of all oversets of $\Delta$. The definition of "uniform convergence space" given above differs from that of [2] only in Condition (1) which, in [2], is replaced by the stronger condition: $\dot{\Delta} \in I$. Virtually all of the theorems of [2] appear to be valid using the weaker axiom system of Wyler.

The abbreviation "u.c.s." will be used both for "uniform convergence space" and "uniform convergence structure"; it should be obvious from the context which meaning is intended.

For any convergence space $X$, let "$\mathrm{cl}_X$" be the closure operator for $X$, and $\lambda X$ the *topological modification* of $X$. A u.c.s. $(X, I)$ is said to be *regular* if $\mathrm{cl}_{X\times X}\Phi \in I$ whenever $\Phi \in I$. The goal of this paper is to obtain a regular completion for a class of u.c.s.'s. A summary of our results follows.

Let $(X, I)$ be a u.c.s., $U(X)$ the set of all uniformly continuous functions from $(X, I)$ to $(R, U)$, and $J$ the coarsest u.c.s. on $U(X)$ relative to which the evaluation map $\omega$ (defined above) is uniformly continuous. In Section 2, we show that $(U(X), J)$ is a complete u.c.s. Let $(U^2(X), J^2)$ be the dual space obtained by a repetition of the previous construction; for notational convenience, we will sometimes use the symbol "$D$" for dual space, especially in conjunction with the closure operator. If the natural map $i : (X, I) \to (U^2(X), J^2)$, defined by $i(x)(f) = f(x)$ for all $f \in U(X)$, is a uniform embedding, then $(X, I)$ is said to be *u-embedded*. The $u$-embedded spaces form a productive and hereditary class of u.c.s.'s which include, as a subclass, the Hausdorff uniform spaces. Some other interesting facts about this class are:

(a) $u$-embedded spaces are regular and Hausdorff;

(b) $(U(X), \mathcal{J})$ is always $u$-embedded;

(c) a totally bounded $u$-embedded space is a uniform space;

(d) a $u$-embedded space can be non-topological.

A regular completion of a $u$-embedded space $(X, I)$ is obtained by forming $X^* = \mathrm{cl}_{\lambda D} iX$ ; the latter set, being a closed subspace of the complete space $D$ , is complete. This completion is shown to have the universal property relative to the class of $u$-embedded u.c.s.'s, and is equivalent to the usual uniform completion if $(X, I)$ is a Hausdorff uniform space. Consequently, the standard completion of a Hausdorff uniform space is unique in the larger class of all $u$-embedded u.c.s.'s.

It should be noted that in the completion described above, $iX$ is dense in $X^*$ relative to $\lambda X^*$ , not relative to $X^*$ itself. This does not appear to be a serious flaw, and we conjecture that $\mathrm{cl}_D iX = \mathrm{cl}_{\lambda D} iX$ for any $u$-embedded space $(X, I)$ . A similar completion theory for Cauchy spaces has been obtained by two of the authors (see [4]).

One of the more interesting features of the construction described above is that it yields a natural external completion of a Hausdorff uniform space which could not be obtained without introducing, as an intermediate step, the concept of a uniform convergence space.

## 2. The space $U(X)$

Throughout this section, it will be assumed that $(X, I)$ is an arbitrary u.c.s. For basic definitions and other information about u.c.s.'s not already provided in the Introduction, see [2].

If $A \subset U(X) \times U(X)$ and $F \subset X \times X$ , then $A(F)$ denotes the set $\{(f(x), g(y)) : (f, g) \in A, (x, y) \in F\}$ . If $\Phi$ is a filter on $U(X) \times U(X)$ , $\mathsf{F}$ a filter on $X \times X$ , then $\Phi(\mathsf{F})$ designates the filter generated by $\{A(F) : A \in \Phi, F \in \mathsf{F}\}$ . Let $\mathcal{J}$ be the collection of filters on $U(X) \times U(X)$ defined by: $\Phi \in \mathcal{J}$ if and only if, for each $\mathsf{F} \in I$ , $\Phi(\mathsf{F}) \geq \mathsf{U}$ , where $\mathsf{U}$ is the usual uniformity on $R$ .

THEOREM 2.1. $(U(X), J)$ *is a u.c.s.*

Proof. Let $f \in U(X)$ and $F \in I$. Since $(f\times f)F \geq U$, $(f, f) \in J$, and (1) is established. Condition (5) can be obtained with the help of the following inequality: $(\Phi\circ\Psi)(F) \geq \Phi(F) \circ \Psi(F\circ F)$, for any symmetric filter $F$ on $X \times X$. The proofs for Conditions (2), (3), and (4) are trivial.

The following example shows that $(U(X), J)$ may fail to be a u.c.s. in the sense of [2].

EXAMPLE 2.2. Let $X = R$, and let $I$ be the usual u.c.s. for $R$. Let $\Delta_1$ be the diagonal in $U(X) \times U(X)$, and let $f$ be any member of $U(X)$ such that $f(0) \neq f(1)$. For any positive real number $k$, let $kf$ be the scalar multiple of $f$. Note that $(kf, kf) \in \Delta_1$ for all $k > 0$, so it is clear that $\dot{\Delta}_1(U) \not\geq U$. Thus $\dot{\Delta}_1 \notin J$, and $(U(X), J)$ is not a u.c.s. as defined in [2].

For any product space $X \times Y$, we will use $P_1 : X \times Y \to X$ and $P_2 : X \times Y \to Y$ as the two projection maps. If $F$ and $G$ are two filters on the same set; the coarsest filter finer than $F$ and $G$ (if it exists) is denoted by $F \vee G$.

THEOREM 2.3. *$J$ is the coarsest u.c.s. on $U(X)$ relative to which the evaluation map $\omega$ is continuous.*

Proof. A basic element in the product u.c.s. on $(U(X), J) \times (X, I)$ is of the form $(P_1\times P_1)^{-1}\Phi \circ (P_2\times P_2)^{-1}F = A$, where $\Phi \in J$ and $F \in I$. But $(\omega\times\omega)A = \Phi(F) \geq U$, and so $\omega$ is uniformly continuous relative to $(U(X), J)$. If $J_1$ is another admissible u.c.s. on $U(X)$, then $\Phi(F)$ must be finer than $U$ whenever $\Phi \in J_1$ and $F \in I$, so $\Phi \in J$ and $J \leq J_1$.

THEOREM 2.4. *$U(X)$ is complete.*

Proof. Let $G$ be a Cauchy filter on $U(X)$; that is, $G \times G \in J$. For each $x \in X$, $\dot{x}\times\dot{x} \in I$, and $G(\dot{x})$ is a Cauchy filter on $R$. Let $f(x)$ be the limit of $G(\dot{x})$ in $R$.

First we show that $f \in U(X)$. Let $F \in I$; then $(G\times G)F \geq U$. If

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$V \in \mathbf{U}$ , then let $G \in \mathbf{G}$ , $A \in \mathbf{F}$ be such that $(G\times G)A \subset V$ . It suffices to show that $(f\times f)A \subset \mathrm{cl}_{R\times R}V$ . Let $(x, y) \in A$ ; then

$$\mathrm{cl}_R\big(G(x)\big) \times \mathrm{cl}_R\big(G(y)\big) \subset \mathrm{cl}_{R\times R}(G\times G)A \subset \mathrm{cl}_{R\times R}V .$$

Since $f(x) \in \mathrm{cl}_R G(x)$ , $\big(f(x), f(y)\big) \in \mathrm{cl}_{R\times R}V$ , and so $f \in U(X)$ .

Finally, we show that $\mathbf{G} \times f^{\bullet} \in J$ $\big($that is, $G \to F$ in $U(X)$ $\big)$. For $\mathbf{F} \in I$ , $V \in \mathbf{U}$ , there are sets $G$ in $\mathbf{G}$ and $F$ in $\mathbf{F}$ such that $(G\times G)F \subset V$ . Then $(G\times\{f\})F \subset \mathrm{cl}_{R\times R}V$ follows as in the preceding paragraph, and the proof is complete.

Let $\alpha X$ be the space consisting of the set $X$ supplied with the weak topology induced by the set of functions $U(X)$ . Thus $\alpha X$ is a completely regular topological space whose topology is coarser than the convergence structure which $I$ induces on $X$ .

THEOREM 2.5. *$U(X)$ separates points and $\alpha X$-closed subsets of $X$ .*

Proof. Let $A \subset X$ be $\alpha X$-closed, and $x \notin A$ . Then there is a basic open set $V = \cap\left\{f_i^{-1}(V_i) : i = 1, \ldots, n\right\}$ containing $x$ and contained in $X - A$ , where $V_i$ is a subbasic open set in $R$ of the form $(a_i, \infty)$ or $(-\infty, b_i)$ . Indeed, if $f_i$ is replaced by $-f_i$ whenever necessary, we can assume with no loss of generality that each $V_i$ is of the form $(a_i, \infty)$ for $i = 1, \ldots, n$ . Letting $g_i(x) = \sup\{f_i(x)-a_i, 0\}$ for all $x \in X$ , we obtain that $V = \cap\left\{g_i^{-1}(0, \infty) : i = 1, \ldots, n\right\}$ . Finally, let $g$ be the product function $g_1g_2 \cdots g_n$ . Then $g \in U(X)$ , $g(x) \neq 0$ , and $g(A) = \{0\}$ , which establishes the desired result.

## 3. $u$-embedded spaces

There is a close analogy between the function algebra $C(X)$ with the continuous convergence structure, which is studied in [1], and the space $\big(U(X), J\big)$ . Also, the notion of a *c-embedded space*, [1], corresponds to our "$u$-embedded space" in an obvious way. Some of the theorems of [1] pertaining to $c$-embedded spaces extend without difficulty to $u$-embedded spaces. In particular, Theorems 3.1 and 3.2 below have proofs which are

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similar to those of Lemma 16 and Satz 21, respectively of [1]; the proofs of the former theorems will therefore be omitted.

**THEOREM 3.1.** *For any u.c.s.* $(X, I)$, $(U(X), J)$ *is u-embedded.*

**THEOREM 3.2.** *(a) A product of u-embedded spaces is u-embedded.*

*(b) A subspace of a u-embedded space is u-embedded.*

From Theorem 3.1 and Example 2.2, we conclude that a $u$-embedded space may fail to be a "uniform convergence space" according to the definition of that term in [2].

Recall the notation $\alpha X$ (introduced in Section 2) for the weak topological space on the set $X$ generated by the set of functions $U(X)$. For notational convenience, let $Y = \alpha X \times \alpha X$. A u.c.s. $(X, I)$ is said to be *α-regular* if $\mathrm{cl}_Y \mathsf{F} \in I$ whenever $\mathsf{F} \in I$. An α-regular u.c.s. is clearly regular. Also, the underlying convergence space of an α-regular u.c.s. has the property: $\mathrm{cl}_{\alpha X} \mathsf{F} \to x$ whenever $\mathsf{F} \to x$.

**THEOREM 3.3.** *A u-embedded space* $(X, I)$ *is Hausdorff and α-regular.*

Proof. The map $i : X \to U^2(X)$ is injective if and only if $U(X)$ separates points of $X$. Thus a $u$-embedded space $X$ has the property that $\alpha X$ is Hausdorff, and hence $X$ is also Hausdorff.

To show that $X$ is α-regular, let $\mathsf{F} \in I$. Since $(X, I)$ is $u$-embedded, it is sufficient to show that $\Phi(\mathrm{cl}_Y \mathsf{F}) \geq \mathsf{U}$ for all $\Phi \in J$. Given $\Phi \in J$, let $U$ be a closed entourage in $\mathsf{U}$, and choose $A \in \Phi$ and $F \in \mathsf{F}$ such that $A(F) \subset U$. If $(x, y) \in \mathrm{cl}_Y F$, then there is a filter $\mathsf{G} \to (x, y)$ in $Y$ such that $F \in \mathsf{G}$. For any pair $(f, g) \in A$, $(f \times g)\mathsf{G} \to (f(x), g(y))$ in $R$, and $(f \times g)\mathsf{G}$ contains $U$. But $U$ is closed, and so $A(\mathrm{cl}_Y F) \subset U$. This establishes that $\Phi(\mathrm{cl}_Y \mathsf{F}) \geq \mathsf{U}$.

A u.c.s. $(X, I)$ will be called a *pseudo-uniformity* if $\mathsf{F} \in I$ whenever $\mathsf{G} \in I$ for each ultrafilter $\mathsf{G} \geq \mathsf{F}$.

**THEOREM 3.4.** *A u-embedded space* $(X, I)$ *is a pseudo-uniformity.*

Proof. Let $\mathsf{F}$ be a filter on $X \times X$ such that $\mathsf{G} \in I$ for each ultrafilter $\mathsf{G} \geq \mathsf{F}$. Let $\Phi$ be an arbitrary member of $J$. Since $X$ is

$u$-embedded, it is sufficient to show that $\Phi(\mathsf{F}) \geq \mathsf{U}$ . We will do this by showing that if $\mathsf{A}$ is an ultrafilter on $R$ such that $\mathsf{A} \geq \Phi(\mathsf{F})$ , then there is an ultrafilter $\mathsf{G} \geq \mathsf{F}$ such that $\mathsf{A} \geq \Phi(\mathsf{G})$ .

Let $\mathsf{Z} = \{\mathsf{H} : \mathsf{H}$ a filter on $X \times X$ , $\mathsf{A} \geq \Phi(\mathsf{H}), \mathsf{H} \geq \mathsf{F}\}$ . $\mathsf{Z}$ contains $\mathsf{F}$ , and so is non-empty. By Zorn's Lemma, $\mathsf{Z}$ contains a maximal element $\mathsf{G}$ . One can show by a straightforward argument that $\mathsf{G}$ is an ultrafilter, which completes the proof.

Let $(X, \mathcal{I})$ and $(X_1, \mathcal{I}_1)$ be arbitrary u.c.s.'s, and $\phi : (X, \mathcal{I}) \to (X_1, \mathcal{I}_1)$ a uniformly continuous function. The function $\phi_1 : \mathcal{U}(X_1) \to \mathcal{U}(X)$ , defined by $\phi_1(f) = f \circ \phi$ , will be called the *transpose map of* $\phi$ . It is easy to show that $\phi_1$ us uniformly continuous whenever $\phi$ is. Let $\phi_2 : \mathcal{U}^2(X) \to \mathcal{U}^2(X_1)$ denote the transpose of $\phi_1$ . The following diagram is easily seen to be commutative;

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X_1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_1} \\ \mathcal{U}^2(X) & \xrightarrow{\ \phi_2\ } & \mathcal{U}^2(X_1) \end{array} .$$

We now obtain a completion for an arbitrary u.c.s. $(X, \mathcal{I})$ . Recall the notation $D = \mathcal{U}^2(X)$ for the dual space. Let $X^* = \mathrm{cl}_{\lambda D} iX$ , and assume that $X^*$ has the u.c.s. $\mathcal{I}^*$ inherited from $(\mathcal{U}^2(X), \mathcal{J}^2)$ .

THEOREM 3.5. *For any u-embedded space* $(X, \mathcal{I})$ *, the space* $(X^*, \mathcal{I}^*)$ *, along with the natural injection* $i$ *, is a Hausdorff,* $\alpha$*-regular, pseudo-uniform completion of* $(X, \mathcal{I})$ . *If* $(X_1, \mathcal{I}_1)$ *is any complete u-embedded space, and* $\phi : (X, \mathcal{I}) \to (X_1, \mathcal{I}_1)$ *is uniformly continuous, then* $\phi$ *has a unique extension* $\phi^* : (X^*, \mathcal{I}^*) \to (X_1, \mathcal{I}_1)$ .

Proof. By assumption, $i$ is an embedding. $(X^*, \mathcal{I}^*)$ is a closed subspace of a complete space, and hence complete. $i(X)$ is dense in $X^*$ in the weak sense mentioned in the introduction (that is, the closure is taken with respect to $\lambda X^*$ rather than $X^*$ itself).

To establish the universal property, first note that, since $(X_1, I_1)$ is complete, $(X_1^*, I_1^*)$ is uniformly isomorphic to $(X_1, I_1)$ under the natural injection $i_1$. Thus the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X_1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_1} \\ X^* & \xrightarrow{\psi_2} & X_1^* \end{array}$$

is commutative, where $\psi_2$ is the restriction of $\phi_2$ to $X^*$, and $i_1$ is a uniform isomorphism. We obtain the desired extension by setting $\phi^* = i_1^{-1} \circ \psi_2$. The uniqueness of the extension is clear.

## 4. Uniform and topological $u$-embedded spaces

It was shown in Example 2.2 that a $u$-embedded space can fail to satisfy the most basic property of a uniform space. In this section we show that all Hausdorff uniform spaces are $u$-embedded, and that all totally bounded $u$-embedded spaces are uniform. We also obtain a characterization of those u.c.s.'s $(X, I)$ for which the underlying convergence structure of $(U(X), \mathcal{J})$ is topological. This leads to an example of a non-topological $u$-embedded space.

**THEOREM 4.1.** *A Hausdorff uniform space is $u$-embedded.*

Proof. Let $(X, \mathcal{W})$ be a Hausdorff uniform space. It is clear that $U(X)$ separates points, and so $i : X \to U^2(X)$ is injective. Since $i$ is always uniformly continuous, it remains to show that $i^{-1}$ is uniformly continuous. Assume the contrary; that is, there is a filter $\mathcal{F}$ on $X \times X$ such that $(i \times i)\mathcal{F} \in \mathcal{J}^2$, but $\mathcal{F} \not\geq \mathcal{W}$. Then among the pseudo-metrics that generate $\mathcal{W}$, there is a pseudo-metric $d$ and a positive real number $\varepsilon$ such that $V_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\} \notin \mathcal{F}$. For each $F \in \mathcal{F}$, choose an element $(x_F, y_F) \in F - V_\varepsilon$. Let $\mathcal{G}$ denote the filter associated with the net $(x_F, y_F)_{F \in \mathcal{F}}$. Then $\mathcal{G} \geq \mathcal{F}$, and so $(i \times i)\mathcal{G} \in \mathcal{J}^2$.

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For each $F \in \mathsf{F}$, define $f_F : X \to R$ as follows: $f_F(z) = d(x_F, z)$ for all $z \in X$. Then $f_F(x_F) = 0$, and $f_F(y_F) \geq \varepsilon$. If $V_\delta = \{(x, y) : d(x, y) < \delta\}$, then

$$(f_F \times f_F) V_\delta \subset \{(r, s) \in R \times R : |r-s| < \delta\} ,$$

and so $f_F \in U(X)$. Moreover, if $\Phi$ denotes the filter associated with the net $(f_F, f_F)_{F \in \mathsf{F}}$, then $\Phi \in J$. Hence $\Phi(\mathsf{G}) \geq \mathsf{U}$, which contradicts $|f_F(x_F) - f_F(y_F)| \geq \varepsilon$, for each $F \in \mathsf{F}$.

Recall that a u.c.s. is *totally bounded* if each ultrafilter is Cauchy.

THEOREM 4.2. *A Hausdorff, totally bounded u.c.s. is u-embedded if and only if it is a uniformity.*

Proof. Let $(X, I)$ be a Hausdorff, totally bounded u.c.s. which is *u*-embedded. Let $\mathsf{G}$ be an ultrafilter finer than $\mathsf{W} = \cap\{\mathsf{F} : \mathsf{F} \in I\}$ and let $\mathsf{G}_1 = (i \times i)\mathsf{G}$. Assume $\mathsf{G} \notin I$; then $\mathsf{G}_1 \notin J^2$.

Let $X' = \mathrm{cl}_D iX$ be a subspace of $U^2(X)$; we will first show that $X'$ is compact. Let $\mathsf{K}$ be an ultrafilter containing $X'$. By Lemma 2.1, [5], there is an ultrafilter $\mathsf{M}$ containing $iX$ such that $\mathsf{K} \geq \mathrm{cl}_D \mathsf{M}$. Since $iX$ is totally bounded and $U^2(X)$ is complete, $\mathsf{M}$ converges to an element of $X'$. But $D$ is regular, and so $\mathsf{K}$ converges to the same element. Thus $X'$ is compact; indeed, it is easy to see that $X' = X^*$, the completion space of Theorem 3.5.

From the results of the preceding paragraph, there are elements $r, s \in X'$ such that $\mathsf{G}_1 \to (r, s)$ in $X' \times X'$. Since $\mathsf{G}_1 \notin J^2$, $r \neq s$. (For otherwise, $P_1\mathsf{G}_1 \times r^\cdot \in J^2$ and $r^\cdot \times P_2\mathsf{G}_1 \in J^2$, which would imply $\mathsf{G}_1 \geq (P_1\mathsf{G}_1 \times r^\cdot) \circ (r^\cdot \times P_2\mathsf{G}_1) \in J^2$.) Since $r \neq s$, there is $g \in U(X')$ such that $g(r) \neq g(s)$; let $f \in U(X)$ be given by $f = g \circ i$. Then, $(f \times f)\mathsf{G} \geq \cap\{(f \times f)\mathsf{F} : \mathsf{F} \in I\} \geq \mathsf{U}$, which contradicts the previous assertion that $g(r) \neq g(s)$. Thus $\mathsf{G} \in I$, and since $I$ is a pseudo-uniformity by Theorem 3.4, it follows that $\mathsf{W} \in I$. Thus $I$ is generated by a single

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filter, and by Theorem 6, [2], $I$ is a uniformity.

Let $(X, I)$ be a u.c.s., $\mathcal{S}$ a cover of $X$. $\mathcal{S}$ will be called a *uniform cover* if for each $\mathcal{F} \in I$, there is $A \in \mathcal{F}$ such that $P_1 A \in \mathcal{S}$. (Recall that $P_1$ is the first projection map.) A subset $A$ of $X$ is said to be *α-bounded* if each uniform cover of $X$ contains a finite subcollection $\{A_i\}$ such that $A \subset \mathrm{cl}_Y(\cup A_i)$.

**THEOREM 4.3.** *Let $(X, I)$ be a u.c.s. Then $J$ induces a topology on $U(X)$ if and only if, for each $\mathcal{F} \in I$, $P_1\mathcal{F}$ contains an α-bounded subset of $X$.*

*Proof.* First, note that $\Phi \to 0$ in $U(X)$ if and only if $\Phi(P_1\mathcal{F}) \to 0$ in $R$ for each $\mathcal{F} \in I$.

Assume that $(U(X), J)$ is topological, and let $\mathcal{F} \in I$. Let $W_0$ be an open neighborhood of 0 in $R$ not containing 1, and let $U_0$ be a neighborhood of 0 in $U(X)$, and $F_0 \in \mathcal{F}$ such that $U_0(P_1F_0) \subset W_0$. Given a uniform cover $\mathcal{S}$ of $X$, let $(B, W) = \{f \in U(X) : f(B) \subset W\}$, where $B \in \mathcal{S}$, and $W$ is an open neighborhood of 0 in $R$. Let $\psi$ be the filter on $U(X)$ generated by sets of the form $(B, W)$ as described above. Then $\psi \to 0$ in $U(X)$, and there exists a set of the form $U_1 = (K, W_1) \subset U_0$, where $K = \cup\{B_i : i = 1, \ldots, n\}$, each $B_i \in \mathcal{S}$, and $W_1$ is an open neighborhood of 0 in $R$. If $x \in P_1F_0 - \mathrm{cl}_Y K$, then by Theorem 2.5 there exists $f \in U(X)$ such that $f(x) = 1$ and $f(\mathrm{cl}_Y K) = \{0\}$. Thus $f \in U_1 \subset U_0$, which contradicts the assertion that $U_0(P_1F_0) \subset W_0$. Therefore, $P_1F_0$ is α-bounded.

Conversely, let $\Phi \to 0$ in $U(X)$, let $\mathcal{F} \in I$, and assume that there is $F_0 \in \mathcal{F}$ such that $P_1F_0$ is α-bounded. Let $W$ be any closed neighborhood of 0 in $R$. For each $\mathcal{G} \in I$, there are sets $G \in \mathcal{G}$ and $A_{\mathcal{G}} \in \Phi$ such that $A_{\mathcal{G}}(P_1G) \subset W$. Let $\mathcal{S}$ denote the collection of all such sets $P_1G$. Then $\mathcal{S}$ is a uniform covering of $X$. Using the fact that $P_1F_0$ is α-bounded, it follows that there is some $A_\Phi \in \Phi$ such that

$A_\Phi(P_1F_0) \subset W$ . Since $\Phi$ is arbitrary, $\cap\{\Phi : \Phi \to 0\}$ also converges to 0 in $U(X)$ . Thus $U(X)$ is a pretopology (or principal convergence space). But $U(X)$ is also a convergence group (it is easy to verify that the group operations on $U(X)$ are continuous). It is shown in [3] (Satz 5, III.3, p. 294) that a pretopological convergence group is a topology, and the proof is complete.

If $(X, I)$ is a u.c.s. and $\Delta^\cdot \in I$ , then each uniform cover of $X$ contains $X$ . Consequently, $X$ is $\alpha$-bounded, and we obtain the following corollary.

COROLLARY 4.4. *Let* $(X, I)$ *be a u.c.s. such that* $\Delta^\cdot \in I$ *(that is, a u.c.s. in the sense of* Cook *and* Fischer, *[2]). Then* $(U(X), J)$ *is topological.*

We conclude with an example which, along with Theorem 3.1, shows that $u$-embedded spaces can be non-topological.

EXAMPLE 4.5. Let $X = R$ , and for each $\delta > 0$ let $F_\delta = \{(x, x) : x < \delta\}$ . Let F be the filter on $X \times X$ generated by $\{F_\delta : \delta > 0\}$ . Let $I$ be the u.c.s. on $X$ generated by $\{F, x^\cdot \times x^\cdot : x \in X\}$ . Since $P_1F$ contains no $\alpha$-bounded subset, Theorem 4.3 implies that $(U(X), J)$ is non-topological.

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