CERTAIN FUNDAMENTAL CONGRUENCES ON A REGULAR SEMIGROUP†

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In recent developments in the algebraic theory of semigroups attention has been focussing increasingly on the study of congruences, in particular on lattice-theoretic properties of the lattice of congruences. In most cases it has been found advantageous to impose some restriction on the type of semigroup considered, such as regularity, commutativity, or the property of being an inverse semigroup, and one of the principal tools has been the consideration of special congruences. For example, the minimum group congruence on an inverse semigroup has been studied by Vagner [**21**] and Munn [**13**], the maximum idempotent-separating congruence on a regular or inverse semigroup by the authors separately [**9, 10**] and by Munn [**14**], and the minimum semilattice congruence on a general or commutative semigroup by Tamura and Kimura [**19**], Yamada [**22**], Clifford [**3**] and Petrich [**15**]. In this paper we study regular semigroups and our primary concern is with the minimum group congruence, the minimum band congruence and the minimum semilattice congruence, which we shall consistently denote by σ, β and η respectively.

In § 1 we establish connections between β and η on the one hand and the equivalence relations of Green [**7**] (see also Clifford and Preston [**4**, § 2.1]) on the other. If for any relation $\mathfrak{R}$ on a semigroup S we denote by $\mathfrak{R}^*$ the congruence on S generated by $\mathfrak{R}$, then, in the usual notation,

$$\mathscr{H}^* \subseteq \beta \subseteq \mathscr{R}^* \cap \mathscr{L}^*, \quad \eta = \mathscr{D}^* = \mathscr{J}^*.$$

In § 2 we show that the intersection of σ with β is the smallest congruence ρ on S for which S/ρ is a UBG-semigroup, that is, a band of groups [**4**, p. 26] in which the idempotents form a unitary subsemigroup. The structure of such semigroups (and indeed of semigroups more general than this) has been investigated by Fantham [**6**]; his theorem (or rather the special case that is of interest here) is described below. A corollary of our result is that $\sigma \cap \eta$ is the smallest congruence ρ for which S/ρ is a USG-*semigroup*, that is, a semilattice of groups with a unitary subsemigroup of idempotents.

These results lead naturally to a study of RU-*semigroups* (regular semigroups whose idempotents form a unitary subsemigroup), ISBG-*semigroups* (bands of groups whose idempotents form a subsemigroup) and SG-*semigroups* (semilattices of groups), and to the consideration of the minimum RU-congruence κ, the minimum ISBG-congruence ζ and the minimum SG-congruence ξ on a regular semigroup. The principal results of § 3 are that $\sigma \cap \beta = \zeta \vee \kappa$ and $\sigma \cap \eta = \xi \vee \kappa$.

In § 4 we show that any UBG-congruence on a regular semigroup can be expressed in a unique way as $\tau \cap \gamma$, where τ is a group congruence and γ is a band congruence. A similar result holds for USG-congruences.

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1. Preliminaries; the connection with Green's relations. The notation of Clifford and Preston [4] will be used throughout. In particular we denote Green's relations by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$. If a is an element of a regular semigroup S, then a' will denote any inverse of a, that is, any element of S such that

$$aa'a = a, \quad a'aa' = a'.$$

We shall consistently denote the set of idempotents of a semigroup S by E.

For standard definitions and notations regarding congruences the reader is referred to § 1.5 of [4]. It will be convenient here to regard a binary relation ρ on a semigroup S as a subset of $S \times S$ and to write $(x, y) \in \rho$ rather than $x\rho y$. Thus the notations $\rho \subseteq \rho'$ and $\rho \cap \rho'$ have the usual set-theoretic meanings. We shall denote the identical congruence $\{(x, x) : x \in S\}$ on S by ι_S.

If ρ and ρ' are congruences on S such that $\rho \subseteq \rho'$, then the relation ρ'/ρ on S/ρ defined by

$$\rho'/\rho = \{(x\rho, y\rho) : (x, y) \in \rho'\}$$

is a congruence on S/ρ, and is the identical congruence on S/ρ if and only if $\rho' = \rho$. Moreover,

$$S/\rho' \simeq (S/\rho)/(\rho'/\rho).$$

If γ and δ are congruences on S containing ρ, then $\gamma \cap \delta$ is again a congruence containing ρ, and

$$(\gamma \cap \delta)/\rho = (\gamma/\rho) \cap (\delta/\rho).$$

By the *minimum group congruence* on a semigroup S we mean the smallest congruence ρ on S for which S/ρ is a group. Of course not every semigroup has a minimum group congruence [4, Ex. 6, p. 21], but, if S is regular, then (since the property of regularity is clearly preserved by homomorphisms) a homomorphic image of S is a cancellative semigroup if and only if it is a group. Since it is known [4, p. 18] that there is a minimum cancellative congruence on any semigroup, it follows that there exists on a regular semigroup a minimum group congruence σ. On any semigroup, and so certainly on any regular semigroup, it is known [4, p. 18] that there exists a minimum band congruence β and a minimum semilattice congruence η.

The following two lemmas will be of use later.

Lemma 1.1. *If e and f are idempotents in a regular semigroup S, then ef has an idempotent inverse g in S such that $ge = fg = g$.*

Proof. Certainly ef has an inverse x in S, since S is regular:

$$efxef = ef, \quad xefx = x.$$

Then $(fxe)^2 = f(xefx)e = fxe$, and

$$ef \cdot fxe \cdot ef = efxef = ef,$$
$$fxe \cdot ef \cdot fxe = f(xefx)e = fxe.$$

Thus fxe is the required idempotent inverse.

The other lemma is effectively Lemma 2.2 in [**10**] and is quoted here for convenience.

LEMMA 1.2. *If ρ is a congruence on a regular semigroup S, then every idempotent of S/ρ is a ρ-class of S containing an idempotent.*

We now state the first of the two chief results of this section.

THEOREM 1.3. *If β is the minimum band congruence on a regular semigroup S, then*

$$\mathscr{H}^* \subseteq \beta \subseteq \mathscr{R}^* \cap \mathscr{L}^*.$$

Proof. It is clear from the definition of $\mathscr{H}$ that, if two elements a and b in S are equivalent mod $\mathscr{H}$ in S, then $a\beta$ and $b\beta$ are equivalent mod $\mathscr{H}$ in S/β. Since an $\mathscr{H}$-class cannot contain more than one idempotent [**4**, Lemma 2.15], we must have that $a\beta = b\beta$. Thus $\mathscr{H} \subseteq \beta$. Since β is thus a congruence on S containing $\mathscr{H}$, it must contain the smallest congruence containing $\mathscr{H}$; that is, $\beta \supseteq \mathscr{H}^*$. Notice that this part of the argument does not use any assumption of regularity.

On the other hand, for any a in S and any inverse a' of a, we have that $(a, aa') \in \mathscr{R}$ by [**4**, Lemma 2.12]. It follows that $(a^2, aa'a) \in \mathscr{R}^*$, i.e. that $(a^2, a) \in \mathscr{R}^*$. Thus $S/\mathscr{R}^*$ is a band and so $\beta \subseteq \mathscr{R}^*$. A similar argument establishes that $\beta \subseteq \mathscr{L}^*$ and so the proof is complete.

It is worth remarking that the inclusions in the above theorem can be strict. First, if we consider the bicyclic semigroup B [**4**, pp. 43–45], we find that $\mathscr{H} = \mathscr{H}^* = \iota_B$, while

$$\beta = \eta = \mathscr{R}^* = \mathscr{L}^* = \mathscr{D}^* = \mathscr{J}^* = B \times B.$$

On the other hand, in the free band F on three generators x_1, x_2, x_3 considered by Green and Rees [**8**], we find that, for two elements a and b of F (each being a word in the " letters " x_1, x_2, x_3 in which no letter or sequence of letters ever occurs twice in succession): (i) $(a, b) \in \mathscr{R}$ if and only if a and b have the same letters, the same *initial* and the same *initial mark*, in the sense of [**8**]; (ii) $(a, b) \in \mathscr{R}^*$ if and only if a and b have the same letters and the same order of first appearance of letters. Dual statements apply to $\mathscr{L}$ and $\mathscr{L}^*$. Hence

$$(x_1x_2x_1x_3x_2x_1, x_1x_2x_3x_2x_1) \in \mathscr{R}^* \cap \mathscr{L}^*$$

and so, since clearly $\beta = \iota_F$, we have $\beta \subset \mathscr{R}^* \cap \mathscr{L}^*$.

THEOREM 1.4. *If η is the minimum semilattice congruence on a regular semigroup S, then $\eta = \mathscr{D}^* = \mathscr{J}^*$.*

Proof. It is clear from the definition of $\mathscr{J}$ that if two elements a and b are equivalent mod $\mathscr{J}$ in S, then $a\eta$ and $b\eta$ are equivalent mod $\mathscr{J}$ in S/η. By commutativity in S/η we have that $\mathscr{J} = \mathscr{H} = \iota_{S/\eta}$; thus $a\eta = b\eta$ and so $\mathscr{J} \subseteq \eta$. Since η is a congruence it follows that $\mathscr{J}^* \subseteq \eta$. Again notice that we have used no assumption of regularity in this part of the argument.

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On the other hand, since $S/\mathscr{R}^*$ is a band and since $\mathscr{R}^* \subseteq \mathscr{D}^*$, we have that $S/\mathscr{D}^*$ is a band. To show that $\eta \subseteq \mathscr{D}^*$ we must show that $S/\mathscr{D}^*$ is commutative. As a first step we have

LEMMA 1.5. *If e and f are idempotents of a regular semigroup S, then $(ef, fe) \in \mathscr{D}^*$.*

Proof. By Lemma 1.1 we know that ef has an idempotent inverse g such that $ge = fg = g$. By [**4**, Lemma 2.12] we can deduce that

$$(ef, g) \in \mathscr{D} \subseteq \mathscr{D}^*. \tag{1}$$

Since $\mathscr{D}^*$ is a congruence, it follows that $(f \cdot ef \cdot e, fge) \in \mathscr{D}^*$, i.e. that $((fe)^2, g) \in \mathscr{D}^*$. It follows by the transitivity of $\mathscr{D}^*$ that $(fe, g) \in \mathscr{D}^*$, and this together with (1) gives us the result of the lemma.

Now let a and b be arbitrary elements of S. By the lemma just proved we have that $(aa'bb', bb'aa') \in \mathscr{D}$. But (a, aa') and (b, bb') are both in $\mathscr{D}$ by [**4**, Lemma 2.12] and so both $(ab, aa'bb')$ and $(ba, bb'aa')$ are in $\mathscr{D}^*$, since $\mathscr{D}^*$ is a congruence. It follows that $(ab, ba) \in \mathscr{D}^*$. Thus $S/\mathscr{D}^*$ is a semilattice and so $\eta \subseteq \mathscr{D}^*$.

Now, it is known [**7**] (see also [**4**, p. 48]) that $\mathscr{D} \subseteq \mathscr{J}$. Hence certainly $\mathscr{D}^* \subseteq \mathscr{J}^*$ and so we have

$$\mathscr{J}^* \subseteq \eta \subseteq \mathscr{D}^* \subseteq \mathscr{J}^*,$$

from which it follows that $\eta = \mathscr{D}^* = \mathscr{J}^*$ as required. This completes the proof of Theorem 1.4.

If S is an *inverse semigroup* (a regular semigroup in which idempotents commute), then by a result due to Vagner [**20**] and Preston [**16**] any homomorphic image of S is again an inverse semigroup. In particular, S/β is both a band and an inverse semigroup; that is, S/β is a semilattice. It follows that $\beta = \eta$ and so, using Theorems 1.3 and 1.4, we obtain

THEOREM 1.6. *If η is the minimum semilattice congruence on an inverse semigroup, then*

$$\mathscr{H}^* \subseteq \eta = \mathscr{R}^* = \mathscr{L}^* = \mathscr{D}^* = \mathscr{J}^*.$$

A congruence ρ on a semigroup S is called *idempotent-separating* if every ρ-class contains at most one idempotent. It is known [**10**] that there is a maximum such congruence on any regular semigroup and that this congruence is contained in $\mathscr{H}$. Hence we have the following corollary to Theorem 1.3:

COROLLARY 1.7. *If S is a regular semigroup, and if β and μ are respectively the minimum band congruence and the maximum idempotent-separating congruence on S, then $\mu \subseteq \beta$.*

2. The intersection of σ with β and η. It is clear that, if S is a regular semigroup, then $\sigma \vee \beta$, the smallest congruence containing both σ and β, is the universal congruence $S \times S$, since $S/(\sigma \vee \beta)$ is both a group and a band. Since $\eta \supseteq \beta$ we must also have that $\sigma \vee \eta = S \times S$. The nature of $\sigma \cap \beta$ and $\sigma \cap \eta$ is not quite so easy to determine.

Some preliminaries are needed. By a result of McLean [**11**], which is a special case of a theorem of Clifford [**1**] (see also [**4**, Ex. 1, p. 129]), any band B is a semilattice Y of rectangular bands E_α. The semilattice Y is isomorphic to B/η, the maximum semilattice homomorphic image of B, and the rectangular bands E_α are the η-classes in B. A semigroup which is the direct product of a rectangular band E and a group G will be called a *rectangular group*; such a semigroup is regular (indeed completely simple) and its idempotents form a subsemigroup.

By a theorem of Fantham [**6**], a band of groups in which the idempotents form a subsemigroup (what we are calling an ISBG-semigroup) can be described in terms of a band B with maximum semilattice homomorphic image $B/\eta = Y$ and η-classes E_α ($\alpha \in Y$), a collection of groups G_α indexed by Y, and a system of homomorphisms $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ for all α, β in Y such that $\alpha \geqq \beta$, satisfying the condition that $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \geqq \beta \geqq \gamma$. (The homomorphism $\phi_{\alpha,\alpha}$ is the identical automorphism of G_α for each α in Y.) The semigroup is then the disjoint union of the rectangular groups $E_\alpha \times G_\alpha$, the product in the semigroup of (e_α, a_α) and (f_β, b_β) being $(e_\alpha f_\beta, (a_\alpha\phi_{\alpha,\gamma})(b_\beta\phi_{\beta,\gamma}))$, where γ is the product $\alpha\beta$ of α and β in Y. The product $e_\alpha f_\beta$ is evaluated in the band B, while the product $(a_\alpha\phi_{\alpha,\gamma})(b_\beta\phi_{\beta,\gamma})$ is evaluated in the group G_γ.

If the idempotents of an ISBG-semigroup form a *commutative* subsemigroup, Fantham's structure theorem specializes to the structure theorem for SG-semigroups discovered earlier by Clifford [**1**] (see also [**4**, Theorem 4.11]).

Following Dubreil [**5**], we call a subset U of a semigroup S *left unitary* if $s \in U$ whenever $us \in U$ for some u in U. A *right unitary* subset is defined dually. A *unitary* subset is one which is both left and right unitary. The following lemma is useful.

LEMMA 2.1. *A regular semigroup S is an* RU-*semigroup if and only if E is a left unitary subset of S.*

Proof. We must show that, if E is a left unitary subset of S, then it is also right unitary and a subsemigroup. If $e \in E$ and $xe = f \in E$, then $(efx)^2 = efxefx = ef^3x = efx$ and so $efx \in E$. Applying the left unitary property of E twice, we conclude that $x \in E$. Thus E is right unitary.

To show that E is a subsemigroup of S, consider two elements e, f of E and let y be an inverse of ef. Then $(ef)y \in E$ and so by the left unitary property of E we obtain first that $fy \in E$ and then that $y \in E$. Hence by the right unitary property of E we deduce that $ef \in E$ as required.

An important property of RU-semigroups is contained in the next lemma:

LEMMA 2.2. *If a and b are elements of an* RU-*semigroup S, then $ab \in E$ if and only if $ba \in E$.*

Proof. If $ab \in E$, then, for any inverse b' of b,

$$(babb')^2 = ba(bb'b)abb' = b(ab)^2b' = babb'$$

and so $babb' \in E$, from which it follows that $ba \in E$, since E is right unitary. The other half of the lemma follows by symmetry.

The next theorem identifies those ISBG-semigroups that are also RU-semigroups:

THEOREM 2.3. *An* ISBG-*semigroup is a* UBG-*semigroup if and only if all its structure homomorphisms are one-to-one.*

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Proof. Consider the ISBG-semigroup $S = \bigcup\{E_\alpha \times G_\alpha : \alpha \in Y\}$, with structure homomorphisms $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ $(\alpha \geqq \beta)$. For each α in Y let 1_α be the identity element of G_α. The subsemigroup of idempotents of S is

$$E = \{(e_\alpha, 1_\alpha) : e_\alpha \in E_\alpha, \alpha \in Y\}.$$

Suppose first that E is unitary. If $a_\alpha \phi_{\alpha,\beta} = 1_\beta$, where $a_\alpha \in G_\alpha$ and $\alpha \geqq \beta$, then

$$(e_\beta, 1_\beta)(f_\alpha, a_\alpha) = (e_\beta f_\alpha, 1_\beta(a_\alpha \phi_{\alpha,\beta})) = (e_\beta f_\alpha, 1_\beta) \in E.$$

By the unitary property of E we conclude that $(f_\alpha, a_\alpha) \in E$. Thus $a_\alpha = 1_\alpha$ and so $\phi_{\alpha,\beta}$ is one-to-one.

Conversely, suppose that each structure homomorphism is one-to-one. Then if

$$(e_\beta, 1_\beta)(f_\alpha, a_\alpha) \in E$$

we must have that $a_\alpha \phi_{\alpha,\gamma} = 1_\gamma$, where $\gamma = \alpha\beta$, from which it follows that $a_\alpha = 1_\alpha$. Thus E is left unitary, which by Lemma 2.1 is all we need.

As an obvious corollary we have

COROLLARY 2.4. *An* SG-*semigroup is a* USG-*semigroup if and only if all its structure homomorphisms are one-to-one.*

The next theorem and its corollary give useful information about $\sigma \cap \beta$.

THEOREM 2.5. *The intersection $\sigma \cap \beta$ of the minimum group congruence σ and the minimum band congruence β on a regular semigroup S is equal to ι_S if and only if S is a* UBG-*semigroup.*

Proof. The first part of the argument will be useful again and accordingly is stated as a lemma.

LEMMA 2.6. *If S is an* RU-*semigroup, then E is a σ-class.*

Proof. Certainly E satisfies the condition

$$xEy \cap E \neq \emptyset \Rightarrow xEy \subseteq E$$

of Marianne Teissier [20]; for if $xey \in E$ for some e in E, we can successively deduce, for any f in E, that

$$yxe \in E \text{ (by Lemma 2.2)},$$

$$yx \in E \quad \text{(since } E \text{ is unitary)},$$

$$yxf \in E \text{ (since } E \text{ is a subsemigroup)},$$

and

$$xfy \in E \text{ (by Lemma 2.2)}.$$

Hence, by Teissier's result, there is a congruence ρ on S such that E is a ρ-class. By Lemma 1.2 every idempotent of S/ρ is a ρ-class in S containing an idempotent. Hence S/ρ contains only one idempotent and so (being regular) is a group. If σ is the *minimum* group congruence

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on S, then clearly the identity element of S/σ is a σ-class *containing* E. But $\sigma \subseteq \rho$ and so we conclude that E is a σ-class.

Returning now to the proof of Theorem 2.5, we see that, if S is a UBG-semigroup, then E is a σ-class.

Also, in any semigroup that is a union of groups it is clear that the $\mathscr{H}$-classes are precisely the maximal subgroups. In a band of groups we can also say that $\mathscr{H}$ is a congruence and that $\mathscr{H} = \beta$. Thus in the UBG-semigroup S under consideration two β-equivalent elements are necessarily of the form (e_α, a_α), (e_α, b_α). If the elements are also σ-equivalent, we can deduce that

$$((e_\alpha, a_\alpha b_\alpha^{-1}), (e_\alpha, 1_\alpha)) \in \sigma,$$

and by the lemma it follows that $(e_\alpha, a_\alpha b_\alpha^{-1}) \in E$, i.e., that $a_\alpha = b_\alpha$. We have thus shown that $\sigma \cap \beta = \iota_S$.

Conversely, suppose that S is regular and that $\sigma \cap \beta = \iota_S$. Then, if e and f are idempotents and if $(e, f) \in \beta$, we have that $(e, f) \in \sigma \cap \beta = \iota_S$ and so $e = f$. Thus β is idempotent-separating and so, by [**10**, Theorem 2.3], is contained in $\mathscr{H}$. By Theorem 1.3 it follows that $\beta = \mathscr{H} = \mathscr{H}^*$. Now each $\mathscr{H}$-class is an idempotent element in S/β and so by Lemma 1.2 each $\mathscr{H}$-class contains an idempotent of S. Hence by Green's theorem [**7**] (see also [**4**, Theorem 2.16]) each $\mathscr{H}$-class is a group. Thus S is a union of groups and, since $\mathscr{H}$ is a congruence, is even a band of groups.

It remains to show that E is a unitary subsemigroup. In fact we show—and this is clearly sufficient—that E is a σ-class, which is then necessarily the identity element of S/σ. Certainly the identity element of S/σ is a subset E' of S containing E. If $x \in E'$, then x is in some maximal subgroup H_e (where $e \in E$) and so $(x, e) \in \sigma \cap \mathscr{H} = \sigma \cap \beta = \iota_S$; that is, $x = e$. Hence $E' = E$ and so the proof of Theorem 2.5. is complete.

COROLLARY 2.7. *If S is a regular semigroup, then $\sigma \cap \beta$ is the minimum* UBG-*congruence on S.*

Proof. The minimum group and band congruences on $S/(\sigma \cap \beta)$ are respectively $\sigma/(\sigma \cap \beta)$ and $\beta/(\sigma \cap \beta)$. Their intersection is $(\sigma/(\sigma \cap \beta)) \cap (\beta/(\sigma \cap \beta)) = (\sigma \cap \beta)/(\sigma \cap \beta) = \iota_{S/(\sigma \cap \beta)}$ and so by the theorem $S/(\sigma \cap \beta)$ is a UBG-semigroup.

Now let ρ be a congruence on S such that S/ρ is a UBG-semigroup. If τ' is the minimum group congruence on S/ρ, then $\tau' = \tau/\rho$, where τ is a congruence on S containing ρ. Moreover τ is a group congruence on S, since $S/\tau \simeq (S/\rho)/\tau'$. Similarly γ', the minimum band congruence on S/ρ, is equal to γ/ρ, where γ is a band congruence on S containing ρ. Since S/ρ is a UBG-semigroup, we have that $\tau' \cap \gamma' = \iota_{S/\rho}$; that is,

$$\iota_{S/\rho} = (\tau/\rho) \cap (\gamma/\rho) = (\tau \cap \gamma)/\rho,$$

and so $\tau \cap \gamma = \rho$. But $\tau \supseteq \sigma$ and $\gamma \supseteq \beta$ and hence $\rho = \tau \cap \gamma \supseteq \sigma \cap \beta$. Thus $\sigma \cap \beta$ is the minimum UBG-congruence on S.

Corresponding results concerning η are now obtained fairly easily.

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THEOREM 2.8. *The intersection $\sigma \cap \eta$ of the minimum group congruence σ and the minimum semilattice congruence η on a regular semigroup S is equal to the identical congruence ι_S if and only if S is a* USG-*semigroup.*

Proof. First, if S is a USG-semigroup, then it is a UBG-semigroup in which $\beta = \eta$ and so, by Theorem 2.5, $\sigma \cap \eta = \sigma \cap \beta = \iota_S$.

Conversely, if $\sigma \cap \eta = \iota_S$, then certainly $\sigma \cap \beta = \iota_S$ and so S is a UBG-semigroup. If e and f are idempotents of S, then ef and fe are both idempotent, since the idempotents of S form a (unitary) subsemigroup. Hence $(ef, fe) \in \sigma$. We also have that $(ef, fe) \in \eta$, since S/η is commutative; hence $(ef, fe) \in \sigma \cap \eta = \iota_S$, that is, $ef = fe$. Thus S is a USG-semigroup.

An argument closely similar to that employed in the proof of Corollary 2.7 gives us

COROLLARY 2.9. *If S is a regular semigroup, then $\sigma \cap \eta$ is the minimum* USG-*congruence on S.*

3. The congruences κ, ζ and ξ. We must first show that every regular semigroup has a minimum RU-congruence κ, a minimum ISBG-congruence ζ and a minimum SG-congruence ξ. The first and third of these are consequences of the following theorem:

THEOREM 3.1. *Let S be a regular semigroup. Then*

(i) *S is an* RU-*semigroup if and only if, for all x, y in S,* $[x^2 = x, (xy)^2 = xy] \Rightarrow [y^2 = y]$;

(ii) *S is an* SG-*semigroup if and only if, for all x, y in S,* $[x^2 = x] \Rightarrow [xy = yx]$.

Part (i) is simply a restatement of Lemma 2.1. Part (ii) is due to Clifford [**1**] (see also [**4**, § 4.2]).

It follows that a congruence ρ on a regular semigroup S is an RU-congruence if and only if

$$[(x^2, x) \in \rho, ((xy)^2, xy) \in \rho] \Rightarrow [(y^2, y) \in \rho].$$

A routine argument now shows that the intersection of a non-empty family $\{\rho_i : i \in I\}$ of RU-congruences is again an RU-congruence. Since the universal congruence $S \times S$ is an RU-congruence, it follows that there exists a minimum RU-congruence κ on S. An exactly parallel argument establishes the existence of ξ, the minimum SG-congruence on S.

We have been unable to find a characterisation of ISBG-semigroups comparable to the characterisations of RU-semigroups and SG-semigroups in Theorem 3.1, and so we require a more complicated argument to establish the existence of a minimum ISBG-congruence on an arbitrary regular semigroup. First, let us call a regular semigroup an RIS-*semigroup* if its idempotents form a subsemigroup. Then a regular semigroup S is an RIS-semigroup if and only if, for all x, y in S,

$$[x^2 = x, y^2 = y] \Rightarrow [(xy)^2 = xy].$$

Hence by our previous argument we can deduce the existence of a minimum RIS-congruence λ on S.

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Next, let us call a regular semigroup a BG-*semigroup* if it is a band of groups. We shall show that there exists a minimum BG-congruence on any regular semigroup S. Let β be the minimum band congruence on S and let $\alpha = \beta \cap (E \times E)$, where E is the set of idempotents of S. Let $\pi = \alpha^*$, the congruence on S generated by α. Then π is the minimum BG-congruence on S, as we shall show.

We first establish that S/π is a band of groups. To do this it will be sufficient to show that in S/π the minimum band congruence coincides with $\mathscr{H}$; for we can then deduce by Lemma 1.2 that every $\mathscr{H}$-class contains an idempotent (and so is a group), and that $\mathscr{H}$ is a congruence. Notice first that β/π (which exists since $\pi \subseteq \beta$) is the minimum band congruence on S/π. Next, consider two idempotents $e\pi$, $f\pi$ in S/π; by Lemma 1.2 we can assume that $e, f \in E$. If $(e\pi, f\pi) \in \beta/\pi$, then $(e,f) \in \beta$ (by definition of β/π). Thus $(e,f) \in \beta \cap (E \times E) = \alpha \subseteq \pi$ and so $e\pi = f\pi$. It follows that β/π is an idempotent-separating congruence on S/π and so, by Theorem 1.3 and [**10**, Theorem 2.3],

$$\beta/\pi \subseteq \mathscr{H} \subseteq \mathscr{H}^* \subseteq \beta/\pi,$$

which gives us the result we require.

Suppose now that π' is a BG-congruence on S; we must show that $\pi \subseteq \pi'$. For this it will be sufficient to show that $\alpha \subseteq \pi'$. If $(a, b) \in \alpha$, then in particular a and b are idempotents of S and so $a\pi'$ and $b\pi'$ are idempotents of S/π'. If β' is the minimum band congruence on S/π', then $\beta' = \gamma/\pi'$, where γ is a band congruence on S containing π'. Certainly $\gamma \supseteq \beta$, the minimum band congruence on S. Now $(a, b) \in \alpha \subseteq \beta \subseteq \gamma$ and so $(a\pi', b\pi') \in \gamma/\pi' = \beta'$. But S/π' is a band of groups by hypothesis and so β' coincides with $\mathscr{H}$. We thus have that $(a\pi', b\pi') \in \mathscr{H}$ and so, since $a\pi'$ and $b\pi'$ are idempotents, it follows by [**4**, Lemma 2.15] that $a\pi' = b\pi'$. Thus $(a, b) \in \pi'$ and so $\pi \subseteq \pi'$ as required.

It follows fairly easily from Lemma 1.2 that the property of being an RIS-semigroup is inherited by homomorphic images. The corresponding fact for BG-semigroups seems to follow most easily from the characterisation of such semigroups due to Clifford [**2**], who showed that a semigroup S is a BG-semigroup if and only if (1) $a \in Sa^2 \cap a^2S$ for every a in S; (2) $Sba = Sba^2$, $abS = a^2bS$ for every a, b in S. It is clear that any homomorphic image of S also has these two properties. If S is an arbitrary regular semigroup we can therefore conclude that $S/(\lambda \vee \pi)$ is both an RIS-semigroup and a BG-semigroup: that is, $S/(\lambda \vee \pi)$ is an ISBG-semigroup. Also, if ρ is any congruence on S for which S/ρ is an ISBG-semigroup, then clearly $\rho \supseteq \lambda$ and $\rho \supseteq \pi$, and so $\rho \supseteq \lambda \vee \pi$. Thus $\lambda \vee \pi$ is the minimum ISBG-congruence on S. For convenience we shall write ζ instead of $\lambda \vee \pi$.

We have already seen that the property of being an RIS-semigroup and the property of being a BG-semigroup are inherited by homomorphic images. It follows immediately that any homomorphic image of an ISBG-semigroup is again an ISBG-semigroup. The corresponding fact for SG-semigroups follows from Lemma 1.2 and from the characterization of SG-semigroups given by Theorem 3.1 (ii).

Remark. We have shown that any congruence containing ζ is an ISBG-congruence. A consequence of this, which will be of use later, is that the intersection ρ of a non-empty family of ISBG-congruences ρ_i is again an ISBG-congruence; for each ρ_i contains ζ and so ρ contains ζ.

It is important to notice that a homomorphic image of an RU-semigroup need not be an RU-semigroup. Consider for example the USG-semigroup S which is the disjoint union of two non-trivial isomorphic groups G_1 and G_0, the semilattice Y being $\{0, 1\}$ and $\phi_{1,0}$ being any isomorphism from G_1 onto G_0. Then G_0 is a two-sided ideal of S and so an obvious homomorphic image of S is S/G_0, the Rees factor semigroup of S modulo G_0, defined in [**18**] (see also [**4**, § 1.5]). It is immediate that S/G_0 is not an RU-semigroup, since it contains a zero.

It is thus not immediately obvious that on an RU-semigroup the minimum SG-congruence and the minimum USG-congruence will coincide. In fact they do, as the next theorem shows.

THEOREM 3.2. *If ξ is the minimum* SG-*congruence on an* RU-*semigroup S, then S/ξ is a* USG-*semigroup.*

Proof. Let $\mathfrak{X} = \{(ea, ae)\colon e \in E, a \in S\}$, where E is the subsemigroup of idempotents of S. Now, for every e in E and every a in S we have that $(e\xi)(a\xi) = (a\xi)(e\xi)$, since the idempotents of S/ξ are central. Hence $(ea, ae) \in \xi$. Thus $\mathfrak{X} \subseteq \xi$ and so $\mathfrak{X}^* \subseteq \xi$ since ξ is a congruence.

Conversely, by Lemma 1.2, every idempotent of $S/\mathfrak{X}^*$ is of the form $e\mathfrak{X}^*$, where e is an idempotent of S. Now clearly

$$(e\mathfrak{X}^*)(a\mathfrak{X}^*) = (a\mathfrak{X}^*)(e\mathfrak{X}^*)$$

for every e in E and a in S. Thus the idempotents of $S/\mathfrak{X}^*$ are central and so $S/\mathfrak{X}^*$ is an SG-semigroup. Hence $\mathfrak{X}^* \supseteq \xi$ and, combining this with the conclusion of the previous paragraph, we obtain that $\mathfrak{X}^* = \xi$.

Now, if a and b are elements of S, then $(a, b) \in \mathfrak{X}^*$ if and only if b can be obtained from a by a finite sequence of *elementary $\mathfrak{X}$-transitions* (in the sense of Clifford and Preston [**4**, p. 18]). The proof of Theorem 3.2 depends on the following lemma.

LEMMA 3.3. *If S is an* RU-*semigroup, then any element obtained from an idempotent of S by means of an elementary $\mathfrak{X}$-transition is itself idempotent.*

Proof. It will be sufficient to consider the case where f, an idempotent of S, is transformed by an elementary $\mathfrak{X}$-transition thus:

$$f = peaq \rightarrow paeq,$$

where $p, q \in S$. The cases where one or both of p and q are absent, or where $paeq$ changes to $peaq$, can all be dealt with in a similar way.

We have that $peaq \in E$. Using Lemma 2.2 and the hypothesis that E is a unitary subsemigroup, we successively deduce that $eaqp \in E$, $aqp \in E$, $qpa \in E$, $qpae \in E$ and $paeq \in E$. Thus the lemma is proved.

Returning to the proof of Theorem 3.2, we note again that every idempotent of S/ξ can be expressed as $e\xi$, where e is an idempotent of S. Suppose therefore that $(e\xi)(a\xi) = f\xi$, where $a \in S$ and e, f are idempotents; if we can show that $a\xi$ is an idempotent of S/ξ, then the desired result will follow by Lemma 2.1.

Since $\xi = \mathfrak{X}^*$, there is a finite sequence

$$f \to \ldots \to ea$$

of elementary $\mathfrak{X}$-transitions connecting f to ea. Applying Lemma 3.3 repeatedly to this sequence, we find that ea is an idempotent of S. But by hypothesis S is an RU-semigroup and so it follows that a is an idempotent of S. Thus certainly $a\xi$ is an idempotent of S/ξ, which was what we required.

An analogous result holds for ISBG-congruences as follows.

THEOREM 3.4. *If ζ is the minimum* ISBG-*congruence on an* RU-*semigroup S, then S/ζ is a* UBG-*semigroup.*

Proof. We must show that the set of idempotents of S/ζ is left unitary. If e and f are idempotents of S and $(e\zeta)(a\zeta) = (f\zeta)$, then certainly $(ea, f) \in \xi$ (for $\zeta \subseteq \xi$) and so, by Lemma 3.3, ea is idempotent. It follows that a is an idempotent of S and so certainly $a\zeta$ is an idempotent of S/ζ. This completes the proof.

Now let S be an arbitrary regular semigroup. It is clear that the congruence $\zeta \vee \kappa$ on S is an ISBG-congruence, since it contains ζ. It is not, however, immediately obvious that $\zeta \vee \kappa$ is an RU-congruence. This is in fact the case, and will follow from the next lemma.

LEMMA 3.5. *The minimum* ISBG-*congruence on S/κ is $(\zeta \vee \kappa)/\kappa$.*

Proof. We have already observed that $S/(\zeta \vee \kappa)$ is an ISBG-semigroup. Hence $(\zeta \vee \kappa)/\kappa$ is an ISBG-congruence on S/κ, since

$$S/(\zeta \vee \kappa) \simeq (S/\kappa)/((\zeta \vee \kappa)/\kappa). \tag{2}$$

Suppose now that ρ' is an ISBG-congruence on S/κ. Then $\rho' = \rho/\kappa$, where ρ is an ISBG-congruence on S containing κ. Thus $\rho \supseteq \zeta$ and $\rho \supseteq \kappa$ and hence $\rho \supseteq \zeta \vee \kappa$. Thus $\rho' \supseteq (\zeta \vee \kappa)/\kappa$, so that $(\zeta \vee \kappa)/\kappa$ is indeed the minimum ISBG-congruence on S/κ.

It now follows by Theorem 3.4 that $(S/\kappa)/((\zeta \vee \kappa)/\kappa)$ is a UBG-semigroup. Hence $S/(\zeta \vee \kappa)$ is a UBG-semigroup by virtue of (2). Thus we have

THEOREM 3.6. *Let S be a regular semigroup and let ζ and κ be respectively the minimum* ISBG-*congruence and the minimum* RU-*congruence on S. Then $\zeta \vee \kappa$ is the minimum* UBG-*congruence on S.*

A similar argument, depending on Theorem 3.2 rather than Theorem 3.4, gives us

THEOREM 3.7. *Let S be a regular semigroup and let ξ and κ be respectively the minimum* SG-*congruence and the minimum* RU-*congruence on S. Then $\xi \vee \kappa$ is the minimum* USG-*congruence on S.*

Comparing these last two theorems with Corollaries 2.7 and 2.9, we obtain

THEOREM 3.8. *Let S be a regular semigroup and let σ, β, η, ζ, ξ and κ be defined as above. Then $\sigma \cap \beta = \zeta \vee \kappa$ and $\sigma \cap \eta = \xi \vee \kappa$.*

Our results are substantially simplified if we restrict our attention to RU-semigroups (for which κ is the identical congruence); we obtain $\sigma \cap \beta = \zeta$ and $\sigma \cap \eta = \xi$. For such semigroups we also have

THEOREM 3.9. *In an* RU-*semigroup* S, *the intersection* $\sigma \cap \mathscr{H}$ *of the minimum group congruence* σ *with Green's relation* $\mathscr{H}$ *is the identical congruence* ι_S.

Proof. First, by Lemma 2.6, the set E of idempotents of S is a σ-class. Hence, if $(x, e) \in \sigma \cap \mathscr{H}$, then $x = e$; for $x \in E$ by the preceding remark, and then $x = e$ since $\mathscr{H}$ is idempotent-separating [**4**, Lemma 2.15].

If $(a, b) \in \sigma \cap \mathscr{H}$ and a' is any inverse of a, then $(aa', a) \in \mathscr{R}$. By Green's Lemma [**4**, Lemma 2.2], $x \to xa'$ and $y \to ya$ are mutually inverse one-to-one mappings of H_a onto $H_{aa'}$ and $H_{aa'}$ onto H_a respectively. In particular $ba' \in H_{aa'}$ and so $(aa', ba') \in \mathscr{H}$. Since σ is a congruence we certainly have $(aa', ba') \in \sigma$; hence $(aa', ba') \in \sigma \cap \mathscr{H}$. By the remark in the previous paragraph we therefore have

$$aa' = ba'. \tag{3}$$

The next phase in the argument has some claim to independent interest and so is given as a lemma.

LEMMA 3.10. *Let S be an arbitrary regular semigroup. If $(a, b) \in \mathscr{H}$, then for every inverse a' of a there exists an inverse b' of b such that $a'a = b'b$.*

Proof. Let $(a, b) \in \mathscr{H}$ and let a' be an inverse of a. By an argument dual to that used in the proof of formula (3) we obtain

$$(a'a, a'b) \in \mathscr{H}. \tag{4}$$

By a theorem of Miller and Clifford [**12**] (see also [**4**, Theorem 2.18 (ii)]) there exists an inverse b' of b such that $(a', b') \in \mathscr{H}$. Repeating the argument that gave us formulae (3) and (4) we now obtain (for this particular b')

$$(a'b, b'b) \in \mathscr{H} \tag{5}$$

and so, by (4) and (5), we have that $(a'a, b'b) \in \mathscr{H}$. Since $\mathscr{H}$ is idempotent-separating [**4**, Lemma 2.15] we thus obtain that $a'a = b'b$ as required.

From (3) and Lemma 3.10 we now deduce that

$$a = aa'a = ba'a = bb'b = b.$$

Thus $\sigma \cap \mathscr{H} = \iota_S$, and so the proof of Theorem 3.9 is complete.

Since μ, the maximum idempotent-separating congruence on S, is contained in $\mathscr{H}$, and since $\zeta \subseteq \xi \subseteq \sigma$, we have the following corollary to Theorem 3.9.

COROLLARY 3.11. *In an* RU-*semigroup* S,

$$\xi \cap \mathscr{H} = \zeta \cap \mathscr{H} = \sigma \cap \mu = \xi \cap \mu = \zeta \cap \mu = \iota_S.$$

None of these results is true for regular semigroups in general. Consider for example a Brandt semigroup $S = \mathscr{M}^0(G;\ I, I;\ \Delta)$, where $|I| \geqq 2$. (For an explanation of the notation see [4, §§ 3.1, 3.3].) Then S is not a union of groups. Moreover, it has been shown by Preston [17] that any non-trivial homomorphic image of S is a Brandt semigroup $\mathscr{M}^0(G';\ I, I;\ \Delta)$, where G' is a homomorphic image of G; hence no non-trivial homomorphic image of S is a union of groups. It follows that $\sigma = \xi = \zeta = S \times S$. Also, it is easy to show that $\mu = \mathscr{H}$ and that $((a)_{ij}\ (b)_{kl}) \in \mathscr{H}$ if and only if $i = k$ and $j = l$. Thus, if the group G contains more than one element, we have

$$\sigma \cap \mathscr{H} = \xi \cap \mathscr{H} = \zeta \cap \mathscr{H} = \sigma \cap \mu = \xi \cap \mu = \zeta \cap \mu = \mathscr{H} \neq \iota_S.$$

4. UBG-congruences. In this final section we establish the result:

THEOREM 4.1. *If S is a regular semigroup, then the intersection of a group congruence τ and a band congruence γ is a* UBG-*congruence on S. Conversely, any* UBG-*congruence ρ on S can be expressed in this way, and τ and γ are uniquely determined by ρ.*

Proof. We have already observed (at the beginning of § 3) that the intersection of a family of ISBG-congruences [RU-congruences] is another ISBG-congruence [RU-congruence]. Hence the intersection of a family of UBG-congruences is a UBG-congruence. Now, if τ is a group congruence and γ a band congruence, then certainly both τ and γ are UBG-congruences, for the class of UBG-semigroups contains both the class of groups and the class of bands. Hence $\tau \cap \gamma$ is a UBG-congruence.

Conversely, let ρ be a UBG-congruence on S, and let τ' and γ' be respectively the minimum group congruence and the minimum band congruence on S/ρ. Since S/ρ is a UBG-semigroup we must have that $\tau' \cap \gamma' = \iota_{S/\rho}$ by Theorem 2.5. Now $\tau' = \tau/\rho$, where τ is a group congruence on S containing ρ; and similarly $\gamma' = \gamma/\rho$, where γ is a band congruence on S containing ρ. We thus have that $(\tau/\rho) \cap (\gamma/\rho) = \iota_{S/\rho}$, from which it follows that $\tau \cap \gamma = \rho$ as required.

It remains to show that τ and γ are uniquely determined by ρ. Suppose that $\tau_1 \cap \gamma_1 = \tau_2 \cap \gamma_2$, where τ_1, τ_2 are group congruences and γ_1, γ_2 are band congruences. Then $\gamma_1 \cap \gamma_2$ is a band congruence and, by Lemma 1.2, there exists for every a in S an idempotent e such that $(a, e) \in \gamma_1 \cap \gamma_2$. Suppose now that $(a, b) \in \gamma_1$. There exist idempotents e, f such that

$$(a, e) \in \gamma_1 \cap \gamma_2, \quad (b, f) \in \gamma_1 \cap \gamma_2. \tag{6}$$

Hence $(e, f) \in \gamma_1$ and since τ_1 is a group congruence we certainly have that $(e, f) \in \tau_1$. Thus

$$(e, f) \in \tau_1 \cap \gamma_1 = \tau_2 \cap \gamma_2 \subseteq \gamma_2$$

and it follows from (6) that $(a, b) \in \gamma_2$. Thus $\gamma_1 \subseteq \gamma_2$, and by symmetry we also have $\gamma_1 \supseteq \gamma_2$.

We may now assume that $\gamma_1 = \gamma_2 = \gamma$, so that $\tau_1 \cap \gamma = \tau_2 \cap \gamma$. If $(a, b) \in \tau_1$, then

$$(a \cdot a \cdot ba,\ a \cdot b \cdot ba) \in \tau_1.$$

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Since S/γ is a band,

$$(a\gamma)(a\gamma)(b\gamma)(a\gamma) = (a\gamma)(b\gamma)(b\gamma)(a\gamma) \; (= (a\gamma)(b\gamma)(a\gamma))$$

and so $(aaba, abba) \in \gamma$. Thus

$$(aaba, abba) \in \tau_1 \cap \gamma = \tau_2 \cap \gamma \subseteq \tau_2.$$

Since τ_2 is a congruence, it follows that $(eaf, ebf) \in \tau_2$, where $e = a'a$ and $f = (ba)(ba)'$. Now $e\tau_2$ and $f\tau_2$ are both equal to the identity element of the group S/τ_2 and so

$$a\tau_2 = (e\tau_2)(a\tau_2)(f\tau_2) = (eaf)\tau_2 = (ebf)\tau_2 = b\tau_2;$$

that is, $(a, b) \in \tau_2$. Thus $\tau_1 \subseteq \tau_2$, and by symmetry we also have $\tau_1 \subseteq \tau_2$. This completes the proof.

Finally, we have

THEOREM 4.2. *If S is a regular semigroup, then the intersection of a group congruence τ and a semilattice congruence ε is a* USG-*congruence on S. Conversely, any* USG-*congruence ρ can be expressed in this way, and τ and ε are uniquely determined by ρ.*

The proof is closely similar to that for the previous theorem and is omitted. The uniqueness of τ and ε is in fact a corollary of Theorem 4.1.

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