EXPANSION OF CONTINUOUS DIFFERENTIABLE FUNCTIONS IN FOURIER LEGENDRE SERIES

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1. Let

\[ S_n(f, x) = \sum_{k=0}^{n} a_k \bar{P}_k(x) \]

denote the nth partial sum of the Fourier Legendre series of a function \( f(x) \). The references available to us, except (5), prove only that \( S_n(f, x) \) converges uniformly to \( f(x) \) in \([-1, 1]\) if \( f(x) \) has a continuous second derivative on \([-1, 1]\). Very recently Suetin (5) has shown by employing a theorem of A. F. Timan (7) (which is a stronger form of Jackson’s theorem) that \( S_n(f, x) \) converges uniformly to \( f(x) \) if \( f(x) \) belongs to a Lipschitz class of order greater than 1/2 in \([-1, 1]\). More generally he has proved the following theorem.

**Theorem 1** (P. K. Suetin (5)). If \( f(x) \) has \( p \) continuous derivatives on \([-1, 1]\) and \( f^{(p)}(x) \in \text{Lip } \alpha \), then

\[
(1.2) \quad \left| f(x) - \sum_{k=0}^{n} a_k \bar{P}_k(x) \right| \leq \frac{c_1 \log n}{n^{p+\alpha-1/2}}, \quad x \in [-1, 1],
\]

for \( p + \alpha \geq \frac{1}{2} \).

In the course of his proof it is shown (as is mentioned by him), without using the theorem of Timan, that the uniform convergence of \( S_n(f, x) \) to \( f(x) \) holds in \([-1, 1]\) if \( f'(x) \) is continuous in \([-1, 1]\).

In this paper we shall supplement the above theorem by proving the following theorem.

**Theorem 2.** If \( f(x) \) has \( p \) continuous derivatives on \([-1, 1]\) and \( f^{(p)}(x) \in \text{Lip } \alpha \), then together with (1.2) the following inequalities hold:

\[
(1.3) \quad (1 - x^2)^{1/2} |f'(x) - S'_n(x)| \leq c_2 \frac{\log n}{n^{p+\alpha-1}} \quad (0 < \alpha < 1, \ p > 1),
\]

\[
(1.4) \quad (1 - x^2)^{3/2} |f''(x) - S''_n(x)| \leq c_3 \frac{\log n}{n^{p+\alpha-3/2}} \quad (\frac{1}{2} < \alpha < 1, \ p > 1),
\]

and

\[
(1.5) \quad |f'(x) - S'_n(x)| \leq c_4 \frac{\log n}{n^{p+\alpha-b/2}} \quad (\frac{1}{2} < \alpha < 1, \ p > 2)
\]

uniformly in \([-1, 1]\).
2. To prove the above theorem we shall require a number of well-known results on Legendre polynomials.

The orthonormalized Legendre polynomial $\tilde{P}_n(x)$ is given by (2)

$$\tilde{P}_n(x) = \sqrt{\left\{\frac{1}{2}(n + 1)\right\}} P_n(x),$$

where $P_n(x)$ denotes the $n$th Legendre polynomial with the normalization $P_n(1) = 1$.

For the $\tilde{P}_n(x)$ we have the uniform estimations (2, 3, 6)

$$|\tilde{P}_n(x)| \leq c_5 \sqrt{n}, \quad x \in [-1, 1],$$

and the inequality

$$ (1 - x^2)^{1/2} |\tilde{P}_n(x)| \leq c_6, \quad x \in [-1, 1].$$

For the derivatives $\tilde{P}_n'(x)$ we have the following Bernstein inequality:

$$ |\tilde{P}_n'(x)| \leq c_7 n^{3/2},$$

the Stieltjes inequality

$$ |\tilde{P}_n'(x)| \leq c_8 n,$$

and Markov’s inequality

$$ |\tilde{P}_n'(x)| \leq c_9 n^{5/2}$$

for $x \in [-1, 1]$.

3. In order to prove Theorem 2 we need the following two lemmas.

**Lemma 3.1.** For $-1 < x < 1$ we have

$$ (1 - x^2)^{1/2} \int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t) \tilde{P}_k'(x) \right| dt \leq c_{10} n^{3/2},$$

$$ (1 - x^2)^{1/2} \int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t) \tilde{P}_k'(x) \right| dt \leq c_{11} n^2,$$

and

$$ \int_{-1}^{+1} \left| \sum_{k=1}^{n} \tilde{P}_k(t) \tilde{P}_k'(x) \right| dt \leq c_{12} n^3.$$

**Proof.** We give here only the proof for (3.1). In fact we have

$$ (1 - x^2)^{3/2} \int_{-1}^{+1} \left( \sum_{k=1}^{n} \tilde{P}_k(t) \tilde{P}_k'(x) \right)^2 dt = \sum_{k=1}^{n} |(1 - x^2)^{3/2} \tilde{P}_k'(x)|^2$$

which, owing to the inequality (2.5), gives (3.1).
Lemma 3.2. For \(-1 \leq x \leq 1\) we have

\[(3.4) \quad (1 - x^2) \frac{1}{2} \int_{-1}^{1} (1 - t^2)^{\frac{1}{2}} \left| \sum_{k=1}^{n} \bar{P}_k(t) \bar{P}'_k(x) \right| dt \leq c_{13} n \log n,
\]

\[(3.5) \quad (1 - x^2) \frac{1}{2} \int_{-1}^{1} (1 - t^2)^{\frac{1}{2}} \left| \sum_{k=1}^{n} \bar{P}_k(t) \bar{P}'_k(x) \right| dt \leq c_{14} n^{3/2} \log n,
\]

and

\[(3.6) \quad \int_{-1}^{1} (1 - t^2)^{\frac{1}{2}} \left| \sum_{k=1}^{n} \bar{P}_k(t) \bar{P}'_k(x) \right| dt \leq c_{15} n^{5/2} \log n.
\]

Proof. We shall confine ourselves to the proof of (3.4). We denote by \(\Delta_n(x)\) the part of \([-1, 1]\) on which \(|x - t| \leq 1/n\) and by \(\lambda_n(x)\) the rest of the interval. Thus taking account of (2.3) and (2.5) we have

\[(3.7) \quad (1 - x^2)^{\frac{1}{4}} \int_{\Delta_n(x)} (1 - t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \bar{P}_k(t) \bar{P}'_k(x) \right| dt
\]

\[\leq \int_{\Delta_n(x)} \left[ \sum_{k=1}^{n} (1 - t^2)^{\frac{1}{4}} |\bar{P}_k(t)| (1 - x^2)^{\frac{1}{4}} |\bar{P}'_k(x)| \right] dt
\]

\[\leq c_{16} \frac{1}{n} \sum_{k=1}^{n} k \leq c_{17} n, \quad x \in [-1, 1].
\]

To estimate the integral over \(\lambda_n\) we use the Christoffel–Darboux formula (6).

\[(3.8) \quad \sum_{k=0}^{n} \bar{P}_k(t) \bar{P}_k(x) = \theta_n \frac{\bar{P}_{n+1}(x) \bar{P}_n(t) - \bar{P}_n(x) \bar{P}_{n+1}(t)}{x - t}, \quad 0 < \theta_n < 1.
\]

Differentiating the above relation with respect to \(x\) we have

\[(3.9) \quad \sum_{k=0}^{n} \bar{P}_k(t) \bar{P}'_k(x) = \theta_n \frac{\bar{P}'_{n+1}(x) \bar{P}_n(t) - \bar{P}_n(x) \bar{P}'_{n+1}(t)}{x - t}
\]

\[+ \theta_n \frac{\bar{P}_{n+1}(x) \bar{P}_n(t) - \bar{P}_n(x) \bar{P}_{n+1}(t)}{(x - t)^2}.
\]

Then we have

\[(3.10) \quad (1 - x^2)^{\frac{1}{4}} \int_{\lambda_n(x)} (1 - t^2)^{\frac{1}{4}} \left| \sum_{k=1}^{n} \bar{P}_k(t) \bar{P}'_k(x) \right| dt
\]

\[\leq (1 - x^2)^{\frac{1}{4}} \int_{\lambda_n(x)} (1 - t^2)^{\frac{1}{4}} \left| \frac{\bar{P}'_{n+1}(x) \bar{P}_n(t) - \bar{P}_n(x) \bar{P}'_{n+1}(t)}{x - t} \right| dt
\]

\[+ (1 - x^2)^{\frac{1}{4}} \int_{\lambda_n(x)} (1 - t^2)^{\frac{1}{4}} \left| \frac{\bar{P}_{n+1}(x) \bar{P}_n(t) - \bar{P}_n(x) \bar{P}_{n+1}(t)}{(x - t)^2} \right| dt
\]

\[= I_1 + I_2.
\]

Since \(|x - t| > 1/n\) for \(t \in \lambda_n(x)\), we find by using (2.3) and (2.5) that

\[(3.11) \quad I_1 \leq c_{18} n \int_{\lambda_n(x)} (1 - t^2)^{\frac{1}{4}} \left[ |\bar{P}_n(t)| + |\bar{P}_{n+1}(t)| \right] \frac{dt}{|x - t|}
\]

\[\leq c_{19} n \int_{|x - t| \leq 1} \frac{dt}{|x - t|} \leq c_{19} n \log n, \quad x \in [-1, 1].
\]
For $I_2$ we have, on using (2.3),

$$I_2 \leq c_2 \int_{h(t)} (1 - t^2)^{\frac{3}{2}} \left[ |P_n(t)| + |P_{n+1}(t)| \right] \frac{dt}{(x - t)^2} \leq c_{20} n, \quad x \in [-1, 1].$$

Thus (3.7), (3.10), (3.11), and (3.12) complete the proof of (3.1).

4. Let $Q_n(x)$ be an algebraic polynomial of degree not greater than $n$; then we have the following theorem of A. F. Timan (7) on the order of approximation of the function $f(x)$.

**THEOREM 3 (A. F. Timan).** If $f(x)$ has $p$ continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \text{Lip } \alpha$, then there is a sequence of polynomials $\{Q_n(x)\}$ such that

$$|f(x) - Q_n(x)| \leq \frac{c_{21}}{n^{p+\alpha}} \left( \sqrt{1 - x^2} + \frac{1}{n} \right)^{p+\alpha}, \quad x \in [-1, 1].$$

From this theorem, on using the Dzyadyk inequality (1), we have the following lemma.

**LEMMA 4.1.** Let $f^{(r)}(x) \in \text{Lip } \alpha$ ($0 < \alpha < 1, r > 1$) in $[-1, 1]$; then there is a polynomial $p_n(x)$ of degree at most $n$ possessing the following properties:

$$|f(x) - p_n(x)| \leq \frac{c_{22}}{n^{p+\alpha}} \left[ \left( \sqrt{1 - x^2} \right)^{p+\alpha} + \frac{1}{n^{p+\alpha}} \right]$$

and

$$|f'(x) - p'_n(x)| \leq \frac{c_{23}}{n^{p+\alpha-1}} \left[ \left( \sqrt{1 - x^2} \right)^{p+\alpha-1} + \frac{1}{n^{p+\alpha-1}} \right]$$

uniformly in $[-1, 1]$.

The author has proved this lemma for $r = 1$ in (4). For general $r$ the lemma can be proved in the same manner.

We now complete the proof of Theorem 2. We shall confine ourselves to proving (1.3).

We write

$$|f'(x) - S'_n(x)| = |f'(x) - p'_n(x) + p'_n(x) - S'_n(x)|$$

$$\leq |f'(x) - p'_n(x)| + \left| \int_{-1}^{+1} p_n(t) - f(t) \right| \left| \sum_{k=1}^{n} P_k(t) P_k'(x) \right| dt.$$

Now using Lemma 4.1 we have

$$|f'(x) - S'_n(x)| \leq \frac{c_{22}}{n^{p+\alpha-1}} \left[ \left( \sqrt{1 - x^2} \right)^{p+\alpha-1} + \frac{1}{n^{p+\alpha-1}} \right]$$

$$+ \frac{c_{23}}{n^{p+\alpha}} \int_{-1}^{+1} \left( 1 - t^2 \right)^{p+\alpha/2} + \frac{1}{n^{p+\alpha}} \left| \sum_{k=1}^{n} P_k(t) P_k'(x) \right| dt.
so that

\[
(1 - x^2)^{3/2} |f'(x) - S'_n(x)|
\leq \frac{c_{24}}{n^{p+\alpha-1}} + \frac{c_{22}}{n^{p+\alpha}} \left(1 - x^2\right)^{3/2} \int_{-1}^{+1} \left(1 - t^2\right)^{p+\alpha/2} \left|\sum_{k=1}^{n} \hat{P}_k(t) \hat{P}'_k(x)\right| dt
\]

\[
+ \frac{c_{22}}{n^{p+2\alpha}} \left(1 - x^2\right)^{3/2} \int_{-1}^{+1} \left|\sum_{k=1}^{n} \hat{P}_k(t) \hat{P}'_k(x)\right| dt
\]

which, by the help of (3.4) and (3.1), gives

\[
(1 - x^2)^{3/2} |f'(x) - S'_n(x)| \leq \frac{c_{24}}{n^{p+\alpha-1}} + \frac{c_{22}}{n^{p+\alpha}} c_{13} n \log n + \frac{c_{22}}{n^{p+2\alpha}} c_{10} n^{3/2}
\]

\[
\leq c_{25} \log n, \quad p \geq 1.
\]

This completes the proof of (1.3). The proof of (1.4) and (1.5) can be obtained similarly.

References


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