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CRITERIA FOR THE BOUNDEDNESS AND COMPACTNESS OF INTEGRAL TRANSFORMS WITH POSITIVE KERNELS

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Abstract The necessary and sufficient conditions that guarantee the boundedness and compactness of integral operators with positive kernels from $L^p(a,b)$ to $L^q_\nu(a,b)$, where $p,q \in (1,\infty)$ or $0 < q \leqslant 1 < p < \infty$, for a non-negative Borel measure ν on (a,b) are found.

Keywords: boundedness; compactness; weight; operators with positive kernels; measure of non-compactness

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1. Introduction

In the present work we find the necessary and sufficient conditions for the boundedness and compactness of the operator

$$K(f)(x) = \int_a^x k(x,y) f(y) \, \mathrm{d}y$$

from $L^p(a,b)$ to $L^q_\nu(a,b)$ ($p,q \in (1,\infty)$ or $0 < q \leqslant 1 < p < \infty$, $-\infty < a < b \leqslant \infty$ and ν is a non-negative σ-finite Borel measure on (a,b)).

Analogous problems for the Riemann–Liouville type operator

$$R_\alpha f(x) = \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}} \, \mathrm{d}y,$$

with $a = 0$, $b = +\infty$, $p,q \in (1,\infty)$ and $\alpha > 1/p$ are solved in [**13, 14**] (for the case where $p = q = 2$ and ν is absolute continuous see [**15**]). For the boundedness and compactness criteria of operators with power-logarithmic kernels

$$I_{\alpha,\beta}(f)(y) = \int_0^x (x-y)^{\alpha-1} \ln^\beta \left(\frac{\gamma}{x-y} \right) f(y) \, \mathrm{d}y$$

with $0 < b \leqslant \gamma < \infty$, $\alpha > 1/p$ and $\beta \geqslant 0$ see [**10**].

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A complete description of the weight pairs (v, w), which guarantee the boundedness of the operators with positive kernels from L^p_w to L^q_v when $1 < p < q < \infty$, is given in [**6**] (see also [**7**, Chapter 3]).

Two-weight criteria for the boundedness of the operator R_α from $L^p_w(0,\infty)$ to $L^q_v(0,\infty)$ for $\alpha > 1$ were found in [**11**] for $1 < p \leqslant q < \infty$ and in [**19**] for $1 < p, q < \infty$. An analogous problem for the Hardy operator,

$$Hf(x) = \int_0^x f(t)\,\mathrm{d}t,$$

was solved in [**2**, **9**, **12**] for $1 < p \leqslant q < \infty$, and in [**12**] for $1 < q < p < \infty$.

In the non-compact case we give the upper and the lower bound for the distance of K from the subspace of compact operators from $L^p(a,b)$ to $L^q_v(a,b)$ when $1 < p \leqslant q < \infty$.

2. Preliminaries

Let ν be a non-negative σ-finite Borel measure on (a,b). Denote by $L^q_\nu(a,b)$ $(0 < q < \infty)$ a class of all ν-measurable functions $g : (a,b) \to \mathbb{R}^1$ for which

$$\|g\|_{L^q_\nu(a,b)} = \left(\int_{(a,b)} |g(x)|^q\,\mathrm{d}\nu\right)^{1/q} < \infty.$$

If ν is absolutely continuous (i.e. $\mathrm{d}\nu = v(x)\,\mathrm{d}x$, where v is a positive Lebesgue-measurable function on (a,b)), then the symbol $L^q_v(a,b)$ is used instead of $L^q_\nu(a,b)$. If ν is the Lebesgue measure, then we shall use the symbol $L^q(a,b)$.

The following lemma is known for the case $a = 0$ and $b = \infty$ (see [**12**, § 1.3]), but we give the proof in the case where $-\infty < a < b \leqslant +\infty$ for completeness.

Lemma 2.1. *Let $-\infty < a < b \leqslant +\infty$, $1 < p \leqslant q < \infty$ and let μ be a non-negative Borel measure on (a,b). The inequality*

$$\left(\int_{(a,b)} \left|\int_a^x f(y)\,\mathrm{d}y\right|^q \mathrm{d}\mu\right)^{1/q} \leqslant c\left(\int_a^b |f(y)|^p\,\mathrm{d}y\right)^{1/p}, \tag{2.1}$$

where the positive constant c does not depend on f, holds if and only if

$$A = \sup_{a<t<b} (\mu([t,b)))^{1/q}(t-a)^{1/p'} < \infty,$$

where $p' = p/(p-1)$. Moreover, if c is the best constant in (2.1), then $A \leqslant c \leqslant 4A$.

Proof. Let $f \geqslant 0$, $f \in L^p(a,b)$ and let

$$\int_a^b f(y)\,\mathrm{d}y \in (2^m, 2^{m+1}]$$

for some integer m. Denote

$$\int_a^x f(y)\,\mathrm{d}y \equiv I(x),$$

then for every $x \in (a,b)$ we have $I(x) \leqslant \|f\|_{L^p(a,b)}(x-a)^{1/p'} < \infty$. The function I is continuous on (a,b). Therefore, for every $k \in \mathbb{Z}$, with $k \leqslant m$, there exists t_k such that $2^k = I(t_k) = \int_{t_k}^{t_{k+1}} f(y)\,\mathrm{d}y$ for $k \leqslant m-1$ and $2^m = I(t_m)$.

It is easy to verify that the sequence $\{t_k\}$ is increasing. Let $\alpha = \lim_{k\to-\infty} t_k$. Then we have $(a,b) = (a,\alpha] \cup (\cup_{k\leqslant m} E_k)$, where $E_k = [t_k, t_{k+1})$ and $t_{m+1} = b$. When

$$\int_a^b f(y)\,\mathrm{d}y = \infty$$

we have $(a,b) = (a,\alpha] \cup (\cup_{k=-\infty}^{+\infty} E_k)$ (i.e. $m = +\infty$). If $t \in (a,\alpha)$, then $I(t) = 0$ and if $t \in E_k$, then $I(t) \leqslant I(t_{k+1}) \leqslant 2^{k+1}$.

We have

$$\begin{aligned}
&\biggl(\int_{(a,b)} \biggl(\int_a^x f(y)\,\mathrm{d}y\biggr)^q \mathrm{d}\mu\biggr)^{p/q} \\
&\qquad = \biggl(\sum_{k\leqslant m} \int_{E_k} (I(x))^q\,\mathrm{d}\mu\biggr)^{p/q} \\
&\qquad \leqslant \sum_{k\leqslant m} \biggl(\int_{E_k} (I(x))^q\,\mathrm{d}\mu\biggr)^{p/q} \leqslant \sum_{k\leqslant m} 2^{(k+1)p} \biggl(\int_{E_k} \mathrm{d}\mu\biggr)^{p/q} \\
&\qquad = 4^p \sum_{k\leqslant m} 2^{(k-1)p} (\mu(E_k))^{p/q} = 4^p \sum_{k\leqslant m} \biggl(\int_{t_{k-1}}^{t_k} f(y)\,\mathrm{d}y\biggr)^p (\mu(E_k))^{p/q} \\
&\qquad \leqslant 4^p \sum_{k\leqslant m} \biggl(\int_{t_{k-1}}^{t_k} (f(y))^p\,\mathrm{d}y\biggr) (t_k - t_{k-1})^{p-1} (\mu(E_k))^{p/q} \\
&\qquad \leqslant 4^p A^p \|f\|^p_{L^p(a,b)}.
\end{aligned}$$

To prove the necessity, we put $f(y) = \chi_{(a,t)}(y)$ in (2.1), where $t \in (a,b)$. Then we have $\|f\|_{L^p(a,b)} = (t-a)^{1/p}$. On the other hand,

$$\biggl(\int_{(a,b)} \biggl(\int_a^x f(y)\,\mathrm{d}y\biggr)^q \mathrm{d}\mu\biggr)^{1/q} \geqslant (\mu([t,b)))^{1/q}(t-a),$$

and consequently we obtain $A \leqslant c$. □

We also need the following lemma.

Lemma 2.2. *Let $-\infty < a < b \leqslant +\infty$, $0 < q < p < \infty$ and let $p > 1$. Then the inequality*

$$\biggl(\int_a^b \biggl|\int_a^x f(y)\,\mathrm{d}y\biggr|^q v(x)\,\mathrm{d}x\biggr)^{1/q} \leqslant c\biggl(\int_a^b |f(y)|^p\,\mathrm{d}y\biggr)^{1/p}, \tag{2.2}$$

where the positive constant c does not depend on f, is fulfilled if and only if

$$\bar{A} = \biggl(\int_a^b \biggl(\int_x^b v(t)\,\mathrm{d}t\biggr)^{p/(p-q)} (x-a)^{p(q-1)/(p-q)}\,\mathrm{d}x\biggr)^{(p-q)/pq} < \infty.$$

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Moreover, there exist positive constants c_1 and c_2 depending only on p and q such that if c is the best constant in (2.2), then

$$c_1\bar{A} \leqslant c \leqslant c_2\bar{A}.$$

This lemma can be proved in the same way as Lemma 1.3.2 of [**12**] (for the case $0<q<1<p<\infty$, see, for example, [**18**]).

We also need the following theorem, which can be obtained, for example, from Lemma 2 in Chapter XI of [**8**].

Theorem A. *Let $1<p,q<\infty$ and let $-\infty<a<b\leqslant+\infty$. Suppose that $T : L^p(a,b)\to L^q_\nu(a,b)$ is an integral operator of the type $Tf(x)=\int_a^b T_1(x,y)f(y)\,\mathrm{d}y$, where ν is a σ-finite, separable measure on (a,b) (i.e. $L^q_\nu(a,b)$ is separable). If*

$$\bar{A} = \| \|T_1(x,\cdot)\|_{L^{p'}(a,b)}\|_{L^q_\nu(a,b)} < \infty,$$

then the operator T is compact from $L^p(a,b)$ to $L^q_\nu(a,b)$.

Definition 2.3. Let $-\infty<a<b\leqslant+\infty$. A kernel $k : \{(x,y) : a<y<x<b\}\to(0,\infty)$ belongs to $V(k\in V)$ if there exists a positive constant d_1 such that for all x,y,z with $a<y<z<x<b$ the inequality

$$k(x,y)\leqslant d_1k(x,z)$$

holds.

Definition 2.4. Let $-\infty<a<b\leqslant+\infty$. We say that k belongs to $V_\lambda(k\in V_\lambda)$ $(1<\lambda<\infty)$ if there exists a positive constant d_2 such that for all $x,x\in(a,b)$ the inequality

$$\int_{a+(x-a)/2}^{x} k^{\lambda'}(x,y)\,\mathrm{d}y \leqslant d_2(x-a)k^{\lambda'}(x,a+(x-a)/2),$$

is fulfilled, where $\lambda'=\lambda/(\lambda-1)$.

Let k_1 be a positive measurable function on $(0,b-a)$ (if $b=\infty$, then we assume that $b-a=\infty$).

Definition 2.5. Let $-\infty<a<b\leqslant+\infty$. We say that k_1 belongs to $V_{1\lambda}(k_1\in V_{1\lambda})$ $(1<\lambda<\infty)$ if there exists a positive constant d_3 such that the inequality

$$\int_0^{(x-a)/2} k_1^{\lambda'}(y)\,\mathrm{d}y \leqslant d_3(x-a)k_1^{\lambda'}((x-a)/2), \quad \lambda'=\lambda/(\lambda-1),$$

is fulfilled for all $x,x\in(a,b)$.

It is easy to verify that if k_1 is a non-increasing function on $(0,b-a)$ and $k_1\in V_{1\lambda}$, then the kernel $k(x,y)\equiv k_1(x-y)$ belongs to $V\cap V_\lambda$.

Now we give some examples of kernels satisfying the above-mentioned conditions.

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Let $-\infty < a < b \leqslant +\infty$ and let $k(y) = y^{\alpha-1}$, where $\alpha > 0$. If $1 < \lambda < \infty$ and $1/\lambda < \alpha \leqslant 1$, then $k_1 \in V_{1\lambda}$, and, consequently, the kernel $k(x,y) \equiv k_1(x-y)$ belongs to $V \cap V_\lambda$.

Assume that $-\infty < a < b < +\infty$, $b - a \leqslant \gamma < \infty$, $1/\lambda < \alpha \leqslant 1$ and $\beta \geqslant 0$. Let $k_1(y) = y^{\alpha-1}\ln^\beta(\gamma/y)$. Then $k_1 \in V_{1\lambda}$ and, therefore, $k(x,y) \equiv k_1(x-y)$ belongs to $V \cap V_\lambda$.

Now suppose that $-\infty < a < b \leqslant +\infty$,

$$k(x,y) = (x-y)^{\alpha-1}\ln^{\beta-1}\left(\frac{x-a}{y-a}\right),$$

where $1/\lambda < \alpha \leqslant 1$ and $1 - \alpha + 1/\lambda < \beta \leqslant 1$. Then $k \in V \cap V_\lambda$.

Let $a = 0$, $0 < b \leqslant +\infty$ and let $k(x,y) = x^{-\sigma(\alpha+\eta)}(x^\sigma - y^\sigma)^{\alpha-1}y^{\sigma\eta+\sigma-1}$ be the Erdelyi–Kober kernel, where $\sigma > 0$ and $0 < \alpha \leqslant 1$. It easy to see that if $1/\lambda < \alpha \leqslant 1$ and $\eta > 1/\sigma - 1$, then $k \in V \cap V_\lambda$.

Some results about integral transforms with the above-mentioned kernels can be found in [**17**].

3. The boundedness criteria

In this section we find the boundedness criteria for the integral operators with positive kernels.

Theorem 3.1. *Let $-\infty < a < b \leqslant +\infty$. Suppose that $1 < p \leqslant q < \infty$ and $k \in V \cap V_p$. Then the operator K is bounded from $L^p(a,b)$ to $L^q_\nu(a,b)$ if and only if*

$$B \equiv \sup_{a<t<b}\left(\int_{[t,b)} k^q(x, a + (x-a)/2)\,\mathrm{d}\nu\right)^{1/q}(t-a)^{1/p'} < \infty.$$

Moreover, there exist positive constants b_1 and b_2 depending only on d_1, d_2, p and q such that the inequality

$$b_1 B \leqslant \|K\| \leqslant b_2 B$$

is fulfilled. (If the constants d_1 and d_2 from Definitions 2.3 and 2.4 do not depend on a and b, then the constants b_1, b_2 are independent of a and b.)

Proof. First we prove the theorem when $b = \infty$. Let $f \geqslant 0$. Then we have

$$\|Kf\|_{L^q_\nu(a,\infty)} \leqslant \left(\int_{(a,\infty)}\left(\int_a^{a+(x-a)/2} k(x,y)f(y)\,\mathrm{d}y\right)^q \mathrm{d}\nu\right)^{1/q} + \left(\int_{(a,\infty)}\left(\int_{a+(x-a)/2}^x k(x,y)f(y)\,\mathrm{d}y\right)^q \mathrm{d}\nu\right)^{1/q} \equiv I_1 + I_2.$$

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If $a < y < a + (x-a)/2$, then $k(x,y) \leqslant k(x, a + (x-a)/2)$, and, consequently, using Lemma 2.1, we obtain

$$\begin{aligned} I_1 &\leqslant c_1 \biggl(\int_{(a,\infty)} k^q(x, a + (x-a)/2) \biggl(\int_a^x f(y)\,\mathrm{d}y \biggr)^q \mathrm{d}\nu \biggr)^{1/q} \\ &\leqslant c_2 B \|f\|_{L^p(a,\infty)}. \end{aligned}$$

Using Hölder's inequality, the condition $k \in V_p$ and the notation $s_j \equiv a + 2^j$, we find that

$$\begin{aligned} I_2^q &\leqslant \int_{(a,\infty)} \biggl(\int_{a+(x-a)/2}^x (f(y))^p\,\mathrm{d}y \biggr)^{q/p} \biggl(\int_{a+(x-a)/2}^x k^{p'}(x,y)\,\mathrm{d}y \biggr)^{q/p'} \mathrm{d}\nu \\ &\leqslant c_3 \int_{(a,\infty)} \biggl(\int_{a+(x-a)/2}^x (f(y))^p\,\mathrm{d}y \biggr)^{q/p} (x-a)^{q/p'} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu \\ &\leqslant c_3 \sum_{j\in\mathbb{Z}} \int_{[s_j, s_{j+1})} \biggl(\int_{a+(x-a)/2}^x (f(y))^p\,\mathrm{d}y \biggr)^{q/p} (x-a)^{q/p'} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu \\ &\leqslant c_3 \sum_{j\in\mathbb{Z}} \biggl(\int_{s_{j-1}}^{s_{j+1}} (f(y))^p\,\mathrm{d}y \biggr)^{q/p} \int_{[s_j, s_{j+1})} (x-a)^{q/p'} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu \\ &\leqslant c_4 B^q \sum_{j\in\mathbb{Z}} \biggl(\int_{s_{j-1}}^{s_{j+1}} (f(y))^p\,\mathrm{d}y \biggr)^{q/p} \leqslant c_5 B^q \|f\|^q_{L^p(a,\infty)}. \end{aligned}$$

Now we prove the necessity. First we show that from the boundedness of the operator K the following condition can be obtained:

$$\tilde{B} \equiv \sup_{j\in\mathbb{Z}} \biggl(\int_{[s_j, s_{j+1})} k^q(x, a+(x-a)/2)(x-a)^{q/p'}\,\mathrm{d}\nu \biggr)^{1/q} < \infty. \tag{3.1}$$

Let $f_j(y) = \chi_{(a, s_{j+1})}(y)$, where $j \in \mathbb{Z}$. Then we have that

$$\begin{aligned} \|Kf_j\|_{L^q_\nu(a,\infty)} &\geqslant \biggl(\int_{[s_j, s_{j+1})} (Kf_j(x))^q\,\mathrm{d}\nu \biggr)^{1/q} \\ &\geqslant \biggl(\int_{[s_j, s_{j+1})} \biggl(\int_{a+(x-a)/2}^x f_j(y) k(x,y)\,\mathrm{d}y \biggr)^q \mathrm{d}\nu \biggr)^{1/q} \\ &\geqslant c_6 \biggl(\int_{[s_j, s_{j+1})} k^q(x, a+(x-a)/2)(x-a)^q\,\mathrm{d}\nu \biggr)^{1/q}. \end{aligned}$$

Consequently, using the boundedness of K, we obtain $\tilde{B} < \infty$. Now we show that $B \leqslant c_7 \tilde{B}$. Denote

$$\biggl(\int_{[t,\infty)} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu \biggr)^{1/q} (t-a)^{1/p'} \equiv B(t).$$

Let $t \in (a,\infty)$; then $t \in [s_m, s_{m+1})$ for some $m \in \mathbb{Z}$.

We have

$$B^q(t) \leqslant \biggl(\int_{[s_m,\infty)} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu\biggr) 2^{(m+1)q/p'}$$
$$= c_8 2^{mq/p'} \sum_{j=m}^{+\infty} \int_{[s_j,s_{j+1})} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu$$
$$\leqslant c_9 \tilde{B}^q 2^{mq/p'} \sum_{j=m}^{+\infty} 2^{-jq/p'} = c_{10}\tilde{B},$$

where c_{10} depends only on q and p.

The case $b \leqslant \infty$ can be proved analogously. In this case we take $s_j = a+(b-a)2^j$. (It is clear that $(a,b) = \cup_{j\leqslant 0}[s_{j-1}, s_j)$.) □

Remark 3.2. There exist positive constants a_1, a_2, a_3 and a_4 depending only on p and q such that

$$a_1 B \leqslant \tilde{B} \leqslant a_2 B$$

if $b = \infty$, where $\tilde{B}$ is from (3.1) and

$$a_3 B \leqslant \bar{B} \leqslant a_4 B$$

if $b < \infty$, where

$$\bar{B} = \sup_{j\leqslant 0}\biggl(\int_{[a+(b-a)2^{j-1},\,a+(b-a)2^j)} k^q(x, a+(x-b)/2)(x-a)^{q/p'}\,\mathrm{d}\nu\biggr)^{1/q}.$$

Indeed, let $b = \infty$. Then the inequality $a_1 B \leqslant \tilde{B}$ follows from the proof of Theorem 3.1. Moreover,

$$\biggl(\int_{[a+2^j,\,a+2^{j+1})} k^q(x, a+(x-a)/2)(x-a)^{q/p'}\,\mathrm{d}\nu\biggr)^{1/q}$$
$$\leqslant c_1\biggl(\int_{[a+2^j,\,a+2^{j+1})} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu\biggr)^{1/q} 2^{j/p'} \leqslant c_1 B$$

for every $j \in \mathbb{Z}$. Consequently, $\tilde{B} \leqslant a_2 B$, where a_2 depends only on p and q. We have an analogous result for $\bar{B}$.

Let g be a ν-measurable positive function on (a,b) and let

$$K'g(y) = \int_y^b k(x,y)g(x)\,\mathrm{d}\nu,$$

where $y \in (a,b)$.

From the duality arguments we can derive the following result.

Theorem 3.3. *Let $-\infty < a < b \leqslant +\infty$ and let $1 < p \leqslant q < \infty$. Suppose that $k \in V \cap V_{q'}$. Then the operator K' is bounded from $L^p_\nu(a,b)$ to $L^q(a,b)$ if and only if*

$$B' = \sup_{a<t<b} \biggl(\int_{[t,b)} k^{p'}(x, a + (x-a)/2) \, \mathrm{d}\nu \biggr)^{1/p'} (t-a)^{1/q} < \infty.$$

Moreover, there exist positive constants b_1 and b_2 depending only on d_1, d_2, p and q such that

$$b_1 B' \leqslant \|K'\| \leqslant b_2 B'.$$

Now we consider the case $q < p$. We shall assume that v and w are Lebesgue-measurable, a.e. positive functions on (a,b).

Theorem 3.4. *Let $-\infty < a < b \leqslant +\infty$, $0 < q < p < \infty$ and let $p > 1$. Suppose that $k \in V \cap V_p$. Then the operator K is bounded from $L^p(a,b)$ to $L^q_v(a,b)$ if and only if*

$$B_1 = \biggl(\int_a^b \biggl(\int_x^b k^q(t, a + (t-a)/2) v(t) \, \mathrm{d}t \biggr)^{p/(p-q)} (x-a)^{p(q-1)/(p-q)} \, \mathrm{d}x \biggr)^{(p-q)/pq} < \infty.$$

Moreover, there exist positive constants b_1 and b_2 such that

$$b_1 B_1 \leqslant \|K\| \leqslant b_2 B_1.$$

Proof. We prove the theorem when $b = \infty$. The case $b < \infty$ can be proved similarly. Let $f \geqslant 0$. Then we have

$$\begin{aligned} \|Kf\|^q_{L^q_v(a,\infty)} \leqslant c_1 \int_a^\infty \biggl(\int_a^{a+(x-a)/2} f(y) k(x,y) \, \mathrm{d}y \biggr)^q v(x) \, \mathrm{d}x \\ + c_1 \int_a^\infty \biggl(\int_{a+(x-a)/2}^x f(y) k(x,y) \, \mathrm{d}y \biggr)^q v(x) \, \mathrm{d}x = \bar{I}_1 + \bar{I}_2. \end{aligned}$$

Using Lemma 2.2, we obtain $\bar{I}_1 \leqslant c_2 B_1^q \|f\|^q_{L^p(a,\infty)}$, where c_2 depends only on p, q and d_1. By Hölder's inequality and the condition $k \in V_p$ we find that

$$\begin{aligned} \bar{I}_2 &\leqslant c_3 \int_a^\infty \biggl(\int_{a+(x-a)/2}^x (f(y))^p \, \mathrm{d}y \biggr)^{q/p} (x-a)^{q/p'} k^q(x, a + (x-a)/2) v(x) \, \mathrm{d}x \\ &= c_3 \sum_{j \in \mathbb{Z}} \int_{s_j}^{s_{j+1}} \biggl(\int_{a+(x-a)/2}^x (f(y))^p \, \mathrm{d}y \biggr)^{q/p} (x-a)^{q/p'} k^q(x, a + (x-a)/2) v(x) \, \mathrm{d}x \\ &\leqslant c_3 \sum_{j \in \mathbb{Z}} \biggl(\int_{s_{j-1}}^{s_{j+1}} (f(y))^p \, \mathrm{d}y \biggr)^{q/p} \int_{s_j}^{s_{j+1}} (x-a)^{q/p'} k^q(x, a + (x-a)/2) v(x) \, \mathrm{d}x, \end{aligned}$$

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where $s_j = a + 2^j$. Using Hölder's inequality again, we have

$$\begin{aligned}\bar{I}_2 &\leqslant c_3\biggl(\sum_{j\in\mathbb{Z}}\int_{s_{j-1}}^{s_{j+1}}(f(y))^p\,\mathrm{d}y\biggr)^{q/p}\\&\qquad\times\biggl(\sum_{j\in\mathbb{Z}}\biggl(\int_{s_j}^{s_{j+1}}(x-a)^{q/p'}k^q(x,a+(x-a)/2)v(x)\,\mathrm{d}x\biggr)^{p/(p-q)}\biggr)^{(p-q)/p}\\&\leqslant c_4\bar{B}_1^q\|f\|^q_{L^p(a,\infty)},\end{aligned}$$

where

$$\bar{B}_1 \equiv \biggl(\sum_{j\in\mathbb{Z}}\biggl(\int_{s_j}^{s_{j+1}}(x-a)^{q/p'}k^q(x,a+(x-a)/2)v(x)\,\mathrm{d}x\biggr)^{p/(p-q)}\biggr)^{(p-q)/pq}.$$

Moreover,

$$\begin{aligned}&\bar{B}_1^{pq/(p-q)}\\&\qquad\leqslant c_5\sum_{j\in\mathbb{Z}}2^{jq(p-1)/(p-q)}\biggl(\int_{s_j}^{s_j+1}k^q(x,a+(x-a)/2)v(x)\,\mathrm{d}x\biggr)^{p/(p-q)}\\&\qquad\leqslant c_5\sum_{j\in\mathbb{Z}}\int_{s_{j-1}}^{s_j}(y-a)^{p(q-1)/(p-q)}\biggl(\int_y^{s_{j+1}}k^q(x,a+(x-a)/2)v(x)\,\mathrm{d}x\biggr)^{p/(p-q)}\,\mathrm{d}y\\&\qquad\leqslant c_5\int_a^\infty(y-a)^{p(q-1)/(p-q)}\biggl(\int_y^\infty k^q(x,a+(x-a)/2)v(x)\,\mathrm{d}x\biggr)^{p/(p-q)}\,\mathrm{d}y\\&\qquad= c_5B_1^{pq/(p-q)}.\end{aligned}$$

Consequently, $\bar{I}_2 \leqslant c_6B_1^q\|f\|^q_{L^p(a,\infty)}$, where the positive constant c_6 depends only on d_2, p and q.

Now we prove the necessity. Let the operator K be bounded from $L^p(a,\infty)$ to $L^q_v(a,\infty)$. If we repeat the arguments used in the proof of Theorem 3.1, then we can obtain that, for every $x \in (a,\infty)$,

$$\int_x^\infty v(t)k^q(t,a+(t-a)/2)\,\mathrm{d}t < \infty.$$

Let $v_n(t) = v(t)\chi_{(a+1/n,a+n)}(t)$, where n is an integer with $n \geqslant 2$. Suppose that

$$f_n(x) = \biggl(\int_x^\infty v_n(t)k^q(t,a+(t-a)/2)\,\mathrm{d}t\biggr)^{1/(p-q)}(x-a)^{(q-1)/(p-q)}.$$

Then, by integration by parts we obtain

$$\begin{aligned}\|f_n\|_{L^p(a,\infty)} &= \left(\int_a^\infty \left(\int_x^\infty v_n(t)k^q(t,a+(t-a)/2)\,\mathrm{d}t\right)^{p/(p-q)} (x-a)^{(q-1)p/(p-q)}\,\mathrm{d}x\right)^{1/p}\\ &= c_7\left(\int_a^\infty \left(\int_x^\infty v_n(t)k^q(t,a+(t-a)/2)\,\mathrm{d}t\right)^{q/(p-q)}\right.\\ &\qquad \left.\times (x-a)^{(p-1)q/(p-q)} v_n(x)k^q(x,a+(x-a)/2)\,\mathrm{d}x\right)^{1/p} < \infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\|Kf_n\|_{L^q_v(a,\infty)}\\ &\quad\geqslant c_8\left(\int_a^\infty v(x)\left(\int_{a+(x-a)/2}^x f_n(t)k(x,t)\,\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q}\\ &\quad\geqslant c_9\left(\int_a^\infty v_n(x)k^q(x,a+(x-a)/2)\left(\int_x^\infty v_n(t)k^q(x,a+(t-a)/2)\,\mathrm{d}t\right)^{q/(p-q)}\right.\\ &\qquad\left.\times\left(\int_{a+(x-a)/2}^x (t-a)^{(q-1)/(p-q)}\,\mathrm{d}t\right)^q \mathrm{d}x\right)^{1/q}\\ &\quad\geqslant c_{10}\left(\int_a^\infty v_n(x)k^q(x,a+(x-a)/2)\right.\\ &\qquad\left.\times\left(\int_x^\infty v_n(t)k^q(t,a+(t-a)/2)\,\mathrm{d}t\right)^{q/(p-q)} (x-a)^{(p-1)q/(p-q)}\,\mathrm{d}x\right)^{1/q}\\ &\quad= c_{11}\left(\int_a^\infty \left(\int_x^\infty v_n(t)k^q(t,a+(t-a)/2)\,\mathrm{d}t\right)^{p/(p-q)} (x-a)^{p(q-1)/(p-q)}\,\mathrm{d}x\right)^{1/q}.\end{aligned}$$

From the boundedness of the operator K we get

$$\left(\int_a^\infty \left(\int_x^\infty k^q(t,a+(t-a)/2)v_n(t)\,\mathrm{d}t\right)^{p/(p-q)} (x-a)^{p(q-1)/(p-q)}\,\mathrm{d}x\right)^{(p-q)/pq} \leqslant c,$$

where the positive constant c does not depend on n. By Fatou's Lemma we finally obtain $B_1 < \infty$. □

Now let

$$\tilde{K}f(x) = \int_x^b f(y)k(y,x)w(y)\,\mathrm{d}y,$$

where w is a Lebesgue-measurable a.e. positive function on (a,b). From the duality arguments and from Theorem 3.4 we obtain the following theorem.

Theorem 3.5. *Let* $-\infty < a < b \leqslant +\infty$ *and let* $1 < q < p < \infty$. *Suppose that* $k \in V \cap V_{q'}$. *Then the operator* $\tilde{K}$ *is bounded from* $L^p_w(a,b)$ *to* $L^q(a,b)$ *if and only if*

$$\tilde{B}_1 = \left(\int_a^b \left(\int_x^b k^{p'}(t,(t-a)/2)w(t)\,\mathrm{d}t\right)^{q(p-1)/(p-q)} (x-a)^{q/(p-q)}\,\mathrm{d}x\right)^{(p-q)/pq} < \infty.$$

Moreover, there exist positive constants $\tilde{b}_1$ *and* $\tilde{b}_2$ *such that*

$$\tilde{b}_1 \tilde{B}_1 \leqslant \|\tilde{K}\| \leqslant \tilde{b}_2 \tilde{B}_1.$$

4. The compactness criteria

In this section we investigate the compactness of the operators K and K'. The following theorem is true.

Theorem 4.1. *Let* $-\infty < a < b < +\infty$, $1 < p \leqslant q < \infty$ *and let* $k \in V \cap V_p$. *Suppose that* ν *is a separable measure (i.e.* $L^q_\nu(a,b)$ *is a separable space). Then the following statements are equivalent:*

(i) *the operator* K *is compact from* $L^p(a,b)$ *to* $L^q_\nu(a,b)$*;*

(ii) $B < \infty$ *and* $\lim_{c\to a+} B_c = 0$, *where*

$$B_c \equiv \sup_{a<t<c} \left(\int_{[t,c)} k^q(x, a + (x-a)/2) \, \mathrm{d}\nu \right)^{1/q} (t-a)^{1/p'};$$

(iii) $\bar{B} < \infty$ *and* $\lim_{j\to-\infty} \bar{B}(j) = 0$, *where*

$$\bar{B}(j) = \left(\int_{[s_{j-1}, s_j)} k^q(x, a + (x-a)/2)(x-a)^{q/p'} \, \mathrm{d}\nu \right)^{1/q}$$

and $s_j = a + (b-a)2^j$.

Proof. First we prove that (ii) implies (i). Let $c \in (a,b)$ and represent K as follows:

$$K = \chi_{(a,c)} K + \chi_{[c,b)} K = P_{1c} + P_{2c}.$$

For P_{2c} we have

$$P_{2c} f(x) = \chi_{[c,b)}(x) \int_a^b T_1(x,y) \, \mathrm{d}y,$$

where $T_1(x,y) = k(x,y)$ when $a < y < x < b$ and $T_1(x,y) = 0$ if $a < x \leqslant y < b$. Consequently,

$$\begin{aligned} S &\equiv \int_{[c,b)} \left(\int_a^b (T_1(x,y))^{p'} \, \mathrm{d}y \right)^{q/p'} \mathrm{d}\nu \\ &= \int_{[c,b)} \left(\int_a^x (k(x,y))^{p'} \, \mathrm{d}y \right)^{q/p'} \mathrm{d}\nu \\ &\leqslant c_1 \int_{[c,b)} \left(\int_a^{a+(x-a)/2} (k(x,y))^{p'} \, \mathrm{d}y \right)^{q/p'} \mathrm{d}\nu \\ &\qquad + c_1 \int_{[c,b)} \left(\int_{a+(x-a)/2}^x (k(x,y))^{p'} \, \mathrm{d}y \right)^{q/p'} \mathrm{d}\nu \\ &\equiv S_1 + S_2. \end{aligned}$$

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If $a < y < a + (x-a)/2$, then $k(x,y) \leqslant d_1 k(x, a+(x-a)/2)$ and therefore we have

$$S_1 \leqslant c_2 \int_{[c,b)} k^q(x, a+(x-a)/2)((x-a)/2)^{q/p'}\,\mathrm{d}\nu$$
$$\leqslant c_2 \biggl(\int_{[c,b)} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu\biggr)((b-a)/2)^{q/p'} < \infty.$$

Using the condition $k \in V_p$, for S_2 we obtain

$$S_2 \leqslant c_3 \int_{[c,b)} k^q(x, a+(x-a)/2)(x-a)^{q/p'}\,\mathrm{d}\nu < \infty.$$

Finally, we have $S < \infty$ and, by Theorem A, we conclude that P_{2c} is compact. Moreover, by virtue of Theorem 3.1 we have $\|P_{1c}\| \leqslant c_4 B_c$, where the positive constant c_4 does not depend on c. Consequently,

$$\|K - P_{2c}\| \leqslant c_4 B_c \to 0$$

as $c \to a$ and the operator K is compact as a limit of compact operators. Now we prove that (i) implies (iii). Let $j \in \mathbb{Z}$, $j \leqslant 0$ and let

$$f_j(y) = \chi_{(a, a+(b-a)2^j)}(y)((b-a)2^j)^{-1/p}.$$

Then for $\varphi \in L^{p'}(a,b)$ we have

$$\biggl|\int_a^b f_j(y)\varphi(y)\,\mathrm{d}y\biggr| \leqslant \biggl(\int_a^{s_j} |f_j(y)|^p\,\mathrm{d}y\biggr)^{1/p}\biggl(\int_a^{s_j} |\varphi(y)|^{p'}\,\mathrm{d}y\biggr)^{1/p'}$$
$$= \biggl(\int_a^{s_j} |\varphi(y)|^{p'}\,\mathrm{d}y\biggr)^{1/p'} \to 0$$

as $j \to -\infty$ (here $s_j = a + (b-a)2^j$). On the other hand,

$$\|Kf_j\|_{L^q_\nu(a,b)} \geqslant \biggl(\int_{[s_{j-1},s_j)} (Kf_j(x))^q\,\mathrm{d}\nu\biggr)^{1/q}$$
$$\geqslant \biggl(\int_{[s_{j-1},s_j)} k^q(x, a+(x-a)/2)\biggl(\int_{a+(x-a)/2}^{x} f_j(y)\,\mathrm{d}y\biggr)^q \mathrm{d}\nu\biggr)^{1/q}$$
$$\geqslant c_5 \biggl(\int_{[s_{j-1},s_j)} k^q(x, a+(x-a)/2)(x-a)^q\,\mathrm{d}\nu\biggr)^{1/q}((b-a)2^j)^{-1/p}$$
$$\geqslant c_6 \bar{B}(j).$$

As a compact operator maps a weakly convergent sequence into a strongly convergent form, we have that $\lim_{j\to-\infty} \bar{B}(j) = 0$. The fact that $\bar{B} < \infty$ follows from Remark 3.2 and Theorem 3.1.

Now we prove that (ii) follows from (iii). Let $c \in (a,b)$. Then there exists an integer m with $m \leqslant 0$ such that $c \in [s_{m-1}, s_m)$. We have

$$B_c \leqslant \sup_{a<t<s_m} \biggl(\int_{[t,s_m)} k^q(x, a+(x-a)/2)\,\mathrm{d}\nu\biggr)^{1/q}(t-a)^{1/p'} = B_{s_m}.$$

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Denote

$$B_{s_m}(t) \equiv \left(\int_{[t,s_m)} k^q(x, a + (x-a)/2)\,\mathrm{d}\nu\right)^{1/q} (t-a)^{1/p'}.$$

Let $t \in (a, s_m)$, then $t \in [s_{n-1}, s_n)$ for some integer $n \leqslant m$. We obtain

$$\begin{aligned}
B^q_{s_m}(t) &\leqslant \left(\int_{[s_{n-1},s_m)} k^q(x, a + (x-a)/2)\,\mathrm{d}\nu\right)[(b-a)2^n]^{q/p'} \\
&= [(b-a)2^n]^{q/p'} \sum_{j=n}^{m} \int_{[s_{j-1},s_j)} k^q(x, a + (x-a)/2)\,\mathrm{d}\nu \\
&\leqslant c_7[(b-a)2^n]^{q/p'} \sum_{j=n}^{m} [(b-a)2^j]^{-q/p'} \\
&\qquad \times \int_{[s_{j-1},s_j)} k^q(x, a + (x-a)/2)(x-a)^{q/p'}\,\mathrm{d}\nu \\
&\leqslant c_7(\sup_{j\leqslant m} \bar{B}(j))^q [(b-a)2^n]^{q/p'} \sum_{j=n}^{m} [(b-a)2^j]^{-q/p'} \\
&\leqslant c_8(\sup_{j\leqslant m} \bar{B}(j))^q \equiv c_8 \bar{B}^q_m.
\end{aligned}$$

Consequently,

$$B_{s_m} \leqslant c_9 \bar{B}_m.$$

If $c \to a$, then $s_m \to a$. Therefore $\bar{B}_m \to 0$ as $\lim_{j\to-\infty} \bar{B}(j) = 0$. Finally, we get $\lim_{c\to a+} B_c = 0$. The condition $B < \infty$ follows from Remark 3.2. So we conclude that (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii). □

From the duality argument we obtain the following theorem.

Theorem 4.2. *Let $-\infty < a < b < +\infty$ and let $1 < p \leqslant q < \infty$. Suppose that ν is a separable measure (i.e. $L^{p'}_\nu(a, b)$ is separable) and $k \in V \cap V_{q'}$. Then the following statements are equivalent:*

(i) *the operator K' is compact from $L^p_\nu(a, b)$ to $L^q(a, b)$;*

(ii) *$B' < \infty$ and $\lim_{c\to a+} B'_c = 0$, where*

$$B'_c = \sup_{a<t<c} \left(\int_{[t,c)} k^{p'}(x, a + (x-a)/2)\,\mathrm{d}\nu\right)^{1/p'} (t-a)^{1/q};$$

(iii)

$$\bar{B}' \equiv \sup_{j\leqslant 0} \left(\int_{[s_{j-1},s_j)} k^{p'}(x, a + (x-a)/2)(x-a)^{p'/q}\,\mathrm{d}\nu\right)^{1/p'} < \infty$$

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and $\lim_{j\to-\infty} \bar{B}'(j) = 0$, *where*

$$\bar{B}'(j) = \left(\int_{[s_{j-1},s_j)} k^{p'}(x, a + (x-a)/2)(x-a)^{p'/q}\,\mathrm{d}\nu\right)^{1/p'}$$

and

$$s_j = a + (b-a)2^j.$$

Theorem 4.3. *Let* $-\infty < a < b \leqslant +\infty$, $0 < q < p < \infty$ *and let* $p > 1$. *Suppose that* $k \in V \cap V_p$. *Then the operator* K *is compact from* $L^p(a,b)$ *to* $L^q_v(a,b)$ *if and only if* $B_1 < \infty$.

Proof. The sufficiency of the theorem can be derived in the same way as in the proof of Theorem 4.1. (It also follows from the well-known Ando's Theorem [**1**].) Theorem 3.4 implies the necessity. □

The following theorem can be derived from Theorem 4.3.

Theorem 4.4. *Let* $-\infty < a < b \leqslant +\infty$ *and let* $1 < q < p < \infty$. *Suppose that* $k \in V \cap V_{q'}$. *Then the operator* $\tilde{K}$ *is compact from* $L^p_w(a,b)$ *to* $L^q(a,b)$ *if and only if* $\tilde{B}_1 < \infty$.

5. The measure of non-compactness

In the non-compact case it is useful to estimate the distance of the operator K from the space of compact operators.

Let X and Y be Banach function spaces. Denote by $\mathcal{B}(X,Y)$ the space of all linear bounded operators from X to Y. Let $\mathcal{K}(X,Y)$ be a class of all linear compact operators from X to Y. Suppose that $\mathcal{F}_r(X,Y)$ is a space of operators with finite rank.

We shall assume that v is a Lebesgue-measurable a.e. positive function on (a,b), where $-\infty < a < b \leqslant +\infty$.

The following lemma is true (see [**16**] and [**3**, Corollary V.5.4]).

Lemma 5.1. *Let* $1 \leqslant p < \infty$, $-\infty < a < b \leqslant +\infty$ *and let* $P \in \mathcal{B}(X,Y)$, *where* $Y = L^p_v(a,b)$. *Then*

$$\operatorname{dist}(P, \mathcal{K}(X,Y)) = \operatorname{dist}(P, \mathcal{F}_r(X,Y)).$$

We also need the following lemma (see [**16**] and [**3**, Lemma V.5.6]).

Lemma 5.2. *Let* $1 \leqslant p < \infty$, $-\infty < a < b \leqslant +\infty$ *and let* $Y = L^p_v(a,b)$. *Suppose that* $P \in \mathcal{F}_r(X,Y)$ *and* $\epsilon > 0$. *Then there exist* $T \in \mathcal{F}_r(X,Y)$ *and* $[\alpha,\beta] \subset (a,b)$ *such that*

$$\|P - T\| < \epsilon$$

and

$$\operatorname{supp} Tf \subset [\alpha,\beta]$$

for every $f \in X$.

Theorem 5.3. *Let $1 < p \leqslant q < \infty$, $-\infty < a < b < +\infty$ and let $k \in V \cap V_p$. Suppose that K is bounded from X to Y, where $X = L^p(a,b)$ and $Y = L^q_v(a,b)$. Then the inequality*

$$b_1 J \leqslant \operatorname{dist}(K, \mathcal{K}(X,Y)) \leqslant b_2 J \tag{5.1}$$

is fulfilled, where the positive constants b_1 and b_2 depend only on p, q, d_1 and d_2, $J = \lim_{c\to a+} R_c$ and

$$R_c = \sup_{a<t<c} \biggl(\int_t^c k^q(x, a + (x-a)/2) v(x)\,\mathrm{d}x \biggr)^{1/q} (t-a)^{1/p'}$$

(d_1 and d_2 are from Definitions 2.3 and 2.4).

Proof. As we know from the proof of Theorem 4.1,

$$\|K - \bar{P}_c\| \leqslant c_1 R_c,$$

where $\bar{P}_c$ is a compact operator for every c. From the last inequality we can obtain

$$\operatorname{dist}(K, \mathcal{K}(X,Y)) \leqslant c_1 J,$$

where c_1 depends only on p, q, d_1 and d_2. Now we show that

$$\operatorname{dist}(K, \mathcal{K}(X,Y)) \geqslant b_1 J. \tag{5.2}$$

Let $\lambda > \operatorname{dist}(K, \mathcal{K}(X,Y))$. Then by Lemma 5.1 there exists $P \in \mathcal{F}_r(X,Y)$ such that $\|K - P\| < \lambda$. On the other hand, using Lemma 5.2, for $\epsilon = (\lambda - \|K-P\|)/2$ there exist $T \in \mathcal{F}_r(X,Y)$ and $[\alpha,\beta] \subset (a,b)$ such that

$$\|P - T\| < \epsilon \tag{5.3}$$

and

$$\operatorname{supp} Tf \subset [\alpha,\beta]. \tag{5.4}$$

From (5.3) we obtain

$$\|Kf - Tf\|_Y \leqslant \lambda \|f\|_X$$

for every $f \in X$. Consequently, we have

$$\int_a^\alpha |Kf(x)|^q v(x)\,\mathrm{d}x + \int_\beta^b |Kf(x)|^q v(x)\,\mathrm{d}x \leqslant \lambda^q \|f\|_X^q \tag{5.5}$$

for every $f \in X$.

Let us choose $n \in \mathbb{Z}$ such that $a + (b-a)2^n < \alpha$. Assume that $j \in \mathbb{Z}$, $j \leqslant n$ and $f_j(y) = \chi_{(a,s_j)}(y)$, where $s_j = a + (b-a)2^j$. Then we obtain

$$\begin{aligned}\int_{s_{j-1}}^{s_j} |Kf_j(x)|^q v(x)\,\mathrm{d}x &\geqslant \int_{s_{j-1}}^{s_j} \biggl(\int_{a+(x-a)/2}^x k(x,y) f(y)\,\mathrm{d}y \biggr)^q v(x)\,\mathrm{d}x \\ &\geqslant c_2 \int_{s_{j-1}}^{s_j} k^q(x, a+(x-a)/2)(x-a)^q v(x)\,\mathrm{d}x.\end{aligned}$$

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On the other hand,

$$\|f_j\|_X^q = ((b-a)2^j)^{q/p},$$

and by (5.5) we find

$$c_3 R(j) \equiv c_3 \biggl(\int_{s_{j-1}}^{s_j} k^q(x, a + (x-a)/2)(x-a)^{q/p'} v(x)\,\mathrm{d}x \biggr)^{1/q} \leqslant \lambda$$

for every integer j, $j \leqslant n$. Consequently, $\sup_{j \leqslant n} R(j) \leqslant c_4 \lambda$ for every integer n with the condition $a + (b-a)2^n < \alpha$. Therefore we have

$$\lim_{n \to -\infty} \sup_{j \leqslant n} R(j) \leqslant c_4 \lambda.$$

Let $c \in (a, \alpha)$; then $c \in [s_{m-1}, s_m)$ for some $m = m(c)$, $m \in \mathbb{Z}$. We obtain (see the proof of Theorem 4.1)

$$R_c \leqslant c_5 \sup_{n \leqslant m} R(n) \equiv c_5 \bar{R}_m.$$

From the last inequality we have

$$\lim_{c \to a+} R_c \leqslant c_5 \lim_{m \to -\infty} \bar{R}_m \leqslant c_6 \lambda,$$

where c_6 does not depend on a and b. Finally, we obtain inequality (5.2) and consequently (5.1) is fulfilled. □

Now we give the estimate of measure of non-compactness for the Riemann–Liouville operator R_α. The following theorem is true.

Theorem 5.4. *Let $1 < p \leqslant q < \infty$ and let $\alpha > 1/p$. Suppose that R_α is bounded from X to Y, where $X = L^p(0,\infty)$, $Y = L^q_v(0,\infty)$. Then the inequality*

$$b_1 I \leqslant \operatorname{dist}(R_\alpha, \mathcal{K}(X,Y)) \leqslant b_2 I$$

holds, where $I = \lim_{c \to 0} I_c + \lim_{d \to \infty} I_d$,

$$I_c = \sup_{0<t<c} \biggl(\int_t^c \frac{v(x)}{x^{(1-\alpha)q}}\,\mathrm{d}x \biggr)^{1/q} t^{1/p'},$$

$$I_d = \sup_{t>d} \biggl(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}\,\mathrm{d}x \biggr)^{1/q} (t-d)^{1/p'},$$

and the positive constants b_1 and b_2 depend only on p, q and α.

Proof. If we repeat the arguments used in the proof of Theorem 5 in [**13**], then we can obtain

$$\operatorname{dist}(R_\alpha, \mathcal{K}(X,Y)) \leqslant b_2 I.$$

Now let $\lambda > \operatorname{dist}(R_\alpha, \mathcal{K}(X,Y))$. Then by Lemma 5.1 there exists $P \in \mathcal{F}_r(X,Y)$ such that $\|R_\alpha - P\| < \lambda$. By virtue of Lemma 5.2 for $\epsilon = (\lambda - \|R_\alpha - P\|)/2$ there are $T \in \mathcal{F}_r(X,Y)$ and $[\alpha,\beta] \subset (0,\infty)$ such that (5.3) and (5.4) hold. From (5.3) we obtain

$$\|R_\alpha f - Tf\|_Y \leqslant \lambda \|f\|_X \tag{5.6}$$

for every $f \in X$. Further, from (5.3), (5.4) and (5.6) we can obtain

$$\int_0^\alpha |R_\alpha f(x)|^q v(x)\,\mathrm{d}x + \int_\beta^\infty |R_\alpha f(x)|^q v(x)\,\mathrm{d}x \leqslant \lambda^q \|f\|^q_{L^p(0,\infty)}.$$

Let $d \geqslant \beta$ and let $t \in (d,\infty)$. Then for $f_t(y) = \chi_{(d/2,t/2)}(y)$ we have

$$\begin{aligned}\int_t^\infty |R_\alpha f_t(x)|^q v(x)\,\mathrm{d}x &\geqslant \int_t^\infty \biggl(\int_{d/2}^{t/2} \frac{f_t(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y\biggr)^q v(x)\,\mathrm{d}x \\ &\geqslant c_1 \biggl(\int_t^\infty x^{(\alpha-1)q} v(x)\,\mathrm{d}x\biggr)(t-d)^q.\end{aligned}$$

On the other hand,

$$\|f\|^q_{L^p(0,\infty)} = c_2 (t-d)^{q/p},$$

whence

$$\lambda \geqslant c_3 \biggl(\int_t^\infty x^{(\alpha-1)q} v(x)\,\mathrm{d}x\biggr)^{1/q} (t-d)^{1/p'}$$

for $t > d$. Consequently, $\lambda \geqslant c_3 I_d$ for every d, $d > \beta$. From the last inequality we have

$$c_3 \lim_{d\to\infty} I_d \leqslant \lambda.$$

As λ is an arbitrary number greater than $\operatorname{dist}(R_\alpha, \mathcal{K}(X,Y))$, we conclude that

$$c_3 \lim_{d\to\infty} I_d \leqslant \operatorname{dist}(R_\alpha, \mathcal{K}(X,Y)).$$

Analogously we can show that

$$c_4 \lim_{c\to 0} I_c \leqslant \operatorname{dist}(R_\alpha, \mathcal{K}(X,Y)).$$

Consequently,

$$b_1 I \leqslant \operatorname{dist}(R_\alpha, \mathcal{K}(X,Y)).$$

□

An analogous theorem with two weights for the Hardy operator is proved in [**5**], while the similar problem for the Riemann–Liouville transforms R_α with $\alpha > 1$ and for more general operators was solved in [**4**], [**16**].

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